1. (a) The tree level Feynman diagram for $\phi \to \phi \phi$ is

\[ \begin{array}{c}
\text{Diagram}
\end{array} = ig. \]

(b) The one loop correction is

\[ i\mathcal{M} = (ig)^3 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + m^2 - i\epsilon} \frac{-i}{(k-p_1)^2 + m^2 - i\epsilon} \frac{-i}{(k-p_1-p_2)^2 + m^2 - i\epsilon} \]

(all momenta flowing out of diagram).

(c) Recall that, in position space, one computes

\[ \langle \Omega | T \phi(x_1)\phi(x_2)\phi(x_3) | \Omega \rangle = \frac{\langle 0 | T \phi_1\phi_2\phi_3 \exp(i \int d^4x \mathcal{L}_I) | 0 \rangle}{\langle 0 | \exp(i \int d^4x \mathcal{L}_I) | 0 \rangle}, \]

where we use the shorthand $\phi_j = \phi(x_j)$ for interaction picture fields, and $\mathcal{L}_I = (g/3!)\phi_x^3$. As discussed in the text, the quotient factors out bubble diagrams, so we should consider only graphs without bubbles. Expanding in orders of $g$, we obtain

\[ \langle \Omega | T \phi(x_1)\phi(x_2)\phi(x_3) | \Omega \rangle = \langle 0 | \phi_1\phi_2\phi_3 \exp(i \int d^4x \mathcal{L}_I) | 0 \rangle \]

Next, we contract the fields in bubble-free ways. The terms with odd numbers of fields are zero, and the term of order $g^4$ is

\[ 3! \frac{ig}{3!} \int d^4x \phi_1 \phi_2 \phi_3 \phi_x \phi_x \phi_x = ig \int d^4x D_{x1} D_{x2} D_{x3}, \]

which is simply the diagram in (a). The term corresponding to the diagram in (b) is of order $g^3$, and is given by

\[ (3!)^4 \frac{1}{3!} i \int d^4x d^4y d^4z \phi_1 \phi_2 \phi_3 \phi_x \phi_x \phi_y \phi_y \phi_x \phi_z \phi_z \phi_z = (ig)^3 \int d^4x d^4y d^4z D_{x1} D_{y2} D_{z3} D_{xy} D_{yz} D_{zx}. \]

The $(3!)^4$ combinatorial factor comes from the fact that there are 3! ways to pair up $(1, 2, 3)$ with $(x, y, z)$, and for each of $x, y$ and $z$, there are 3! permutations of the 3 $\phi$ legs.

(d) The LSZ formula states

\[ \langle f | S | i \rangle = \prod_{j=1}^3 i \int d^4x_j e^{ip_j x_j} (-\partial_j^2 + m^2) \langle \Omega | T(\phi(x_1)\phi(x_2)\phi(x_3)) | \Omega \rangle. \]

Using $(-\partial_j^2 + m^2)D_{jx} = -i\delta(x-x_j)$, for the order $g$ term we see that

\[ \langle f | S | i \rangle = ig \int d^4x e^{ip_1 p_2 p_3 x} = ig(2\pi)^4 \delta^4(p_1 + p_2 + p_3) + O[g]^3 \]
(with the convention that $p_j$ is the momentum flowing out of the diagram, so for $\phi \to \phi\phi$ we have $p_1$ being the momentum of the incoming particle and $-p_2, -p_3$ the momenta of the outgoing particles).

