PHY 610 QFT, Spring 2015

HW2 Solutions

1. (a) The equations of motion are

$$\pi_n = \frac{\partial H}{\partial \pi_n} = \dot{\varphi}_n,$$

$$\ddot{\varphi}_n = \dot{\pi}_n = -\frac{\partial H}{\partial \varphi_n} = -(\varphi_n - \varphi_{n-1}) - (\varphi_n - \varphi_{n+1}) - m^2\varphi_n.$$

Equating the mode expansions of the left and right hand sides of the second equation yields the dispersion relation

$$\omega_k^2 = 2 + m^2 - e^{ik} - e^{-ik} = 2(1-\cos k) + m^2.$$

(Notice that ω_k depends only on $|k|$, as before.)

(b) Since the positions of the atoms are discrete, the momentum is a periodic function. A more quantitative way of seeing this is as follows: the position n takes integer values, so $e^{i(-\omega_k t + kn)} = e^{i(-\omega_{k+2\pi}t + (k+2\pi m)n)}$ for any integer m, so k and $k+2\pi m$ describe the same configuration. The interval $[-\pi, \pi]$ in which k takes its values is known as the Brillouin zone.

(c) First, invert the Fourier expansion,

$$\sum_n \varphi_n e^{ink} = \int \frac{dk'}{(2\pi)2\omega_{k'}} \sum_n (a_k e^{i(-\omega_{k'}t + (k+k')n)} + a_k^\dagger e^{-i(-\omega_{k'}t + (k'-k)n)})$$

$$= \frac{1}{2\omega_k}\left(a_{-k}e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right),$$

$$\sum_n \pi_n e^{ink} = \sum_n \dot{\varphi}_n e^{ink} = \frac{i}{2}(-a_{-k}e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}),$$

where we have used that $\sum_n e^{ink} = (2\pi)\delta(k)$. Thus

$$a_k = \sum_n (\omega_k \varphi_n + i\pi_n)e^{-(ikn - i\omega_k t)}, \qquad a_k^\dagger = \sum_n (\omega_k\varphi_n - i\pi_n)e^{ikn - i\omega_k t}.$$

It is then straightforward to show that $[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0$ and $[a_k, a_{k'}^\dagger] = (2\pi)(2\omega_k)\delta(k-k')$.

(d) We are to substitute the mode expansion of φ_n and $\pi_n = \dot{\varphi}_n$ into the hamiltonian. I will be brief here since the algebra is similar to (and less complicated than) problem 4 of homework 1. After substitution, the sum over n yields delta functions which cancel one of the k integrals, leaving us with

$$H = \frac{1}{2}\int_{-\pi}^{\pi} \frac{dk}{(2\pi)2\omega_k^2}\Big(\omega_k^2(-a_k a_{-k}e^{-2i\omega_k t} - a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t} + a_k a_k^\dagger + a_k^\dagger a_k)$$
$$+ ((1-e^{-ik})(1-e^{ik}) + m^2)(a_k a_{-k}e^{-2i\omega_k t} + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t} + a_k a_k^\dagger + a_k^\dagger a_k)\Big).$$

Recognizing that $\omega_k^2 = (1-e^{-ik})(1-e^{ik}) + m^2$, this simplifies to

$$H = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\frac{1}{2}(a_k a_k^\dagger + a_k^\dagger a_k).$$

Now, we normal order using the commutation relation in (c), to yield

$$H = \int_{-\pi}^{\pi} \frac{dk}{2\pi} a_k^\dagger a_k + \Omega_0 V,$$

where $V = 2\pi\delta(0)$ is the "volume of space" (really, the number of particles in this case), and $\Omega_0 = \int dk/2\pi\, \omega_k$ is the zero point energy.

This is the hamiltonian of non-interacting free scalar fields.

