This is the second (and last), part of my notes on path integrals.

In what follows, the contributions from QM to order $\alpha^2$ and order $\beta$ to the ground state energy of the anharmonic oscillator are compared with the corresponding "Feynman diagrams" from path integrals

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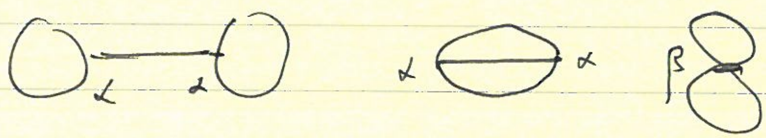



They agree.

Note: some of you asked me whether one should also set $q(t) = 0$ at the endpoints, in addition to the boundary conditions on $x_{cl}(t)$ which we discussed. The answer is no: either one uses $q = 0$ at the endpoints, or one sets the coefficients of the terms linear in $q$ to zero at the end points. I did the second method. You may do the first method. (The results are, of course, the same).

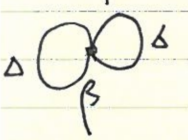
P. van N.

$$= \frac{9\alpha^2}{(2w)^3}\cdot\left(\frac{\hbar}{i}\right) i^3 \frac{1}{iw}\left[t + \frac{1}{iw}\left(e^{-iwt/2} - e^{iwt/2}\right)e^{-iwt/2}\right]$$

$$\left[t + \frac{1}{iw}\left(-1 + e^{-iwt}\right)\right]$$

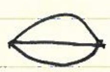
$$= i\hbar\frac{9\alpha^2}{8w^4}\cdot\left(t - \frac{1}{iw} + \frac{1}{iw}e^{-iwt}\right)\cdot e^{-iwt}$$

$\beta$ term : $\Gamma(t) = \frac{i}{\hbar}\int(-\beta)\left(\frac{\hbar}{i}\frac{\partial}{\partial j}\right)^4 d\tau\, \frac{1}{2!}\frac{i}{\hbar}\int\frac{1}{2}j\frac{\Delta}{m}j\,\frac{\partial}{\hbar}\int\frac{1}{2}j\frac{\Delta}{m}j \quad e^{-iwt}$



$$= \frac{\hbar}{i}(-\beta)\cdot\frac{1}{8}\,4\cdot3\cdot2\int_{-t/2}^{t/2}\Delta(\tau_1,\tau_1)^2\,d\tau_1\quad e^{-iwt}$$

$$= i\hbar\beta\,3\,\frac{i^2}{(2w)^2}t. = -i\hbar\beta\frac{3}{(2w)^2}t.$$



$$\Gamma(t) = \frac{1}{2!}\frac{i}{\hbar}\int(-\alpha)\left(\frac{\hbar}{i}\frac{\partial}{\partial j}\right)^3\frac{i}{\hbar}\int(-\alpha)\left(\frac{\hbar}{i}\frac{\partial}{\partial j}\right)^3 \quad < \equiv >$$

$$\frac{1}{3!}\frac{i}{\hbar}\int\frac{1}{2}j\Delta j\;\frac{i}{\hbar}\int\frac{1}{2}j\Delta j\;\frac{i}{\hbar}\int\frac{1}{2}j\Delta j$$

$$= \frac{1}{2!}\frac{\hbar}{i}\alpha^2\frac{1}{3!}\,3\cdot2\cdot3\cdot2\int_{-t/2}^{t/2}dt_1\;2\int_{-t/2}^{t_1}dt_2\;\Delta^3(t_1,t_2)$$

$$= -i\hbar\,6\alpha^2\frac{i^3}{(2w)^3}\int_{-t/2}^{t/2}dt_1\int_{-t/2}^{t_1}e^{-3iw(t_1-t_2)}$$

$$\frac{1}{+3iw}e^{-3iwt_1}\left(e^{3iwt_1} - e^{-3iwt/2}\right)$$

$$\frac{1}{3iw}\left[t + \frac{1}{3iw}\left(e^{-3iwt} - 1\right)\right]$$

$$= \frac{i\hbar\,6\alpha^2}{8w^4}\cdot\left[\frac{1}{3}t + \frac{1}{9iw}\left(e^{-3iwt} - 1\right)\right]$$

Conclusion: We got ~~from~~ $\langle 0|e^{-\frac{i}{\hbar}Ht}|0\rangle$

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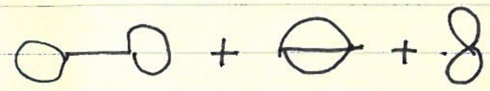

$$= i\hbar\,\frac{9\alpha^2}{8\omega^4}\left(t-\frac{1}{i\omega}+\frac{1}{i\omega}e^{-i\omega t}\right)e^{-\frac{i\omega t}{2}}$$

$$= e^{-\frac{1}{2}\omega t} - i\hbar\beta\,\frac{3}{(2\omega)^2}\,t\,e^{-\frac{i\omega t}{2}} + i\hbar\,\frac{6\alpha^2}{8\omega^4}\left(\frac{1}{3}t-\frac{1}{9i\omega}+\frac{1}{9i\omega}e^{-3i\omega t}\right)e^{-\frac{i\omega t}{2}}$$

$$= |c_0|^2 e^{-\frac{i}{\hbar}E_0 t} + |c_1|^2 e^{-iE_1 t}$$

$$= (1+\dots)(\dots)e^{-\frac{1}{2}\omega t}\left(1-\frac{i}{\hbar}\Delta E_0 t+\dots\right)+\dots$$