We see that the LSZ formula simply removes the propagators attached to the external particles, and replaces them with an overall momentum conserving delta function. For the order $g^3$ term we will similarly have

$$
\langle f|S|i\rangle_{O[g]^3} = (ig)^3 \int d^4xd^4y d^4z \, e^{ip_1x} e^{ip_2y} e^{ip_3z} D_{xy} D_{yz} D_{zx}
$$

$$
= (ig)^3 \int d^4xd^4y d^4z \, e^{ip_1x} e^{ip_2y} e^{ip_3z} \frac{d^4k_1 \, d^4k_2 \, d^4k_3 \, \delta^4(p_1 - k_1 - k_3) \delta^4(p_2 + k_1 - k_2) \delta^4(p_3 + k_2 - k_3)}{(2\pi)^4 \, (2\pi)^4 \, (2\pi)^4 \, k_1^2 + m^2 - i\epsilon \, k_2^2 + m^2 - i\epsilon \, k_3^2 + m^2 - i\epsilon}
$$

$$
= (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k_1^2 + m^2 - i\epsilon} \frac{-i}{k_2^2 + m^2 - i\epsilon} \frac{-i}{k_3^2 + m^2 - i\epsilon},
$$

which is exactly the diagram in (b).

2. The interaction terms are $L_I = (\lambda/2)\phi_1(\partial_\mu\phi_2)^2 + (g/2)\phi_1^2\phi_2$, so there are $\phi_1\phi_2$ and $\phi_1^2\phi_2$ interactions.

The tree level Feynman diagrams for $\phi_1\phi_2 \to \phi_1\phi_2$ are

![diagrams](https://example.com/diagrams)

(with single lines for $\phi_1$ propagators and double lines for $\phi_2$ propagators). The first graph has two $\phi_1^2\phi_2$ interactions, and one $\phi_1$ propagator, so the first graph is

$$
(i\lambda)^2 \frac{-i}{k^2 - i\epsilon} = \frac{i\lambda^2}{(p_1 + p_2)^2}.
$$

The next graph has two $\phi_1(\partial_\mu\phi_2)^2$ interactions and one $\phi_2$ propagator. This means that, at each vertex we are to dot the momenta of the $\phi_2$ particles, yielding

$$
(i\lambda)^2 (-p_2)_\mu(p_1 + p_2)_\nu(-p_4)_\mu(p_3 + p_4)_\nu \frac{-i}{k^2 - i\epsilon} = \frac{i\lambda^2 (p_1 \cdot p_2)(p_3 \cdot p_4)}{(p_1 + p_2)^2},
$$

where we have used that the $p_j$ are massless on shell. A similar calculation yields the third and fourth graphs as

$$
\frac{ig^2}{(p_1 - p_4)^2} + \frac{i\lambda^2 (p_1 \cdot p_4)(p_2 \cdot p_3)}{(p_1 - p_4)^2}.
$$

Therefore, in terms of Mandelstam variables $s = -(p_1 + p_2)^2 = -2p_1 \cdot p_2 = -2p_3 \cdot p_4$, $u = -(p_1 - p_4)^2 = 2p_1 \cdot p_4 = 2p_2 \cdot p_3$, $t = -(p_1 - p_2)^2 = -2p_1 \cdot p_4$, $u = -(p_1 - p_4)^2 = 2p_1 \cdot p_4 = 2p_2 \cdot p_3$,

$$
iM = -i \left( g^2 \left( \frac{1}{s} + \frac{1}{u} \right) + \frac{\lambda^2}{4}(s - u) \right) = -i \left( \frac{g^2 t}{su} + \frac{\lambda^2 t}{4} \right).
$$
where we have used $s + t + u = 0$. Hence
\[
\left( \frac{d\sigma}{d\Omega} \right)_{CM} (\phi_1 \phi_2 \rightarrow \phi_1 \phi_2) = \frac{1}{64\pi^2 E_{CM}^2} \left( \frac{g^2 t}{su} + \frac{\lambda^2 t^2}{4} \right)^2.
\]

3. The solutions are in the text.

4. (a) $L_1 = -Z_\lambda \lambda \phi^4 /4!$, so there is a $\phi^4$ vertex with factor $-iZ_\lambda \lambda$.

