Dear Leonardo,

Please forward to the students in your class

1) the 7 pages of the brief history of path integrals.

2) the 6 pages of hand written notes of my lecture.

At the end of page 6 I suggest a problem, which uses ~~the~~ my lecture on path integrals.

Peter.

Feynman and Hibbs, QM and Path Integrals
L.S. Schulman: Techniques and Applications of Path Integrals.
~~Bast~~ Bastianelli and van Nieuw: Path Integrals and Anomalies in Curved Space.
Shrednicki: Quantum Field Theory.

Path integrals in QM.

QM done in 2 ways:

- operators (Heisenberg)
- wave functions (Schrödinger)
- path integrals (Dirac, Feynman). In Eucl. space: Wiener.

K is called the transformation function by Dirac

History: in 1932 Dirac wrote (Phys. Z. Sov. 3 (1933) 64)

$$\psi(x_2,t_2)=\int K(x_2 t_2|x_1 t_1)\,\psi(x_1,t_1)\,dx_1$$

$$K(x_2 t_2|x_1 t_1)=\langle x_2|e^{-\frac{i}{\hbar}H(t_2-t_1)}|x_1\rangle\,;\quad \psi(x_1,t)=\langle x_1,t|\psi\rangle.$$

Dirac wrote: ... K "corresponds to" $\exp\left(\frac{i}{\hbar}S(x_2t_2|x_1t_1)+\text{more}\right)$.

Feynman wondered: equal to?

In 1941 Feynman tried to check this for small t_2-t_1

$$\psi(x,t+\epsilon)=N\int_{-\infty}^{\infty}e^{\frac{i}{\hbar}L\left(\frac{x-y}{\epsilon},\frac{x+y}{2}\right)\epsilon}\,\psi(y,t)\,dy$$

Setting $y=x+\eta$ and $x-y=-\eta$, and expanding, he got

$$\psi(x)+\epsilon\partial_t\psi(x)=N\int_{-\infty}^{\infty}e^{\frac{i}{\hbar}\left(\frac{1}{2}m\frac{\eta^2}{\epsilon}-V(x)\epsilon\right)}\left(\psi(x,t)+\eta\,\partial_x\psi+\tfrac{1}{2}\eta^2\partial_x^2\psi\right)$$

$\eta^2\sim\epsilon$

$$=\underbrace{N\sqrt{\frac{2\pi\hbar i\epsilon}{m}}}\left(1-\frac{i}{\hbar}V(x)\epsilon\right)\psi(x)+\frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2}.$$

$$\boxed{N=\sqrt{\frac{m}{2\pi i\hbar\epsilon}}}\ :\ \text{the Schrödinger equation!}$$

Then he studied for large t_2-t_1 products of these kernels: path integrals for $\langle x_2 t_2|e^{-\frac{i}{\hbar}H(t_2-t_1)}|x_1 t_1\rangle$.

Program: we consider the anharmonic oscillator, write $\langle 0|e^{-\frac{i}{\hbar}Ht}|0\rangle$ as a path integral, and check that the energy in perturbation theory of the ground state (and other states, and propagators, and ...) agrees with QM.

$$L = \frac{m}{2}\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \alpha x^3 - \beta x^4$$

$$H = \left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right) + \left(\alpha x^3 + \beta x^4\right) = H_{(0)} + H_{int}.$$

Consider

$$T(t) \equiv \langle 0 | e^{-\frac{i}{\hbar}Ht} | 0 \rangle = |c_0|^2 e^{-\frac{i}{\hbar}\bar{E}_0 t} + |c_1|^2 e^{-\frac{i}{\hbar}\bar{E}_1 t} + \cdots$$

$$|0\rangle = c_0 |\bar{0}\rangle + c_1 |\bar{1}\rangle + \cdots \text{ where } H_0 |0\rangle = E_0 |0\rangle.$$
$$\text{and } \bar{H} |\bar{0}\rangle = \bar{E}_0 |\bar{0}\rangle.$$

$$T(t) = \iint_{-\infty}^{\infty} \langle 0|z\rangle \langle z| e^{-\frac{i}{\hbar}Ht} |y\rangle \langle y|0\rangle \, dy\, dz$$

