PHY 610 QFT, Spring 2015

HW10 Solutions

1. (a) \[ \Psi = (d - 1)/2 \]
   (b) \[ g_n = d - n(d - 1) = n - (n - 1)d \]
   (c) A scalar field has dimension \[ [\phi] = d/2 - 1 \] from the kinetic term \( (\partial_\mu \phi)^2 \), so \( [g_{m,n}] = d - n(d - 1) - m(d/2 - 1) = n + m - (n + m/2 - 1)d \).
   (d) The only renormalizable interaction in \( d = 4 \) is \( g_{1,1} \phi \Psi \Psi \) (the Yukawa interaction).

2. We have computed the one loop corrections to the propagator and vertex in \( \phi^4 \) theory in the previous homework; they are, in the MS scheme,

\[
\begin{align*}
Z_\phi &= 1 + O(\lambda)^2, \\
Z_m &= 1 + \frac{\alpha}{\epsilon} + O(\lambda)^2, \\
Z_\lambda &= 1 + \frac{3\alpha}{\epsilon} + O(\lambda)^2,
\end{align*}
\]

where \( \alpha = \lambda/(4\pi)^2 \).

Recall that the beta function is defined as the dependence of the coupling \( \alpha \) on the renormalization scale \( \mu \). In terms of bare parameters, \( Z_\lambda \lambda^\prime \phi^4 = \lambda_0 \phi_0^4 \), with \( \phi_0 = Z_{\phi}^{1/2} \phi \), so the bare coupling is \( \alpha_0 = Z_\lambda Z_{\phi}^{-2/3} \lambda^\prime \alpha \). The bare coupling is \( \mu \)-independent, so

\[
0 = \frac{d}{d \log \mu} \log \alpha_0 = \left( \frac{\partial \log Z_\lambda}{\partial \alpha} - 2 \frac{\partial \log Z_{\phi}}{\partial \alpha} + \frac{1}{\alpha} \right) \frac{d \log \alpha}{d \log \mu} + \epsilon.
\]

Multiplying both sides by \( \alpha \), we have

\[
\left( \frac{\partial \log Z_\lambda}{\partial \alpha} - 2 \frac{\partial \log Z_{\phi}}{\partial \alpha} + 1 \right) \frac{d \alpha}{d \log \mu} + \epsilon \alpha = 0.
\]

In a renormalizable theory, \( d\alpha/d \log \mu \) is an affine function of \( \epsilon \). The linear piece is fixed by the above renormalization group equation to be \( -\epsilon \alpha \); while the \( \epsilon \)-independent piece is the \( \beta \) function:

\[
\left( \frac{\partial \log Z_\lambda}{\partial \alpha} - 2 \frac{\partial \log Z_{\phi}}{\partial \alpha} + 1 \right) (-\epsilon \alpha + \beta(\alpha)) + \epsilon \alpha = 0
\]

Comparing both sides at order \( \epsilon^0 \) gives the one loop beta function

\[
\beta(\alpha) = 3\alpha^2 + O(\alpha)^3.
\]

(If you had worked with \( \lambda \) instead of \( \alpha = \lambda/(4\pi)^2 \), you would have obtained \( \beta(\lambda) = 3\lambda^2/(4\pi)^2 + O(\lambda)^3 \).)

Similarly, the anomalous dimension of \( m \) is defined as \( \gamma_m = d \log m/d \log \mu \), the dependence of the mass on scale \( \mu \). This may be computed using the \( \mu \)-independence of the bare mass \( m_0^2 = Z_m Z_{\phi}^{-1} m^2 \), so that

\[
0 = \frac{d}{d \log \mu} \log m_0 = \left( \frac{1}{2} \frac{\partial \log Z_m}{\partial \alpha} - \frac{1}{2} \frac{\partial \log Z_{\phi}}{\partial \alpha} \right) \frac{d \alpha}{d \log \mu} + \frac{d \log m}{d \log \mu}.
\]
Hence
\[ \gamma_\mu (\alpha) = \frac{1}{2} \alpha + O[\alpha]^2. \]

