HW3 Solutions

PHY 610 QFT, Spring 2015

1. (a) With $H = \frac{1}{2}P^2 + \frac{1}{2}\omega^2Q^2$, and canonical quantization $[Q, P] = i$, the Heisenberg equations of motion are

$$\begin{align*}
\dot{Q} &= \frac{i}{\hbar}[P^2, Q] = P, \\
\dot{P} &= \frac{i}{\hbar}\omega^2[Q^2, P] = -\omega^2 Q.
\end{align*}$$

Therefore $\ddot{Q} = -\omega^2 Q$, and solving the differential equation we obtain

$$\begin{align*}
Q(t) &= Q(0) \cos(\omega t) + \frac{1}{\omega} P(0) \sin(\omega t), \\
P(t) &= P(0) \cos(\omega t) - \omega Q(0) \sin(\omega t).
\end{align*}$$

(b) Recall that the standard definitions of the raising and lowering operators are $Q = (1/\sqrt{2\omega})(a + a^\dagger)$, $P = (i\sqrt{\omega/2})(a^\dagger - a)$, so substitution into the above result yields

$$\begin{align*}
Q(t) &= \frac{1}{\sqrt{2\omega}}(a^\dagger e^{i\omega t} + e^{-i\omega t}), \\
P(t) &= i\frac{\omega}{2}(a^\dagger e^{i\omega t} - ae^{-i\omega t}).
\end{align*}$$

(c) We are to verify the expression for the propagator, as well as Wick’s theorem, which was derived in the text using the path integral approach. For the propagator,

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{2\omega} \langle 0| (a^\dagger e^{i\omega t_1} + ae^{-i\omega t_1})(a^\dagger e^{i\omega t_2} + ae^{-i\omega t_2})|0\rangle \theta(t_1 - t_2) + (t_1 \leftrightarrow t_2)$$

$$= \frac{1}{2\omega} \langle 0| aa^\dagger |0\rangle e^{i\omega(t_2 - t_1)} \theta(t_1 - t_2) + (t_1 \leftrightarrow t_2)$$

$$= -iG(t_1 - t_2).$$

For Wick’s theorem, let us assume that $t_1 > t_2 > t_3 > t_4$, then the only nonzero contributions to the correlation function comes from the $aa^\dagger aa^\dagger$ and $aaa^\dagger a^\dagger$ terms,

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{(2\omega)^2} \left( e^{i\omega(-t_1+t_2-t_3+t_4)} \langle 0|aa^\dagger aa^\dagger|0\rangle + e^{i\omega(-t_1-t_2+t_3+t_4)} \langle 0|aaa^\dagger a^\dagger|0\rangle \right)$$

$$= \frac{1}{(2\omega)^2} \left( e^{i\omega(-t_1-t_2-t_3+t_4)} + 2e^{i\omega(-t_1-t_2+t_3+t_4)} \right)$$

$$= (-i)^2 (G(t_1 - t_2)G(t_3 - t_4) + G(t_1 - t_3)G(t_2 - t_4) + G(t_1 - t_4)G(t_2 - t_3)).$$

Now, for other orderings of $t_1, t_2, t_3, t_4$, we may simply relabel the times in the intermediate steps above. Since the final expression is independent of relabellings, the equality between the first and last expressions holds regardless of time orderings.

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2. We are to take the magnitude of (7.10). First, for real \( f(t) \), note that \( \tilde{f}(-E) = \tilde{f}(E)^* \), so

\[
\left| \langle 0|0 \rangle_f \right|^2 = \exp \left( \frac{i}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{\left| \tilde{f}(E) \right|^2}{-E^2 + \omega^2 - i\epsilon} \right)^2
\]

\[
= \exp \left( -\text{Im} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \left| \tilde{f}(E) \right|^2 \frac{\epsilon}{(-E^2 + \omega^2)^2 + \epsilon^2} \right)
\]

Now, notice that as \( \epsilon \to 0 \), \( \epsilon/((-E^2 + \omega^2)^2 + \epsilon^2) \) is an increasingly sharper function of \( E^2 \), peaked at \( E^2 = \omega^2 \). The area under the graph is

\[
\int dE^2 \frac{\epsilon}{(-E^2 + \omega^2)^2 + \epsilon^2} = \pi,
\]

independent of \( \epsilon \). This means that

\[
\lim_{\epsilon \to 0} \frac{\epsilon}{(-E^2 + \omega^2)^2 + \epsilon^2} = \pi \delta(-E^2 + \omega^2) = \frac{\pi}{2\omega} \left( \delta(E - \omega) + \delta(E + \omega) \right).
\]

We therefore have

\[
\left| \langle 0|0 \rangle_f \right|^2 = \exp \left( -\frac{\left| \tilde{f}(\omega) \right|^2}{2\omega} \right).
\]

3. In this problem, we use derive, from the amplitude, the famous Klein-Nishina formula for the cross section of Compton scattering.