(e) Restoring factors of a, the hamiltonian is

$$H = \frac{1}{2}\sum_n \pi_n^2 + a^{-2}(\varphi_n - \varphi_{n-1})^2 + m^2\varphi_n^2.$$

In the continuum limit, this becomes (with $x = na$)

$$H \to \frac{1}{2}\int \frac{dx}{a} \pi^2(x) + (\partial_x \varphi(x))^2 + m^2\varphi^2(x),$$

which is the hamiltonian of a free scalar field. Similarly, the dispersion relation

$$\omega_k^2 = 2(1-\cos(ka))a^{-2} + m^2 \to k^2 + m^2$$

reproduces that of the free scalar in the small a limit. Note also that the Brillouin zone is $[-\pi/a, \pi/a]$, so in the continuum limit we recover that k is allowed to take any value.

2. We are to show that the Noether charge,

$$Q = \int d^3x\, j^0(x) = \int d^3x\, \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_a(x))}\delta\varphi_a(x) = \int d^3x\, \pi^a(x)\delta\varphi_a(x),$$

generates the symmetry transformation, $[Q, \varphi_a] = -i\delta\varphi_a$. This is a straightforward calculation, using the canonical commutation relation $[\varphi_a(x), \Pi^b(y)] = i\delta_a^b \delta^3(x-y)$,[1]

$$\begin{aligned}
[Q, \varphi_a(x)] &= \int d^3y\, [\Pi^b(y)\delta\varphi_b(y), \varphi_a(x)] \\
&= \int d^3y\, \left(\pi^b(y)[\delta\varphi_b(y), \varphi_a(x)] + [\Pi^b(y), \varphi_a(x)]\delta\varphi_b(y)\right) \\
&= -\, i\delta\varphi_a(x),
\end{aligned}$$

where we have assumed that $\delta\varphi_a$ is independent of Π^a in order to set the first term in the integrand to zero.

3. We are to verify that the Noether charge for translations,

$$P^\mu = \int d^3x\, T^{0\mu}(x) = \int d^3x\, (-\Pi^a(x)\partial^\mu\varphi_a(x) + g^{0\mu}\mathcal{L}(x)),$$

indeed generates infinitesimal translations $[P^\mu, \varphi_a(x)] = i\partial^\mu\varphi_a(x)$. For $\mu \neq 0$, the calculation proceeds exactly as in problem 2,

$$[P^i, \varphi_a(x)] = \int d^3y\, -[\Pi^b(y), \varphi_a(x)]\partial^i\varphi_b(y) = i\partial^i\varphi_a(x).$$

[1] Strictly speaking, the canonical commutation relation holds only at equal times $x^0 = y^0$. However, because Q is a conserved charge, it is time independent, so we may choose y^0 to be equal to x^0 in the following calculation.

For $\mu = 0$, notice that the assumption in problem 2, that $\delta\varphi_a$ is independent of Π^a, no longer holds. In this case

$$[P^0, \varphi_a(x)] = \int d^3y \, \frac{1}{2}[\Pi^b(y)\Pi^b(y) + \partial_i\varphi_b(y)\partial_i\varphi_b(y) + 2V(\varphi(y)), \varphi_a(x)]$$
$$= - i\Pi^a(x) = i\partial^0\varphi_a(x).$$

4. (a) We are to derive the algebra satisfied by the currents

$$\begin{cases} T^{00} = & \frac{1}{2}\Pi_a^2 + \frac{1}{2}(\partial_i\varphi_a)^2 + V(\varphi), \\ T^{0j} = & -\Pi_a\partial^j\varphi_a. \end{cases}$$

We begin with the less complicated one,

$$\begin{aligned}[T^{0j}(x), T^{0k}(y)] =& [\Pi_a(x)\partial^j\varphi_a(x), \Pi_b(y)\partial^k\varphi_b(y)] \\ =& \Pi_a(x)[\partial^j\varphi_a(x), \Pi_b(y)]\partial^k\varphi_b(y) + \Pi_b(y)[\Pi_a(x), \partial^k\varphi_b(y)]\partial^j\varphi_a(x) \\ =& \Pi_a(x)\partial^{x_j}[\varphi_a(x), \Pi_b(y)]\partial^k\varphi_b(y) + \Pi_b(y)\partial^{y_k}[\Pi_a(x), \varphi_b(y)]\partial^j\varphi_a(x) \\ =& i\Pi_a(x)\partial^k\varphi_a(y)\partial^{x_j}\delta^3(x-y) - i\Pi_a(y)\partial^j\varphi_a(x)\partial^{y_k}\delta^3(y-x)\end{aligned}$$