$$u_0(z) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}\frac{m\omega}{\hbar}z^2} \qquad \left(H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{1}{2}m\omega^2 z^2\right)$$
$$E_0 = \frac{1}{2}\hbar\omega.$$

$$Z(z,y,t) \equiv \langle z| e^{-\frac{i}{\hbar}Ht} |y\rangle \qquad \Delta t = t/N \qquad \int dq_j\, |q_j\rangle\langle q_j| = 1, \quad \int dp_j\, |p_j\rangle\langle p_j| = 1.$$

$$= \int \langle z|p_N\rangle \langle p_N| e^{-\frac{i}{\hbar}H\Delta t} |\cancel{x}\, q_{N-1}\rangle \qquad \langle q_j|p_j\rangle = \frac{e^{\frac{i}{\hbar}q_j p_j}}{\sqrt{2\pi\hbar}}.$$

$$\langle q_{N-1}|p_{N-1}\rangle \langle p_{N-1}| e^{-\frac{i}{\hbar}H\Delta t} |q_{N-2}\rangle \qquad e^{-\frac{i}{\hbar}(H_0+V)\Delta t} = e^{-\frac{i}{\hbar}H_0\Delta t} e^{-\frac{i}{\hbar}V\Delta t} + O(\Delta t)^2.$$

$$\vdots$$

$$\langle q_j|p_j\rangle \langle p_j| e^{-\frac{i}{\hbar}H\Delta t} |q_{j-1}\rangle$$

$$\vdots$$

$$\langle q_1|p_1\rangle \langle p_1| e^{-\frac{i}{\hbar}H\Delta t} |y\rangle \qquad q_N = z, \quad q_0 = y.$$

$$\int d\mu = \int \prod_{j=1}^{N-1} dq_j \prod_{k=1}^{N} dp_k$$

phase-space path integrals.

$\sum_{row}\sum_{col} =$ Dirac

$\sum_{\Delta t}\sum_{at\ row} =$ Feynman

$$Z(z,y,t) = \int d\mu \prod_{j=1}^{N} \frac{e^{\frac{i}{\hbar}p_j(q_j - q_{j-1})}}{2\pi\hbar} e^{-\frac{i}{\hbar}\frac{p_j^2}{2m}\Delta t - \frac{i}{\hbar}V(q_{j-1})\Delta t}$$

$$\int \prod_{j=1}^{N-1} dq_j \left(\frac{\cancel{2\pi\hbar}\, m}{i\,\Delta t\, 2\pi\hbar}\right)^{N/2} e^{\frac{i}{\hbar}\int_{-t/2}^{t/2}\left[\frac{1}{2}m\left(\frac{q_j - q_{j-1}}{\Delta t}\right)^2 - V(q_{j-1})\right]\Delta t}$$

$$\simeq \oint N \int Dx\; e^{\frac{i}{\hbar}\int_{-t/2}^{t/2}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right)dt}$$

$x(t/2) = z$

$x(-t/2) = y.$

For the Harmonic oscillator $H^{h.o.} = P^2/2m + \frac{1}{2} m w^2 q^2$

$$Z^{H.O.}_{(z,y,t)} = \langle z | e^{-\frac{i}{\hbar}\tau H^{h.o}} | y \rangle = \sqrt{\frac{m w}{2\pi i \hbar \sinh w\tau}}$$

$$\exp\left(\frac{i w m}{2\hbar \sinh w\tau}\right)\left\{\left(x''^2 + x'^2\right)\cosh w\tau - 2x''x'\right\}.$$

Use $\int_{-\infty}^{\infty} du \sqrt{\frac{a}{\pi}}\, e^{-a(x-u)^2} \sqrt{\frac{b}{\pi}}\, e^{-b(u-y)^2}$

$$= \sqrt{\frac{ab}{\pi(a+b)}}\; e^{-\frac{ab}{a+b}(x-y)^2}$$

Free particle ($w \to 0$)

$$Z_0(z,y,t) = \langle z | e^{-\frac{i}{\hbar} t H^{(0)}} | y \rangle = \sqrt{\frac{m}{2\pi i \hbar \tau}}\; e^{\frac{i m}{2\hbar t}(x'' - x')^2}$$

With interaction: add source and use the exact part of the H.O.