Finally, the anomalous dimension of \( \varphi \) is
\[ \gamma_\varphi (\alpha) = \frac{d \log Z_\varphi^{1/2}}{d \log \mu} = \frac{1}{2} \frac{\partial \log Z_\varphi}{\partial \alpha} (-\epsilon \alpha + \beta (\alpha)) = 0 + O[\alpha]^2. \]

3. No counterterm of the form \( Y_\chi \chi \) is needed to cancel \( \chi \) tadpoles due to the \( \mathbb{Z}_2 \) symmetry \( \chi \to -\chi \), which ensures that \( \chi \) tadpoles vanish (assuming that the vacuum does not break this symmetry). Diagrammatically, this can be seen by the fact that it is not possible to draw a diagram for a \( \chi \) tadpole.

(a) • Corrections to \( \varphi \) propagator:
\[
 i\Pi_\varphi (k^2) = \quad \quad + \quad \quad + \quad \quad \quad \bigcirc \\
\]
\[
= \frac{(ig\tilde{\mu}^{\epsilon/2})^2}{2} \int \frac{d^dl}{(2\pi)^d} \frac{-i}{l^2 + m^2} \frac{-i}{(l+k)^2 + m^2} \left( \frac{i\hbar\tilde{\mu}^{\epsilon/2}}{2} \right)^2 \int \frac{d^dk}{(2\pi)^d} \frac{-i}{k^2 + M^2} \frac{-i}{(k+l)^2 + M^2} \left( Z_\varphi - 1 \right)^2 + (Z_m - 1)^2 \\
\]
The general method for computing these integrals were done step by step in homework 9. For the sake of brevity, I will quote the result (14.30)
\[
i\Pi_\varphi (k^2) = -\frac{1}{2} \alpha_g \left( \Gamma(-1 + \epsilon/2)(k^2/6 + m^2) + \int_0^1 dx D_m \log(\mu^2/D_m) \right) \\
- \frac{1}{2} \alpha_h \left( \Gamma(-1 + \epsilon/2)(k^2/6 + M^2) + \int_0^1 dx D_M \log(\mu^2/D_M) \right) - i((Z_\varphi - 1)^2 + (Z_m - 1)^2),
\]
where \( D_m = x(1 - x)k^2 + m^2 \). In the \( \overline{\text{MS}} \) scheme we therefore have, at one loop,
\[
Z_\varphi - 1 = \frac{1}{6\epsilon} (\alpha_g + \alpha_h), \quad Z_m - 1 = -\frac{1}{\epsilon} \left( \alpha_g + \frac{M^2}{m^2} \alpha_h \right).
\]

• Corrections to \( \chi \) propagator:
\[
 i\Pi_\chi (k^2) = \quad \quad + \quad \quad \quad \bigcirc \\
\]
\[
= (i\hbar\tilde{\mu}^{\epsilon/2})^2 \int \frac{d^dl}{(2\pi)^d} \frac{-i}{l^2 + m^2} \frac{-i}{(k+l)^2 + M^2} - i((Z_\chi - 1)^2 + (Z_M - 1)^2) \\
= - \alpha_h \left( \Gamma(-1 + \epsilon/2)(k^2/6 + m^2/2 + M^2/2) + \int_0^1 dx D_{m,M} \log(\mu^2/D_{m,M}) \right) - i((Z_\chi - 1)^2 + (Z_M - 1)^2),
\]
where \( D_{m,M} = x(1 - x)k^2 + xM^2 + (1 - x)M^2 \). Therefore, at one loop,
\[
Z_\chi - 1 = -\frac{\alpha_h}{3\epsilon}, \quad Z_M - 1 = -\frac{1}{\epsilon} \left( 1 + \frac{m^2}{M^2} \right) \alpha_h.
\]