To obtain the first equality, imagine pushing first $\partial^j\varphi_a$, and then Π_a, to the right. Now, what are derivatives of delta functions? Recall that delta functions only make sense inside an integral, $\int dx\ \delta(x-a)f(x) = f(a)$, so we can think of derivatives of delta functions as being defined by partial integration, ie. $\int dx\ \partial_x\delta(x-a)f(x) = -\int dx\ \delta(x-a)f'(x) = -f'(a)$. Hence we partially integrate to get rid of derivatives on the delta functions, yielding[2]

$$\boxed{[T^{0j}(x), T^{0k}(y)] = -i\delta^3(x-y)\left(\partial^{[j}\Pi_a(x)\partial^{k]}\varphi_a(x) + \Pi_a(x)\partial^k\varphi_a(x)\partial^{x_j} - \Pi_a(x)\partial^j\varphi_a(x)\partial^{y_k}\right)}.$$

Note that the last two terms in the parentheses on the right are important if the commutator is multiplying another term inside the integral (the open-ended derivatives will act on the other term).

Similarly,

$$[T^{00}(x), T^{0j}(y)] = -\frac{1}{2}\Pi_b(y)[\Pi^2(x), \partial^j\varphi_b(y)] - \frac{1}{2}[(\partial^k\varphi_a)^2(x), \Pi_b(y)]\partial^j\varphi_b(y) - [V(\varphi(x)), \Pi_b(y)]\partial^j\varphi_b(y).$$

We now know how to deal with the first two terms, but how about $[V(\varphi(x)), \Pi_b(y)]$? The trick is to think of the commutator as a derivative, and use the chain rule to bring down powers of φ from $V(\varphi)$.[3] Therefore

$$[V(\varphi(x)), \Pi_b(y)] = i\frac{\partial V}{\partial\varphi_b(x)}\delta^3(x-y).$$

(Alternatively, this can be seen by Taylor expanding V.) Therefore,

$$[T^{00}(x), T^{0j}(y)] = i\Pi_a(x)\Pi_a(y)\partial^{y_j}\delta^3(y-x) - i\partial^k\varphi_a(x)\partial^{x_k}\delta^3(x-y)\partial^j\varphi_a(y) - i\frac{\partial V}{\partial\varphi_b(x)}\delta^3(x-y)\partial^j\varphi_b(y).$$

[2] $A^{[j}B^{k]}$ is shorthand for $A^jB^k - A^kB^j$.

[3] The operator $[A, \cdot]$ satisfies the Leibniz rule, which is to say, $[A, BC] = B[A, C] + [A, B]C$, so the commutator (with some field A) is in fact a derivation.

This commutator may be written in a neat way as follows. Replace $\partial^{x_k}\delta^3(x-y)$ with $-\partial^{y_k}\delta^3(x-y)$ in the second term, and then partial integrate the first two terms in y, to yield

$$[T^{00}(x), T^{0j}(y)] = \delta^3(x-y)\left(-i\Pi_a(x)\partial^j\Pi_a(y) - i\partial^k\varphi_a(x)\partial^k\partial^j\varphi_a(y) - i\frac{\partial V}{\partial\varphi_b(x)}\partial^j\varphi_b(y)\right) - i\delta^3(x-y)\left(\Pi_a(x)\Pi_a(y)\partial^{y_j} + \partial^k\varphi_a(x)\partial^j\varphi_a(y)\partial^{y_k}\right).$$

Due to the delta function, we may replace y with x in the expression above. We then recognize that the first term in parentheses as $-i\partial^j T^{00}(x)$, so

$$\boxed{[T^{00}(x), T^{0j}(y)] = -i\delta^3(x-y)\partial^j T^{00}(x) - i\delta^3(x-y)(\Pi^2(x)\partial^{y_j} + \partial^k\varphi_a(x)\partial^j\varphi_a(y)\partial^{y_k})}.$$