$$Z(z,y,t) = \left[e^{\frac{i}{\hbar} S_{int}\left(x(t) \to \frac{\hbar}{i}\frac{\delta}{\delta j(t)}\right)} Z_0(z, y, j) \right]_{j=0}.$$

$$Z_0[j, z, y, t] = N \int \mathcal{D}x(t)\; e^{\frac{i}{\hbar}\int_{-t/2}^{t/2}\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m w^2 x^2 + j x\right) dt}$$

Decompose $x(t) = x_{cl}(t) + q(t)$

$x(t/2) = z$; $x(-t/2) = y$.

$$L^{(0)}_{cl} + jx = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 + jx$$

$$= \frac{1}{2} m \dot{x}_{cl}^2 - \frac{1}{2} m \omega^2 x_{cl}^2 + j x_{cl}$$

$$+ m \dot{x}_{cl} \dot{q} - m\omega^2 x_{cl} q + j q$$

$$+ \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$$

A) Removal of terms linear in q (to remove tadpoles):

Bulk: $\boxed{m \ddot{x}_{cl} + m \omega^2 x_{cl} = j}$; ~~Boundary term: $\dot{x}_{cl} q(\frac{t}{2}) - \dot{x}_{cl} q(-\frac{t}{2})$~~

Boundary: $\frac{i}{\hbar} m \dot{x}_{cl} q \Big|_{-t/2}^{t/2} - \frac{m\omega}{\hbar} x_{cl} q \Big|_{-t/2}^{t/2} = 0.$ from $u_0\left(x_{cl}(\pm\frac{t}{2}) + q(\pm\frac{t}{2})\right)$

Two solutions:
- $q(t/2) = q(-t/2) = 0.$
- $\boxed{\frac{i}{\cancel{\hbar}} \cancel{m} \dot{x}_{cl} + \frac{\cancel{m}\omega}{\cancel{\hbar}} x_{cl} = 0 \text{ at } t/2 \text{ and } -t/2.}$

We use the second.

B) q-independent terms:

$$L^{(0)} + jx \to \frac{i}{\hbar}\int \left(\frac{1}{2} m \dot{x}_{cl}^2 - \frac{1}{2} m \omega^2 x_{cl}^2 + j x_{cl}\right) dt$$: partially integrate and use equation of motion for x_{cl}.

$$= \frac{i}{\hbar}\left(\frac{1}{2}\int j x_{cl}\, dt + \frac{1}{2} m x_{cl} \dot{x}_{cl} \Big|_{-t/2}^{t/2}\right)$$ ← use boundary conditions.

$$= \frac{i}{\hbar}\frac{1}{2}\int j x_{cl}\, dt + \frac{1}{2}\frac{m}{\hbar} \omega x_{cl}^2 (t/2) + \frac{1}{2}\frac{m}{\hbar}\omega x_{cl}^2(-t/2).$$

cancels $u_0(x_{cl})\Big|_{-t/2}^{t/2}$ and u_0

$$\boxed{\mathrm{Tr}\langle 0 | e^{-\frac{i}{\hbar} H^{(0)} t} | 0 \rangle = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy\, u_0\left(q(\tfrac{t_0}{2})\right) N \int dx(t)\, e^{\frac{i}{\hbar} S_0(q)}\, u_0\left(q(-\tfrac{t_0}{2})\right) e^{\frac{i}{\hbar}\frac{1}{2}\int_{-t/2}^{t/2} j x_{cl}\, dt}}$$

$x = x_{cl} + q$. x_{cl} fixed so $dx(t) \to dq(t)$.

$q_f \equiv q(t/2) = z - x_{cl}(t/2)$ so $dz \to dq_f$ and $dy \to dq_i$

$$T(t) = \langle 0| e^{-\frac{i}{\hbar} H t} |0\rangle$$

$$= e^{\frac{i}{\hbar} S_{int}\left(x \to \frac{\hbar}{i}\frac{\partial}{\partial j}\right)}\, T Z_0(j=0)\, e^{\frac{i}{\hbar}\frac{1}{2}\int_{-t/2}^{t/2} j\, x_{cl}\, dt} .$$

$$T Z_0(j=0) = N\int_{-\infty}^{\infty} dq_f \int_{-\infty}^{\infty} dq_i\, u_0(q_f) \int Dq(t)\, e^{\frac{i}{\hbar}\int_{-t/2}^{t/2} \left(\frac{m}{2}\dot q^2 - \frac{1}{2} m\omega^2 q^2\right) dt}\, u_0(q_i)$$

$$= \langle 0| e^{-\frac{i}{\hbar} H_0 t} |0\rangle = e^{-\frac{i\omega t}{2}} .$$