• Corrections to \( \varphi^3 \) vertex:
\[ iV_{\phi^3} = \quad + \quad + \]

\[ = \frac{i\alpha_g \tilde{\mu}^{\epsilon/2}}{2} \Gamma(\epsilon/2) \int dF_3 \left( \mu^2/D_m'' \right)^{\epsilon/2} + \frac{i\alpha_h \tilde{\mu}^{\epsilon/2}}{2} \Gamma(\epsilon/2) \int dF_3 \left( \mu^2/D_M'' \right)^{\epsilon/2} + i(Z_g - 1)g\tilde{\mu}^{\epsilon/2}, \]

where \( D_m' = x_1 x_3 k_1^2 + x_3 x_2 k_2^2 + x_1 x_2 k_3^2 + m^2 \), and \( dF_3 = dx_1 \, dx_2 \, dx_3 \, \delta(x_1 + x_2 + x_3 - 1) \).

(These integrals are computed on p.124.) Therefore, in \( \overline{\text{MS}} \), we have

\[ Z_g - 1 = -\frac{1}{\epsilon} \left( \frac{\alpha_g \alpha_{g'}^2}{\alpha_g} \right). \]

- Corrections to \( \phi\chi^2 \) vertex:

\[ iV_{\phi\chi^2} = \quad + \quad + \quad \]

\[ = \frac{i\alpha_h \tilde{\mu}^{\epsilon/2}}{2} \Gamma(\epsilon/2) \int dF_3 \left( \mu^2/D'' \right)^{\epsilon/2} + \frac{i\alpha_h \tilde{\mu}^{\epsilon/2}}{2} \Gamma(\epsilon/2) \int dF_3 \left( \mu^2/D'' \right)^{\epsilon/2} + i(Z_h - 1)h\tilde{\mu}^{\epsilon/2}, \]

so

\[ Z_h - 1 = -\frac{1}{\epsilon} \left( \alpha_h + \frac{\alpha_g^{1/2}}{\alpha_h} \right). \]

(b) Due to the scale independence of the bare couplings, we have

\[ 0 = \begin{pmatrix} \frac{dg}{d\log \mu} \\ \frac{dh}{d\log \mu} \end{pmatrix} = \begin{pmatrix} g \epsilon/2 \\ h \epsilon/2 \end{pmatrix} + \begin{pmatrix} \frac{\partial G}{\partial g} \\ \frac{\partial H}{\partial h} \end{pmatrix} \begin{pmatrix} 1 + g \frac{\partial G}{\partial g} \\ 1 + h \frac{\partial H}{\partial h} \end{pmatrix} \begin{pmatrix} \frac{dg}{d\log \mu} \\ \frac{dh}{d\log \mu} \end{pmatrix}, \]

which we may invert to yield

\[ \begin{pmatrix} \frac{dg}{d\log \mu} \\ \frac{dh}{d\log \mu} \end{pmatrix} = -\begin{pmatrix} 1 + g \frac{\partial G}{\partial g} \\ 1 + h \frac{\partial H}{\partial h} \end{pmatrix}^{-1} \begin{pmatrix} g \epsilon/2 \\ h \epsilon/2 \end{pmatrix}. \]

For a renormalizable theory, \( dg/d\log \mu \) and \( dh/d\log \mu \) are affine functions of \( \epsilon \). The inverse may be computed by inspection in orders of \( \epsilon^{-1} \). At order \( \epsilon \), we see that that pieces linear in \( \epsilon \) are

\[ \frac{dg}{d\log \mu} = -g \epsilon/2 + \ldots, \quad \frac{dh}{d\log \mu} = -h \epsilon/2 + \ldots, \]

and the constant pieces are obtained by substituting the above back in and looking at the order \( \epsilon^0 \) terms, yielding