Finally, a similar calculation shows

$$\begin{aligned}
[T^{00}(x), T^{00}(y)] =& \frac{1}{4}[\Pi^2(x), (\partial^j\varphi(y))^2] + \frac{1}{2}[\Pi^2(x), V(\varphi(y))] - (x \leftrightarrow y)\\
=& \frac{1}{4}[\Pi^2(x), \partial^j\varphi_a(y)]\partial^j\varphi_a(y) + \frac{1}{4}\partial^j\varphi_a(y)[\Pi^2(x), \partial^j\varphi_a(y)] + \frac{1}{2}[\Pi^2(x), V(\varphi(y))] - (x \leftrightarrow y)\\
=& -\frac{i}{2}\Pi_a(x)\partial^j\varphi_a(y)\partial^{y_j}\delta^3(x-y) - \frac{i}{2}\partial^j\varphi_a(y)\Pi_a(x)\partial^{y_j}\delta^3(x-y)\\
&- i\Pi_a(x)\frac{\partial V}{\partial\varphi_a(y)}\delta^3(x-y) - (x \leftrightarrow y)\\
=& \frac{i}{2}\delta^3(x-y)\left(\Pi_a(x)\partial^j\varphi_a(y)\partial^{y_j} + \partial^j\varphi_a(y)\Pi_a(x)\partial^{y_j} - (x \leftrightarrow y)\right)\\
=& \boxed{-\frac{i}{2}\delta^3(x-y)\left(\Pi_a(x)\partial^j\varphi_a(x) + \partial^j\varphi_a(x)\Pi_a(x)\right)(\partial^{x_j} - \partial^{y_j})}.
\end{aligned}$$

(b) Integrating the commutators we just derived over x and y will yield the algebra for the translation generators, $[H,H], [P^j,H]$ and $[P^j,P^k]$. It is clear that $[H,H]=0$, since $[T^{00},T^{00}]$ only involves derivative operators. Next, $[P^i,P^j] = -i\int d^3x\, \partial^{[j}\Pi_a\partial^{k]}\varphi_a$, and partially integrating in x^j, the integrand becomes $\Pi_a\partial^{[j}\partial^{k]}\varphi_a = 0$. Finally, $[H,P^j] = -i\int d^3x\, \partial^j T^{00} = 0$, since $\partial^j T^{00}$ is a total derivative.

For the Lorentz generators $M^{\mu\nu} = \int d^3x\, T^{0[\nu}x^{\mu]}$, we have to integrate the commutators we derived, multiplied by x and y:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \int d^3x\, d^3y\, x^{[\mu}[T^{0\nu]}(x), T^{0[\sigma}(y)]y^{\rho]}.$$

With the expressions for the commutators we have derived, and some tedious but straightforward algebra, which I will not include here, we should be able to verify the algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}),$$

from which $[K^i,K^j] = -i\epsilon_{ijk}J^k, [J^i,J^j] = i\epsilon_{ijk}J^k, [J^i,K^j] = i\epsilon_{ijk}K^k$ follows.

Finally, for the commutators between translation and Lorentz generators, we could integrate the commutators we derived, multiplied by x. However, I will describe a simpler method, using what we proved in problem 3, that P^μ generates translations, ie. $[P^\mu, \varphi_a(x)] = i\partial^i\varphi_a(x)$ and $[P^\mu, \Pi_a(x)] = i\partial^i\Pi_a(x)$. (We did not prove the second statement, but the proof proceeds exactly analogously.) Together with the fact that the commutator is a derivative, it means that for any local operator $f(\varphi(x), \Pi(x))$, we have $[P^\mu, f(\varphi(x), \Pi(x))] = i\partial^\mu f(\varphi(x), \Pi(x))$. In particular, we take f to be $T^{0\mu}$, so

$$[P^\mu, M^{\nu\rho}] = \int d^3x\, [P^\mu, T^{0[\rho}]x^{\nu]} = i\int d^3x\, \partial^\mu T^{0[\rho}x^{\nu]}$$
$$= -i\int d^3x\, T^{0[\rho|}\partial^\mu x^{|\nu]} = -i\int d^3x\, T^{0[\rho}\delta^{\nu]\mu}$$

Therefore, $[P^\mu, M^{\nu\rho}]$ vanishes unless $\mu = \nu$ or ρ. For $\mu = 0 = \rho$, we obtain $[H, K^n] = iP^n$. For $\mu = m = \rho$ and $\nu = n$, we obtain $[P^m, N^{nm}] = iP^n$, ie. $[P^m, J^j] = -i\epsilon_{jnm}P^n$. For $\mu = m = \rho$ and $\nu = 0$, we obtain $[P^j, K^k] = -i\delta^{jk}H$.