<u>Propagator:</u>

$$m\left(\partial_t^2+\omega^2\right)x_{cl} = j \quad \Rightarrow m\, x_{cl}(t) = \frac{1}{(\partial_t^2+\omega^2)}\, j(t)$$

$$m\, x_{cl}(t) = \int_{-t/2}^{t/2} \Delta(t,t')\, j(t')\, dt'$$. Act with $\partial_t^2+\omega^2$; this gives

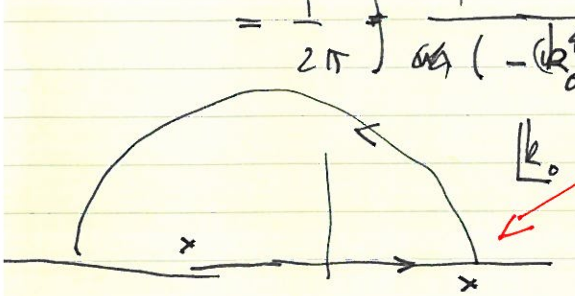
$$m\left(\partial_t^2+\omega^2\right)x_{cl} = j = \int_{-t/2}^{t/2} (\partial_t^2+\omega^2)\Delta(t,t')\, j(t')\, dt' \Rightarrow \boxed{(\partial_t^2+\omega^2)\,\Delta(t,t') = \delta(t-t')} .$$

$$i\dot x_{cl} \mp \omega x_{cl} = 0 \text{ at } \pm t/2 \Rightarrow \boxed{i\frac{\partial}{\partial t}\Delta(t,t') \mp \omega\, \Delta(t,t') = 0}$$

The propagator $\Delta(t,t')$ is completely fixed.

$$\Delta(t,t') = \frac{1}{(\partial_t^2+\omega^2)}\,\delta(t-t') = \frac{1}{(\partial_t^2+\omega^2)}\int \frac{e^{ik_0(t-t')}}{2\pi}\, dk_0$$

$$= \frac{1}{2\pi}\int \frac{1}{(-(k_0)^2+\omega^2 - i\epsilon)}\, e^{ik_0(t-t')}\, dk_0 . \quad (k_0 = -k^0 = -E).$$



For $t-t'>0$: $\frac{i}{2\omega}\, e^{-i\omega(t-t')}$ (close contour in upper half-plane)

Pole at $k_0=-\omega$.

For $t-t'<0$: $+\frac{i}{2\omega}\, e^{i\omega(t-t')}$ (lower half-plane).

Pole at $k_0=\omega$

$$\Delta(t,t') = \frac{i}{2\omega}\, e^{-i\omega(t-t')}\,\theta(t-t') + \frac{i}{2\omega}\, e^{-i\omega(t'-t)}\,\theta(t'-t).$$

Satisfies BC!

$$= \frac{i}{2\omega}\, e^{-i\omega|t-t'|} .$$

(usually without i.)

We have thus found the following result

$$T(t) = \langle 0 | e^{-\frac{i}{\hbar} H t} | 0 \rangle = e^{\frac{i}{\hbar} S_{int}\left(x \to \frac{\hbar}{i} \frac{\delta}{\delta j}\right)} \, e^{-\frac{1}{2}\omega t}$$

$$e^{\frac{i}{\hbar} \int_{-t/2}^{t/2} dt \int_{-t/2}^{t/2} dt' \, \frac{1}{2} j(t) \frac{\Delta(t,t')}{m} j(t')}$$

$$\Delta(t,t') = e^{-i\omega |t-t'|} \frac{i}{2\omega}.$$

~~Since~~ Now consider the anharmonic oscillator with α, β.

Since

$$T(t) = |c_0|^2 e^{-\frac{i}{\hbar} \bar{E}_0 t} + |c_1|^2 e^{-\frac{i}{\hbar} \bar{E}_1 t} + \dots$$

we can determine $\bar{E}_0$ in perturbation theory as follows

1) Expand $e^{\frac{i}{\hbar} S_{int}}$ and consider the first or second term

2) Expand $e^{\frac{i}{\hbar} \iint \frac{1}{2} j \frac{\Delta}{m} j}$ and keep as many terms such that all j are annihilated by the $\frac{\delta}{\delta j}$ in S_{int}.