(a) The Mandelstam variables are Lorentz scalars, so they are frame independent. It is easiest to work in the fixed target frame, with spatial axes chosen such that the initial momentum is along one axis and the collision occurs in one plane. Then the initial momenta are \( k_1 = (\omega, \omega, 0, 0) \) (photon), \( k_2 = (m, 0, 0, 0) \) (electron), and the final momenta are \( k_3 = (\omega', \omega' \cos \theta, \omega' \sin \theta, 0) \) (photon) and \( k_4 = k_1 + k_2 - k_3 \) (electron). Then

\[
s = -(k_1 + k_2)^2 = (m + \omega)^2 - \omega^2 = m^2 + 2m\omega,
\]

\[
u = -(k_2 - k_3)^2 = (m - \omega')^2 - \omega'^2 = m^2 - 2m\omega'.
\]

(b) This may be derived using the mass shell relation for \( k_4 \). Note that \( k_4 = (m + \omega - \omega', \omega - \omega' \cos \theta, -\omega' \sin \theta, 0) \), so

\[
(m + \omega - \omega')^2 - (\omega - \omega' \cos \theta)^2 - \omega'^2 \sin^2 \theta - m^2 = 0.
\]

Simplification yields the relation

\[
\cos \theta = 1 + m(\omega^{-1} - \omega'^{-1}).
\]
(c) This is (mostly) a straightforward but tedious calculation, and I will be brief. Following the procedure in the text, we write

\[ \frac{d\sigma}{d\Omega_{\text{FT}}} = \frac{dt}{d\Omega_{\text{FT}}} \frac{d\sigma}{dt}. \]

The first factor is given by (11.34),

\[ \frac{d\sigma}{dt} = \frac{1}{64\pi s} \left| k_1 \right|^2 |T|^2. \]

Note that \( \left| k_1 \right|^2 \) is to be computed in the center of mass frame, but Srednicki has derived a neat relation (11.9) between \( \left| k_1 \right|^2 \) in the center of mass and fixed target frames. Using that, \( s \left| k_1 \right|^2_{\text{CM}} = m^2 \left| k_1 \right|^2_{\text{FT}} = m^2 \omega^2 \). Meanwhile, use the results of parts (a) and (b) to eliminate \( s, u \) and \( m^2 \) in favor of \( \omega, \omega' \) and \( \theta \). This yields, after some simplification, \( \frac{d\sigma}{dt} = \frac{\pi\alpha^2}{2m^2\omega^2} \left( \frac{\omega'}{\omega} + \frac{\omega'}{\omega} - \sin^2 \theta \right) \).

Next, for the second factor, we differentiate \( t = 2m^2 - s - u = 2m(\omega' - \omega) \) with respect to \( \theta \) (which is related to the solid angle by \( d\Omega_{\text{CM}} = 2\pi d\cos \theta \)), with \( s \) fixed (ie. \( \omega \) fixed). This yields

\[ \frac{dt}{d\cos \theta} = 2m \frac{d\omega'}{d\cos \theta} = 2m \left( \frac{d}{d\cos \theta} \frac{m\omega}{m + \omega(1 - \cos \theta)} \right) = 2\omega'. \]

In the second and third equalities, the solution to part (b) has been used. Putting everything together, we obtain the Klein-Nishina formula for the Compton effect,

\[ \frac{d\sigma}{d\Omega_{\text{FT}}} = \frac{\alpha^2}{2m^2} \frac{\omega^2}{\omega} \left( \frac{\omega'}{\omega} + \frac{\omega'}{\omega} - \sin^2 \theta \right). \]

4. In this problem, we use Fermi’s effective theory of the weak interaction to find an expression for the muon lifetime. In practice, the muon lifetime (which is measured in typical graduate lab courses) provides one of the most precise determinations of the Fermi coupling \( G_F \).

(a) This follows immediately from the definition of \( d\text{LIPS}_n \).