\[ \begin{cases} \frac{dg}{d\log \mu} = -g \epsilon/2 + g \frac{\partial G_1}{\partial \log g} + g \frac{\partial G_1}{\partial \log h}, \\ \frac{dh}{d\log \mu} = -h \epsilon/2 + h \frac{\partial H_1}{\partial \log g} + h \frac{\partial H_1}{\partial \log h}. \end{cases} \]
(c) From the results of part (a), we have
\[
\begin{align*}
G_1 &= \frac{1}{(4\pi)^3} \left[ \left( -\frac{3}{2} - \frac{1}{6} \right) (g^2 + h^2) - \left( g^2 + \frac{h^3}{g} \right) \right], \\
H_1 &= \frac{1}{(4\pi)^3} \left[ -\left( -\frac{1}{6} \right) (g^2 + h^2) + \left( -\frac{1}{2} \right) h^2 - (h^2 + gh) \right],
\end{align*}
\]

so
\[
\begin{align*}
\beta_g(g, h) &= \frac{1}{(4\pi)^3} \left( -\frac{3}{4} g^3 - h^3 + \frac{1}{4} gh^2 \right), \\
\beta_h(g, h) &= \frac{1}{(4\pi)^3} \left( -\frac{7}{12} h^3 - gh^2 + \frac{1}{12} g^2 h \right).
\end{align*}
\]

(d) Redefine \( \varphi \) and \( h \) so that \( g \) is positive, and write the beta functions as
\[
\begin{align*}
\beta_g &= \frac{g^3}{(4\pi)^3} \left( \frac{3}{4} - \frac{h}{g} \right)^3 + \frac{1}{4} \left( \frac{h}{g} \right)^2, \\
\beta_h &= \frac{g^2 h}{(4\pi)^3} \left( -\frac{7}{12} \left( \frac{h}{g} \right)^2 - \frac{h}{g} + \frac{1}{12} \right).
\end{align*}
\]
Both \( \beta_g \) and \( \beta_h/h \) are negative for \( h/g > (-g + \sqrt{13})/7 = 0.08 \). Therefore, in that range, \( \beta_h \) is also negative.

Physically, a negative beta function means that the coupling decreases as the energy scale \( \mu \) increases. This means that perturbation theory (an expansion in the orders of the coupling constants) can be used to probe the short distance behavior of such a theory. This is known as asymptotic freedom.

4. (a) Recall that \( \beta(g) = \lim_{\epsilon \to 0} \frac{dg}{d \log \mu} \), so for the given redefinition of coupling constant we have
\[
\beta(\tilde{g}) = \lim_{\epsilon \to 0} \frac{d\tilde{g}}{d \log \mu} = \frac{d\tilde{g}}{dg} \beta(g).
\]
Now, taking derivative with respect to \( \tilde{g} \) yields
\[
\beta'(\tilde{g}) = \frac{d}{d\tilde{g}} \frac{d\tilde{g}}{dg} \beta(g) + \frac{d\tilde{g}}{dg} \beta'(g) = \beta'(g),
\]
so when evaluating at \( \tilde{g} = g = 0 \) we see that the linear coefficients coincide (and are equal to zero). Taking a second derivative with respect to \( g \) yields
\[
\beta''(\tilde{g}) = \frac{d\tilde{g}}{dg} \beta''(g),
\]
and at \( \tilde{g} = g = 0, dg/d\tilde{g} = 1 \), so the quadratic coefficients coincide. A third derivative yields
\[
\beta'''(\tilde{g}) = \frac{d\tilde{g}}{dg} \left( \frac{d}{dg} \frac{d\tilde{g}}{dg} \beta''(g) + \frac{d\tilde{g}}{dg} \beta'''(g) \right) = \left( \frac{d\tilde{g}}{dg} \right)^2 \beta'''(g),
\]
so the cubic coefficients coincide.