3) collect all terms proportional to $e^{-\frac{1}{2}\omega t}\, t$ ~~for~~ $\bar{E}_0$ (or $e^{-\frac{3}{2}\omega t}\, t$ for $\bar{E}_1$) that are linear in β or quadratic in α.

4) Compute the corrections to $\bar{E}_0$ using QM, and to first order in β and second order in α.

5) Compare the results of 3) and 4) and (hopefully) find agreement.

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Path Integrals and Anomalies in Curved Space

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1.5 A brief history of path integrals

Path integrals yield a third approach to quantum physics, in addition to Heisenberg's operator approach and Schrödinger's wave function approach. They are due to Feynman [45], who in the 1940s developed an approach Dirac had briefly considered in 1932 [44]. In this section we discuss the motivations which led Dirac and Feynman to associate path integrals (with $i/\hbar$ times the action in the exponent) with quantum mechanics. In mathematics Wiener had already studied path integrals in the 1920s but these path integrals contained (-1) times the free action for a point particle in the exponent. Wiener's path integrals were Euclidean path integrals which are mathematically well defined but Feynman's path integrals do not have a similarly solid mathematical foundation. Nevertheless, path integrals have been successfully used in almost all branches of physics: particle physics, atomic and nuclear physics, optics and statistical mechanics [21].

In many applications one uses path integrals for perturbation theory, in particular for semiclassical approximations, and in these cases there are no serious mathematical problems. In other applications one uses Euclidean path integrals, and in these cases they coincide with Wiener's path integrals. However, for the nonperturbative evaluations of path integrals in Minkowski space a completely rigorous mathematical foundation is lacking. The problems increase in dimensions higher than four [52]. Feynman was well aware of these problems, but the physical ideas which stem from path integrals are so convincing that he (and other researchers) considered this not to be worrisome.

Our brief history begins with Dirac who in 1932 wrote an article in a USSR physics journal [44] in which he tried to find a description of quantum mechanics which was based on the Lagrangian instead of the Hamiltonian approach. Dirac was making a trip with Heisenberg around the world, and took the trans-Siberian railway to arrive in Moscow. In those days all work in quantum mechanics (including the work on quantum field theory) started with the Schrödinger equation or operator methods, and in both of these the Hamiltonian played a central role. For quantum mechanics this was fine, but for relativistic field theories an approach based on the Hamiltonian had the drawback that manifest Lorentz invariance was lost (although for QED it had been shown that physical results were nevertheless relativistically invariant). Dirac considered the transition element

$$\langle x_2, t_2 | x_1, t_1 \rangle = K(x_2, t_2 | x_1, t_1) = \langle x_2 | \mathrm{e}^{-\frac{i}{\hbar}\hat{H}(t_2 - t_1)} | x_1 \rangle \tag{1.36}$$

(for time-independent H), and asked whether one could find an expression for this matrix element in which the action was used instead of the Hamiltonian. (The notation $\langle x_2, t_2 | x_1, t_1 \rangle$ is due to Dirac who called this element a transformation function. Feynman introduced the notation $K(x_2, t_2 | x_1, t_1)$ because he used it as the kernel in an integral equation which solved the Schrödinger equation.) Dirac knew that in classical mechanics the time evolution of a system could be written as a canonical transformation, with Hamilton's principal function $S(x_2, t_2 | x_1, t_1)$ as the generating functional [53]. This function $S(x_2, t_2 | x_1, t_1)$ is the classical action evaluated along the classical path that begins at the point x_1 at time t_1 and ends at the point x_2 at time t_2. In his 1932 article Dirac wrote that $\langle x_2, t_2 | x_1, t_1 \rangle$ **corresponds to** $\exp \frac{i}{\hbar} S(x_2, t_2 | x_1, t_1)$. He used the words "corresponds to" to express that at the quantum level there were presumably corrections so that the exact result for $\langle x_2, t_2 | x_1, t_1 \rangle$ was different from $\exp \frac{i}{\hbar} S(x_2, t_2 | x_1, t_1)$. Although Dirac wrote these ideas down in 1932, they were largely ignored until Feynman started his studies on the role of the action in quantum mechanics.