5. (a) We are to find the Noether current corresponding to infinitesimal $SO(N)$ transformation, $\delta\varphi_i = \theta_{ij}\varphi_j = -i\theta^a(T_a)_{ij}\varphi_j$. One may simply use the formula (22.27) (with $K^\mu = 0$), but let us use a different method (which is often easier). Vary the action with respect to a position dependent $SO(N)$ transformation $\theta^a = \theta^a(x)$ (which is no longer a symmetry!). Since the action has only single derivative terms, the variation will contain two terms, one multiplying θ^a, and one multiplying the first derivative, $\partial_\mu\theta^a$. We expect the coefficient of the θ term to be a total derivative, because the transformation with constant θ is indeed a symmetry. Working it out, we find

$$\delta\mathcal{L} = i\partial^\mu\varphi_i(T_a)_{ij}\varphi_j\partial_\mu\theta^a,$$

so the θ term vanishes. Let $j_a^\mu = i\partial^\mu\varphi_i(T_a)_{ij}\varphi_j$ be the coefficient of the $\partial_\mu\theta^a$ term. Partially integrating, we see that $\delta S = -\int dx\, \partial_\mu j_a^\mu\theta^a$. But, once again, this has to vanish for constant θ^a, so we must have $\partial_\mu j_a^\mu = 0$, and j_a^μ is the conserved Noether current.

(b) We are to show that the Noether charge Q_a generates the symmetry transformation, $[Q_a, \varphi_i] = -(T_a)_{ij}\varphi_j$. This is a straightforward calculation, with $Q_a = \int d^3x\, j^0 = -\int d^3x\, i\Pi_i(T_a)_{ij}\varphi_j$ and the canonical commutation relations, yielding

$$[Q_a, \varphi_i(x)] = \int d^3y\, i(T_a)_{jk}[\Pi_j(y), \varphi_i(x)]\varphi_k(y) = -(T_a)_{ij}\varphi_j(x).$$

(This is a special case of problem 2.)

(c) Consider the commutator of two symmetry transformations on φ_i, which may be evaluated using the jacobi identity,

$$\begin{aligned}[[Q_a, Q_b], \varphi_i] &= -[[\varphi_i, Q_a], Q_b] - [[Q_b, \varphi_i], Q_a] \\ &= -(T_a)_{ik}(T_b)_{kj}\varphi_j + (T_b)_{ik}(T_a)_{kj}\varphi_j \\ &= -if_{abc}(T_c)_{ij}\varphi_j = if_{abc}[Q_c, \varphi_i].\end{aligned}$$

(In technical terms, this shows that the φ_i is a representation of the algebra generated by the Q_a, which, of course, is the algebra of $SO(n)$.) The action of $[Q_a, Q_b]$ and $if_{abc}Q_c$ on φ_i coincide. We wish to show that $[Q_a, Q_b]$ in fact is equal to $if_{abc}Q_c$. (In technical terms, we want to show that this representation is faithful.) The most straightforward way of doing this is to note that since $[Q_a, Q_b] - if_{abc}Q_c$ commutes with φ_i and $\Pi_i = \partial_0\varphi_i$ (since charges are time independent), $[Q_a, Q_b] - if_{abc}Q_c$ must in fact be a constant. But there are no $SO(n)$ invariant tensors with two antisymmetric indices for $n > 2$, so in fact $[Q_a, Q_b] - if_{abc}Q_c = 0$.