(b) This generalizes readily to multiple dimensionless couplings by treating \( \tilde{g} \) as a vector, and the redefinitions are such that \( d\tilde{g}/dg|_{\tilde{g}=0} \) is the identity matrix. Then the proof that the beta function has the same form proceeds identically as before.

\[^1\text{Note that there is a typo in earlier versions of the text; (28.44) should read } h_0 = Z^{-1/2}_\varphi Z_{\chi^{-1}} Z_{h} l^2.\]
5. We pick up with the expression for the one loop diagram for the one loop correction to the Yukawa coupling,

\[ iV_{Y}^{\text{one-loop}}(p', p) = \frac{g^3}{8\pi^2} \left[ \left( \frac{1}{\epsilon} - \frac{1}{4} - \frac{1}{2} \int dF_3 \log(D/\mu^2) \right) \gamma_5 + \frac{1}{4} \int dF_3 \frac{N}{D} \right], \]

where \( D = x_1(1 - x_1)p^2 + x_2(1 - x_2)p^2 - 2x_1x_2p^2' + (x_1 + x_2)m^2 + x_3M^2 \) and \( \tilde{N} = -(x_1\phi + (1 - x_2)\phi' + m)(-1 - x_1\phi + x_2\phi' + m)\gamma_5 \). This is to be cancelled by the counterterm \(-N_{\gamma - 1}g\gamma_5\), subject to the renormalization condition \( iV_{Y}(0, 0) = -g\gamma_5 \).

At \( p = p' = 0 \) the expression greatly simplifies, with \( D = (x_1 + x_2)m^2 + x_3M^2 = (1 - x_3)m^2 + x_3M^2 \) and \( N = m^2\gamma_5 \). The integral \( \int dF_3 \) becomes \( 2 \int_0^1 \int_0^{1-x_3} dx_2 dx_3 = 2 \int_0^1 (1 - x_3)dx_3 \). The \( dx_3 \) integral can be done using

\[ \int_0^1 \log(a + bx) \, dx = x \log(a + bx) - x + \frac{a}{b} \log(a + bx) \bigg|_0^1 = -1 + \frac{a}{b} \log a + \frac{a + b}{b} \log(a + b), \]

yielding

\[ iV_{Y}^{\text{one-loop}}(0, 0) = \frac{g^3}{8\pi^2} \gamma_5 \left( \frac{1}{\epsilon} - \frac{1}{2} - \log \frac{M}{\mu} - \frac{m^2}{M^2 - m^2} \log \frac{M}{m} \right), \]

so that

\[ Z_{\phi} - 1 = \frac{g^2}{8\pi^2} \left( \frac{1}{\epsilon} + \frac{1}{2} - \log \frac{M}{\mu} - \frac{m^2}{M^2 - m^2} \log \frac{M}{m} \right) + O[g^4]. \]

This may be substituted back into the vertex to yield

\[ iV_{Y}(p', p) = -g\gamma_5 - \frac{g^3}{8\pi^2} \left[ \left( \frac{3}{4} + \frac{1}{2} \int dF_3 \log \frac{D}{M^2} - \frac{m^2}{M^2 - m^2} \log \frac{M}{m} \right) \gamma_5 + \frac{1}{4} \int dF_3 \frac{\tilde{N}}{D} \right], \]

which is independent of \( \mu^2 \) and \( \epsilon \).

6. A scalar field in \( d \) spacetime dimensions has dimension \([\varphi] = d/2 - 1\), so a \( \varphi^n \) interaction is renormalizable if \( n(d/2 - 1) \leq d \).

For \( d = 2 \), interactions to all orders are renormalizable. A very important class of theories with interactions to all orders is the nonlinear sigma model, where the couplings may be taken to be the series expansion of the metric of some manifold. In \( d = 2 \), such theories are renormalizable.

For \( d = 3 \), \( \varphi^3, \varphi^4, \varphi^5, \varphi^6 \) interactions are renormalizable.

For \( d = 4 \), \( \varphi^3, \varphi^4 \) are renormalizable.