Towards the end of the 1930s Feynman started studying how to formulate an approach to quantum mechanics based on the action. (Here we follow the biography of Feynman by Mehra [54].) The reason he tackled this problem was that with Wheeler he had developed a theory of quantum electrodynamics from which the electromagnetic field had been eliminated. In this way they hoped to avoid the problems of the self-acceleration and infinite self-energy of an electron which are due to the interactions of an electron with the electromagnetic field and which Lienard, Wiechert, Abraham and Lorentz had tried in vain to solve. The resulting "Wheeler–Feynman theory" arrived at a description of the interactions between two electrons in which no reference was made to any field. It is a so-called action-at-a-distance theory. These theories were nonlocal in space and time. (In modern terminology one might say that the fields A_μ had been integrated out from the path integral by completing squares.) Fokker and Tetrode had found a classical action for such a system, given by [54]

$$\begin{aligned} S = & -\sum_i m_{(i)} \int \left[\frac{dx^\mu_{(i)}}{ds_{(i)}} \frac{dx^\nu_{(i)}}{ds_{(i)}} \eta_{\mu\nu} \right]^{1/2} ds_{(i)} \\ & - \frac{1}{2} \sum_{i \neq j} e_{(i)} e_{(j)} \int\!\!\int \delta \left[\left(x^\mu_{(i)} - x^\mu_{(j)} \right)^2 \right] \frac{dx^\rho_{(i)}}{ds_{(i)}} \frac{dx^\sigma_{(i)}}{ds_{(i)}} \eta_{\rho\sigma} \, ds_{(i)} \, ds_{(j)} . \end{aligned} \tag{1.37}$$

Here the sum over (i) denotes a sum over different electrons. So, two electrons only interact when the relativistic four-distance vanishes, and by taking $i \neq j$ in the second sum, the problem of infinite self-energy was eliminated. Integration of the second term over $\frac{dx^0_{(i)}}{ds_{(i)}} ds_{(i)} = dx^0_{(i)}$ yielded an expression of the form $\int A_\mu j^\mu$ where A_μ are one-half times the sum of the retarded and advanced Liénard-Wiechert potentials, generated by the j-th charged particle and acting on the i-th charged particle. Wheeler and Feynman set out to quantize this system, but Feynman noticed that a Hamiltonian treatment was hopelessly complicated.[10] Thus Feynman was looking for an approach to quantum mechanics in which he could avoid the Hamiltonian. The natural object to use was the action.

At this moment in time, an interesting discussion helped him further. A physicist from Europe, Herbert Jehle, who was visiting Princeton, mentioned to Feynman (spring 1941) that Dirac had already (in 1932) studied the problem of how to use the action in quantum mechanics. Together they looked up Dirac's paper, and of course Feynman was puzzled by the ambiguous phrase "corresponds to" in it. He asked Jehle whether Dirac meant that they were equal or not. Jehle did not know, and Feynman decided to take a very simple example and to check. He considered the case where $t_2 - t_1 = \epsilon$ was very small, and wrote the time evolution of the Schrödinger wave function $\psi(x,t)$ as follows:

$$\psi(x, t+\epsilon) = \frac{1}{\mathcal{N}} \int \exp\left[\frac{i}{\hbar}\epsilon L(x, t+\epsilon; y, t)\right] \psi(y,t)\, dy. \tag{1.38}$$

With $L = \frac{1}{2}m\dot{x}^2 - V(x)$ one obtains, as we now know very well, the Schrödinger equation, provided the constant $\mathcal{N}$ is given by

$$\mathcal{N} = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2} \tag{1.39}$$

(the combination $dy/\mathcal{N}$ is nowadays often called the Feynman measure). Thus, as Dirac correctly guessed, $\langle x_2, t_2 | x_1, t_1\rangle$ was analogous to $\exp(\frac{i}{\hbar}\epsilon L)$ for small $\epsilon = t_2 - t_1$; however, they were not equal but rather proportional.

