Homework 2

1. **Phonons**

   Consider a one-dimensional "crystal", described by the Hamiltonian

   $$H = \frac{1}{2}\sum_{-\infty}^{\infty}\left[\pi_n^2 + (\varphi_n - \varphi_{n-1})^2 + m^2\varphi_n^2\right] .$$

   Here φ_n stands for the displacement of the nth atom, at position $x = na$ on the lattice (where the lattice spacing a has been set to one) and π_n is the canonically conjugate variable. The second term in H describes the coupling between nearest neighbors, while the third term is a restoring potential to the equilibrium position. The usual equal time canonical commutation relations (for the Heisenberg picture operators) apply,

   $$[\varphi_m(t), \pi_n(t)] = i\delta_{mn} .$$

   In analogy with our treatment of the continuum scalar field, let us write the Fourier expansion

   $$\varphi_n = \int_{-\pi}^{\pi}\frac{dk}{(2\pi)2\omega_k}\left(a_k\, e^{-i\omega_k t + ikn} + a_k^\dagger\, e^{i\omega_k t - ikn}\right) .$$

   (a) What is ω_k as a function of k and m?

   (b) Why is the momentum taking values in the interval $[-\pi, \pi]$?

   (c) Derive the commutation relations for the a_k and $a_k^\dagger$ oscillators.

   (d) Write the Hamiltonian in terms of a_k and $a_k^\dagger$. Interpret your result physically.

   (e) Finally, use dimensional analysis to restore the lattice spacing a in your formulas, and take the continuum limit $a \to 0$.

2. Srednicki problem 22.1

3. Srednicki problem 22.2

4. Srednicki problem 22.3

5. Srednicki problem 24.3

6. Use Noether's theorem to derive the canonical stress energy tensor $T^{\mu\nu}$ for Maxwell theory coupled to an external conserved current, described by action

   $$S = \int d^4x\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu\right] .$$

   Is $T^{\mu\nu}$ symmetric under exchange of the indices μ and ν? Is it invariant under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda$?