(b) We note that

\[ \int k_1^{\mu} k_2^{\nu} \text{dLIPS}_2(k) = \int k_1^{\mu} k_2^{\nu} (2\pi)^4 \delta^4(k - k_1 + k_2') \frac{d^3k_1'}{(2\pi)^3 2\omega'_1} \frac{d^3k_2'}{(2\pi)^3 2\omega'_2} \]

has dimension 2, is Lorentz covariant with two contravariant indices, and depends only on \( k'^\mu \). From Lorentz covariance, the only possible tensor structures with two contravariant indices are \( k'^\mu k'^\nu \) and \( g'^{\mu\nu} \), and from dimensional considerations, the coefficient of \( k'^\mu k'^\nu \) must be numerical, and the coefficient of \( g'^{\mu\nu} \) must be of dimension 2, and hence a numerical factor multiplying \( k'^2 \).
(c) It is shown in the text that, for the case of two outgoing particles, \(dLIPS_2(k) = \left(\frac{\rho}{|\vec{k}'_1|_{CM}/16\pi^2\sqrt{s}}\right)d\Omega_{CM}\) (11.30). In the case that both outgoing particles are massless, \(|k'_1| = \omega'_1, |k'_2| = \omega'_2\), and in the center of mass frame, both are equal. However, \(\omega'_1 + \omega'_2^2 = 4\omega'_1^2\) is the center of mass energy \(s\), so \(\left|\vec{k}'_1\right|_{CM} = \sqrt{s}/2\), so

\[
\int dLIPS_2(k) = \int \frac{1}{32\pi^2} d\Omega_{CM} = \frac{1}{8\pi}.
\]

(d) Contraction with \(g^{\mu\nu}\) yields

\[
\int k'_1 \cdot k'_2 dLIPS_2(k) = (4A + B)k^2.
\]

The momentum conservation delta function in \(dLIPS_2\) enforces that \(k = k'_1 + k'_2\), and since the outgoing neutrinos are massless, squaring this expression yields \(k^2 = 2k'_1 \cdot k'_2\). Therefore, using the result of part (c),

\[
\int \frac{1}{2}k^2 dLIPS_2(k) = \frac{1}{16\pi} k^2 = (4A + B)k^2.
\]

Contraction with \(k'^\mu k'^\nu\) yields

\[
\int (k \cdot k'_1)(k \cdot k'_2) dLIPS_2(k) = (A + B)k^4.
\]

Similarly, by momentum conservation we have \(k \cdot k'_1 = k'^1_1 k'^2 + k'_2 \cdot k'_1 = 0 + k^2/2\), so

\[
\int \left(\frac{1}{2}k^2\right)^2 dLIPS_2(k) = \frac{1}{32\pi} k^4 = (A + B)k^4.
\]

This yields \(A = 1/48\pi\) and \(B = 1/96\pi\).

(e) Substituting the results of (b) and (d) into (a), we obtain

\[
\Gamma = \frac{G_F^2}{3\pi\alpha m} \int \frac{d^3k'_3}{(2\pi)^3\omega_3} \epsilon_{\mu\nu\rho\sigma} \left( (k'_1 - k'_3)^\mu g_{\nu\sigma} + 2(k'_1 - k'_3)^\nu (k'_1 - k'_3)^\sigma \right).
\]

In the muon mass frame \(k_1 = (m, 0)\), we note that \(k_1 \cdot k'_3 = -m\omega'_3\). We also neglect the electron mass, \(k'^2_3 = 0\). After some work, the decay rate simplifies to

\[
\Gamma = \frac{G_F^2}{\pi} \int \frac{d^3k'_3}{(2\pi)^3\omega_3} \left( m^2 \omega'_3 - \frac{4}{3} m \omega'_3^2 \right).
\]

Notice that in the muon rest frame, the electron decays isotropically. Thus the angular integration simply gives \(d^3k'_3 = 4\pi\omega'_3^2\), and per unit electron energy \(E_e = \omega'_3\), the decay rate is

\[
d\Gamma = \frac{G_F^2}{4\pi^3} \left( m^2 E_e^2 - \frac{4}{3} m E_e^2 \right) dE_e.
\]

The maximum electron energy occurs when it is directed opposite to both the neutrinos. In our approximation that all decay products are massless, this corresponds to \(E_e = m/2\).

(f) Integrate the above result from \(E_e = 0\) to \(E_e = m/2\) to obtain the famous result for the muon decay rate

\[
\Gamma = \frac{m^5 G_F^2}{192\pi^3}.
\]
(g) We should get $G_F = 1.164 \times 10^{-5}$ GeV$^{-2}$. (Actual value $1.16637 \times 10^{-5}$ GeV$^{-2}$)

(h) The energy spectrum is

$$\frac{d\log \Gamma}{dE_e} = \frac{48}{m} \left( \frac{E_e}{m} \right)^2 - \frac{4}{3} \left( \frac{E_e}{m} \right)^3.$$