There is an amusing continuation of this story [54]. In the fall of 1946 Dirac was giving a lecture at Princeton, and Feynman was asked to introduce Dirac and comment on his lecture afterwards. Feynman decided to simplify Dirac's rather technical lecture for the benefit of the audience, but senior physicists such as Bohr and Weisskopf did not much appreciate this watering down of the work of the great Dirac by the young and relatively unknown Feynman. Afterwards people were discussing Dirac's lecture and Feynman who (in his own words) felt a bit let down hap-

[10] By expanding expressions such as $1/(\partial_x^2 + \partial_t^2 - m^2)$ in a power series in ∂_t, and using Ostrogradsky's approach to a canonical formulation of systems with higher-order ∂_t derivatives, one can give a Hamiltonian treatment, but one must introduce infinitely many new fields $B, C, \ldots$ of the form $\partial_t A = B, \partial_t B = C, \ldots$. All of these new fields are, of course, equivalent to the oscillators of the original electromagnetic field.

pened to look out of the window and saw Dirac lying on his back on a lawn and looking at the sky. So Feynman went outside and sitting down near Dirac asked him whether he could ask him a question concerning his 1932 paper. Dirac consented. Feynman said "Did you know that the two functions do not just 'correspond to' each other, but are actually proportional?" Dirac said "Oh, that's interesting". And that was the total reaction that Feynman got from Dirac.

Feynman then asked himself how to treat the case where $t_2 - t_1$ is not small. This Dirac had already discussed in his paper: by inserting a complete set of x-eigenstates one obtains

$$\langle x_\mathrm{f}, t_\mathrm{f} | x_\mathrm{i}, t_\mathrm{i} \rangle = \int \langle x_\mathrm{f}, t_\mathrm{f} | x_{N-1}, t_{N-1} \rangle \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2} \rangle \cdots \\ \cdots \langle x_1, t_1 | x_\mathrm{i}, t_\mathrm{i} \rangle \, dx_{N-1} \cdots dx_1. \tag{1.40}$$

Taking $t_j - t_{j-1}$ small and using the fact that for small $t_j - t_{j-1}$ one can use $\mathcal{N}^{-1} \exp \frac{i}{\hbar}(t_j - t_{j-1})L$ for the transformation function, Feynman arrived at

$$\langle x_\mathrm{f}, t_\mathrm{f} | x_\mathrm{i}, t_\mathrm{i} \rangle = \int \exp\left[\frac{i}{\hbar} \sum_{j=0}^{N-1} (t_{j+1} - t_j) L(x_{j+1}, t_{j+1}; x_j, t_j)\right] \frac{dx_{N-1} \cdots dx_1}{\mathcal{N}^N}. \tag{1.41}$$

At this point Feynman recognized that one obtains the action in the exponent and that by first summing over j and then integrating over x one is summing over paths. Hence $\langle x_\mathrm{f}, t_\mathrm{f} | x_\mathrm{i}, t_\mathrm{i} \rangle$ is equal to a sum over all paths of $\exp(\frac{i}{\hbar}S)$ with each path beginning at x_i, t_i and ending at x_f, t_f.

Of course, only one of these paths is the classical path, but by summing over all other paths (arbitrary paths not satisfying the classical equation of motion) quantum mechanical corrections are introduced. The tremendous result was that all quantum corrections were included if one summed the action over all paths. Dirac had entertained the possibility that in addition to summing over paths one would have to replace the action S by a generalization which contained terms with higher powers in $\hbar$.

Reviewing this development more than half a century later, when path integrals have largely superseded operators methods and the Schrödinger equation for relativistic field theories, one notices how close Dirac came to the solution of using the action in quantum mechanics, and how different Feynman's approach was to solving the problem. Dirac anticipated that the action had to play a role, and by inserting a complete set of states he did obtain (1.41). However, he did not pursue the observation that the sum of terms in (1.41) is the action because he anticipated for large $t_2 - t_1$ a more complicated expression. Feynman, on the other hand, started by working out a few simple examples, curious to see whether Dirac was correct that the complete result would need a more complicated expression than the action, and in this way found that the truth lies in between: Dirac's transformation functions (Feynman's transition kernel K) is equal to the exponent of the action up to a constant. This constant diverges as ϵ tends to zero, but for $N \to \infty$ the result for K (and other quantities) is finite.