For \( d = 5, 6 \), \( \varphi^3 \) is renormalizable.

For \( d > 6 \), only the mass term is renormalizable.
7. (a) The calculations in the $N$ flavor $\varphi^4$ theory will be largely similar to that in the $N = 1 \varphi^4$ theory, which was done in the previous problem set, but with the addition of some flavor factors. I will use the results from the $N = 1 \varphi^4$ theory, which shall be denoted with a subscript $0$.

The Feynman rule for the propagator from flavor $i$ to $j$ is 
\[-i\delta_{ij} / (k^2 + m^2 - i\epsilon),\]
and the Feynman rule attached to the four point vertex is
\[i \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}.\]

The correction to the propagator therefore has the extra flavor factor
\[i \Pi_{i,j}(k^2) = 2(N + 2) \Pi_0(k^2).\]
(The factor of 2 is because there is no longer a symmetry factor of $1/2$, which was present in the $N = 1$ case.) Therefore,
\[Z_m - 1 = 2(N + 2) \frac{\lambda}{(4\pi)^2 \epsilon} + O[\lambda]^2, \quad Z_\varphi - 1 = O[\lambda]^2.\]

The correction to the 4 point vertex similarly acquires flavor factors. For the $s$ channel diagram this goes as
\[2 \delta_{ij} \delta_{mn} \delta_{im} \delta_{jn} + \delta_{im} \delta_{jm} (\delta_{mn} \delta_{kl} + \delta_{mk} \delta_{nl} + \delta_{nk} \delta_{ml}) i V_{4,0}^s,\]
which is
\[2 ((N + 4) \delta_{ij} \delta_{kl} + 2 \delta_{ik} \delta_{jl} + 2 \delta_{il} \delta_{jk}) i V_{4,0}^s.\]
(Once again, the factor of 2 comes from the lack of symmetry factor here compared to the $N = 1$ case.) Adding up the contributions from the $t$ and $u$ channels, we obtain
\[Z_\lambda - 1 = 2(N + 8) \frac{\lambda}{(4\pi)^2 \epsilon}.\]

Notice that these reduce to the correct limits (cf. problem 2) when $N = 1$.

Following problem 2, we find the beta function and anomalous mass dimension
\[\beta(\lambda) = \frac{2(N + 8)\lambda^2}{(4\pi)^2}, \quad \gamma_m(\lambda) = \frac{(N + 2)\lambda}{(4\pi)^2}.\]
(b) The Wilson-Fisher fixed point is a nontrivial solution to $d\lambda/d\log \mu = -\epsilon \lambda + \beta(\lambda) = 0$, $dm^2/d\log \mu = 0$, which is at

$$\lambda_* = \frac{(4\pi)^2\epsilon}{2(N + 8)}, \quad m_*^2 = 0.$$

(c) This scalar field theory may be viewed as an effective description of some statistical model. The massive field $\varphi$ gives rise to a Yukawa potential $-e^{-mr}/r$, which is corrected in interacting theories by the renormalization flow $d\log m/d\log \mu = \gamma_m$, i.e. $(m/m_0) = (\mu/\mu_0)^{\gamma_m} = (r/r_0)^{-\gamma_m}$, where we have taken the mass scale to be $\mu = 1/r$. Therefore, the corrected Yukawa potential is $-e^{-(r/\xi)^{1/2\nu}}$, where $\nu = 1/2(1 - \gamma_m)$ is the critical exponent and $\xi = m_0^{-2\nu} r_0^{1-2\nu}$ is the correlation length.

To obtain the critical exponent in $d = 3$ we can set $\epsilon = 1$ (this is certainly not small, but it seems to give good estimates in certain theories). Then

$$\nu = \frac{1}{2(1 - \gamma_m)} = \frac{1}{2(1 - (N + 2)\lambda_*/(4\pi)^2)} = \frac{N + 8}{N + 14}.$$