(b) A diagram with $E$ external lines and $V$ vertices has $E + 4V$ lines to connect, and $P$ propagators connect two lines. Therefore, $2P = E + 4V$ for each diagram. In particular, $E = 1$ and $E = 3$ are not possible. For $E = 2$, we have the following diagrams:

- $V = 0$:
  \[ S = 2 \]

- $V = 1$:
  \[ S = 4 \]

- $V = 2$:
  \[ S = 8 \]

For $E = 4$:

- $V = 0$: no connected graphs

- $V = 1$:
  \[ S = 4! \]

- $V = 2$:
  \[ S = 16 \] \[ S = 12 \]

(c) As we saw in (b), there are no graphs with $E = 1$, so $\langle \phi \rangle$ is zero.

5. (a) The vertex is $-iZ_\lambda \lambda$, involving two $\phi^\dagger$ and two $\phi$ (i.e. two inward arrows and two outward).

Note that this implies that arrows cannot end or begin except at an external source. (The charge associated with the $U(1)$ complex phase symmetry is conserved.)
(b) As before, we have $2P = E + 4V$. In general, due to the arrows, the symmetry factors in this case are much smaller. For $E_\phi = 1, E_{\phi^\dagger} = 1$:

For $E_\phi = 2, E_{\phi^\dagger} = 2$:

V = 0: no connected graphs

V = 1:

\[ S = 4 \]

V = 2:

\[ S = 8 \]

\[ S = 2 \]

\[ S = 2 \]

\[ S = 2 \]

\[ S = 2 \]

6. Redefining $\varphi \mapsto \varphi + \lambda \varphi^2$ in free field theory yields the lagrangian

\[ \mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - \lambda m^2 \varphi^3 - 2 \lambda \varphi (\partial_\mu \varphi)^2 - \frac{1}{2} \lambda^2 m^2 \varphi^4 - 2 \lambda^2 \varphi^2 (\partial_\mu \varphi)^2. \]

There is a $\varphi^3$ vertex (with outgoing momenta $k_j$) with factor $-i(3!\lambda m^2 + 2!2\lambda (ik_1 \cdot ik_2 + ik_2 \cdot ik_3 + ik_3 \cdot ik_1))$. This may be simplified by noting that $0 = (\sum k_j)^2 = \sum k_j^2 + 2 \sum_{i<j} k_i \cdot k_j$, so the vertex factor is $-i\lambda(6m^2 + 2(k_1^2 + k_2^2 + k_3^2))$.

There is also a $\varphi^4$ vertex with factor $-i(4!2\lambda^2 m^2 + 2!2!2\lambda^2 \sum ik_i \cdot ik_j)$, which may be simplified similarly to yield $-i\lambda^2(12m^2 + 4(k_1^2 + k_2^2 + k_3^2 + k_4^2))$.

At tree level, $\varphi \varphi \rightarrow \varphi \varphi$ consists of the diagrams
\[ i\mathcal{M} = \frac{-i(-\lambda)^2(6m^2 + 2(p_1^2 + p_2^2 + (p_1 + p_2)^2))(6m^2 + 2(p_3^2 + p_4^2 + (p_1 + p_2)^2))}{(p_1 + p_2)^2 + m^2 - i\epsilon} = \frac{4i\lambda^2 (m^2 + (p_1 + p_2)^2)^2}{(p_1 + p_2)^2 + m^2 - i\epsilon} = 4i\lambda^2 (m^2 - s), \]

where we have used that external particles are on shell, \( p_j^2 = -m^2 \). The next two diagrams yield \( 4i\lambda^2 (m^2 - t) \) and \( 4i\lambda^2 (m^2 - u) \) (hence the first three diagrams are known as \( s, t \) and \( u \) channels respectively). Finally, the last diagram is

\[ -i\lambda^2(4(p_1^2 + p_2^2 + p_3^2 + p_4^2) + 12m^2) = 4i\lambda^2 m^2. \]

Hence

\[ i\mathcal{M} = 4i\lambda^2(4m^2 - s - t - u) = 0. \]