Feynman initially believed that in his path integral approach to quantum mechanics ordering ambiguities of the p and x operators of the operator approach would be absent (as he wrote in his PhD thesis of May 1942). However, later in his fundamental 1948 paper in *Review of Modern Physics* [45], he realized that the same ambiguities would be present. For our work the existence of these ambiguities is very important and we shall discuss in great detail how to fix them. Schrödinger [55] had already noticed that ordering ambiguities occur if one tries to promote a classical function $F(x,p)$ to an operator $\hat{F}(\hat{x},\hat{p})$. Furthermore, one can in principle add further terms that are linear and of higher order in $\hbar$ to such operators $\hat{F}$. These are further ambiguities which have to be fixed before one can make definite predictions.

Feynman evaluated the kernels $K(x_{j+1},t_{j+1}|x_j,t_j)$ for small $t_{j+1}-t_j$ by inserting complete sets of **momentum eigenstates** $|p_j\rangle$ in addition to position eigenstates $|x_j\rangle$. In this way he constructed **phase-space path integrals**. We shall follow the same approach for the nonlinear sigma models we consider. It has been claimed in [21] that "...phase space path integrals have more troubles than merely missing details. On this basis they should have been left out [from the book]...". We have instead arrived at a different conclusion: they are well defined and can be used to **derive** the usual configuration-space path integrals from the operatorial approach by adding integrations over intermediate momenta. A continuous source of confusion is the notation $Dx(t)\,Dp(t)$ for these phase-space path integrals. Many authors, who attribute more meaning to this symbol than $dx_1\cdots dx_{N-1}\,dp_1\cdots dp_N$, assume that this measure is invariant under canonical transformations, and apply the powerful methods developed in classical mechanics for the Liouville measure. However, the measure $Dx(t)\,Dp(t)$ in path integrals is not invariant under canonical transformations of the x and the p because there is one more p integration then x integration in $\prod dx_j \prod dp_j$.

Another source of confusion for phase-space path integrals arises if one tries to interpret them as integrals over paths around classical solutions in phase space. Consider Feynman's expression

$$\begin{aligned} K(x_j,t_j|x_{j-1},t_{j-1}) &= \langle x_j|\mathrm{e}^{-\frac{i}{\hbar}\hat{H}(t_j-t_{j-1})}|x_{j-1}\rangle \\ &= \int \frac{dp_j}{2\pi}\langle x_j|\mathrm{e}^{-\frac{i}{\hbar}\hat{H}(t_j-t_{j-1})}|p_j\rangle\langle p_j|x_{j-1}\rangle. \end{aligned} \tag{1.42}$$

For $\langle x_j | e^{-\frac{i}{\hbar}\hat{H}(t_j - t_{j-1})} | x_{j-1} \rangle$ one can substitute $\exp[\frac{i}{\hbar} S(x_j, t_j | x_{j-1}, t_{j-1})]$, where in S one uses the classical path from x_j, t_j to x_{j-1}, t_{j-1}. In a similar way some authors have tried to give meaning to $\langle x_j | e^{-\frac{i}{\hbar}\hat{H}(t_j - t_{j-1})} | p_j \rangle$ by considering a classical path in phase space. However, several proposals have been shown to be inconsistent or impractical [21]. We shall not try to interpret the transition elements in phase space in terms of classical paths, but only do what we are supposed to do: integrate over p_j and x_j.

Yet another source of confusion has to do with path integrals over fermions for which one needs Grassmann numbers and Berezin integration [56]

$$\int d\theta = 0 \, , \qquad \int d\theta \, \theta = 1. \tag{1.43}$$

Some authors claim that the notion of anticommuting classical fields makes no sense and that only quantized fermionic fields are consistent. However, the notion of Grassmann variables is completely consistent if one uses it only at the intermediate stages to construct, for example, fermionic coherent states: all one does is make use of mathematical identities. One begins with fermionic harmonic oscillator operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ and constructs coherent bra and kets states $|\eta\rangle$ and $\langle\bar{\eta}|$ in Hilbert space. In applications traces are taken over these coherent states using Berezin rules for the integrations over η and $\bar{\eta}$. One ends up with physical results which are independent of the Grassmann variables, and since all intermediary steps are mathematical identities [19], defined by Berezin [56], at no point are there any conceptual problems in the treatment of path integrals for fermions.