6. Let us vary the action with respect to a local translation $\delta A_\mu = -a^\nu(x)\partial_\nu A_\mu$, with a infiniestimal. We obtain

$$\begin{aligned}
\delta S &= \int d^4x \; -\frac{1}{2}F^{\mu\nu}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) + J^\mu \delta A_\mu \\
&= \int d^4x \; \frac{1}{4}a^\rho \partial_\rho (F^{\mu\nu}F_{\mu\nu}) + \partial_\mu a^\rho F^{\mu\nu}\partial_\rho A_\nu - a^\rho \partial_\rho (J^\mu A_\mu) \\
&= \int d^4x \; \partial_\mu a^\nu \left(-\frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}\delta^\mu_\nu + F^{\mu\rho}\partial_\nu A_\rho + J^\rho A_\rho \delta^\mu_\nu \right),
\end{aligned}$$

where the assumption that J^μ is a conserved current, $\partial_\mu J^\mu = 0$, has been used in the second equality. According to the prescription in question 5, we can identify the coefficient of $\partial_\mu a^\nu$ as the Noether current,

$$T^{\mu\nu} = -\frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + F^{\mu\rho}\partial^\nu A_\rho + J^\rho A_\rho g^{\mu\nu}.$$

The same result may be obtained using (22.27), noting that $K^\mu = a^\mu(-\frac{1}{4}(F_{\rho\sigma})^2 + J^\rho A_\rho)$.

Notice that this stress tensor, obtained via the Noether method, is neither symmetric (due to $F^{\mu\rho}\partial^\nu A_\rho$ term), nor gauge invariant. The first term is gauge invariant, and the third term is gauge invariant up to a total derivative $\partial_\rho(J^\rho \lambda g^{\mu\nu})$, but the second term is not, since $\delta(F^{\mu\rho}\partial^\nu A_\rho) = F^{\mu\rho}\partial^\nu\partial_\rho\lambda$.

Remark

What's bad about a non-symmetric stress tensor? Consider the angular momentum current $\mathcal{M}^{\mu\nu\rho} = T^{\mu[\rho}x^{\nu]}$. Taking ∂_μ of both sides, we see that $\partial_\mu \mathcal{M}^{\mu\nu\rho} = T^{[\nu\rho]}$, so the angular momentum is not conserved! In the absence of a source, $J^\mu = 0$, we know that the Maxwell theory is Lorentz symmetric, so what is going on here? The resolution is that $\mathcal{M}$ is only the orbital angular momentum, and since the photon has spin, it is the total angular momentum, not just $\mathcal{M}$, that is conserved.

How can we take spin into account in the stress tensor? The solution is to add improvement terms to the Noether stress tensor, of the form $\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}$, where $B^{\rho\mu\nu}$ is some tensor antisymmetric in its first two indices, $B^{\rho\mu\nu} = -B^{\mu\rho\nu}$. Notice that $\tilde{T}^{\mu\nu}$ is still conserved, since $\partial_\rho\partial_\mu B^{\rho\mu\nu} = 0$, and the momenta are unchanged as the improvement term is a total derivative. By considering the spin transformation of the fields in the action, a $B^{\rho\mu\nu}$ can always be found such that $\tilde{T}^{\mu\nu}$ is symmetric.

In the case of the Maxwell field, the improvement term is given by $B^{\rho\mu\nu} = F^{\rho\mu}A^\nu$. Indeed, we have

$$\begin{aligned}
\tilde{T}^{\mu\nu} &= -\frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + F^{\mu\rho}(\partial^\nu A_\rho - \partial_\rho A^\nu) + \partial_\rho F^{\rho\mu}A^\nu \\
&= -\frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + F^{\mu\rho}F^\nu{}_\rho + \partial_\rho F^{\rho\mu}A^\nu
\end{aligned}$$

On shell, the last term vanishes (recall that we have set $J^\mu = 0$), so $\tilde{T}^{\mu\nu}$ is both symmetric and gauge invariant.

Finally, note that there is an alternative definition of the stress tensor, used in general relativity. We place the theory in a curved background, restoring explicit factors of the metric,

$$S = \int d^4x\sqrt{-g} \; -\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + A_\mu J^\mu,$$

and define the stress tensor as the variation with respect to the metric,

$$T_g^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}.$$

Since the metric is symmetric, this produces an off-shell symmetric stress tensor, which coincides with the improved stress tensor on shell. (Try deriving $T_g^{\mu\nu}$! The identity $\delta\sqrt{-g} = -(1/2)\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ might be useful.)