So $-\frac{i}{\hbar}\Delta E_0 t = i\hbar\left(\frac{9\alpha^2}{8\omega^4}+\frac{6\alpha^2}{8\omega^4}\frac{1}{3}t-\frac{3\beta}{4\omega^2}\right)$

$$\boxed{E_0 = \frac{1}{2}\hbar\omega + \hbar^2\left(-\frac{11}{8}\frac{\alpha^2}{\omega^4}+\frac{3\beta}{4\omega^2}\right)+\mathcal{O}(\hbar^3)}$$

QM gives: $E_0^{(1)} = \langle 0|H_{int}|0\rangle = \langle 0|\beta x^4|0\rangle$

$$H_0^{(0)} = p^2/2m + \frac{1}{2}m\omega^2x^2 = \left[\left(\frac{p+i\omega m x}{\sqrt{2\hbar\omega m}}\right)\left(\frac{p-i\omega m x}{\sqrt{2\hbar\omega m}}\right)+\frac{1}{2}\right]\hbar\omega$$

$$= \frac{p^2}{2m}+\frac{\omega^2 m^2x^2}{2m}+\cancel{\left(\frac{-i\omega m}{2\hbar\omega m}\frac{\hbar}{i}+\frac{1}{2}\right)}\hbar\omega$$

$x = (a^\dagger-a)\frac{1}{i}\sqrt{\frac{\hbar}{2m\omega}}$ $\qquad \langle 0|x^4|0\rangle = \langle 0|\cancel{(a^\dagger-a)(a^\dagger-a)(a^\dagger-a)(a^\dagger-a)}|0\rangle\left(\frac{\hbar}{2m\omega}\right)^2$

$p = (a^\dagger+a)\sqrt{\frac{\hbar m\omega}{2}}$ $\qquad = \langle 0|(aa-1)(a^\dagger a^\dagger-1)|0\rangle\left(\frac{\hbar}{2m\omega}\right)^2$

$$= \langle 0|aaa^\dagger a^\dagger+1|0\rangle$$

$$= 3$$

$E_0^{(1)} = \beta\cdot 3\left(\frac{\hbar}{2m\omega}\right)^2.$ AGREES!

Now $E_0^{(2)} = -\sum_n \frac{\langle 0|H_{int}|n\rangle\langle n|H_{int}|0\rangle}{E_n^0-E_0^0} = -\langle 0|\alpha x^3|n\rangle\langle n|\alpha x^3|0\rangle$

$$-\alpha^2\left(\frac{\langle 0|x^3|1\rangle}{\hbar\omega}+\frac{\langle 0|x^3|3\rangle}{3\hbar\omega}\right) = (-\alpha^2)\left(\frac{\hbar}{2m\omega}\right)^3 i\left[\frac{|\langle 0|(a^\dagger-a)^3|1\rangle|^2}{\hbar\omega}+\frac{|\langle 0|(a^\dagger-a)^3|3\rangle|^2}{3\hbar\omega}\right]$$

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Also $E_0^{(2)} = -\sum_n \frac{\langle 0 | H_{int} | n \rangle \langle n | H_{int} | 0 \rangle}{E_n - E_0}$

$= -\sum_n \frac{|\langle 0 | \alpha x^3 | n \rangle|^2}{n\hbar\omega} = -\alpha^2 \left(\frac{\hbar}{2m\omega}\right)^3 \left( \frac{|\langle 0 | x^3 | 1 \rangle|^2}{\hbar\omega} + \frac{\langle 0 | x^3 | 3 \rangle}{3\hbar\omega} \right)$

$= -\alpha^2 \left(\frac{\hbar}{2m\omega}\right)^3 \frac{1}{\hbar\omega} \left[ |\langle 0 | (a^\dagger - a)(a^\dagger - a)(a^\dagger - a) | 1 \rangle|^2 + \frac{1}{3} |\langle 0 | (a^\dagger - a)(a^\dagger - a)(a^\dagger - a) | 3 \rangle|^2 \right]$

$(aa)\, a^\dagger a + a a a^\dagger$

$(1 + 2)^2 \qquad + \frac{1}{3} (\sqrt{3}\sqrt{2}\sqrt{1})^2$

$= -\alpha^2 \frac{\hbar^2}{(2m\omega)^3 \omega} \left( 9 + \frac{1}{3} 6 \right) = -\frac{\alpha^2 \hbar^2}{(m^3\omega^4)} \frac{11}{8}$

agrees!