Midterm Solutions

Problem 1: The ϕ^4 theory has two fixed points, the trivial one at $\lambda = 0$ and the Wilson-Fisher point at $\lambda = \frac{16\pi^2}{3}\epsilon$.

Close to a fixed point the beta function is

$$\beta(\lambda) \approx B(\lambda - \lambda_*) \tag{1}$$

which leads to

$$\lambda(p) - \lambda_* \approx C\left(\frac{p}{\mu}\right)^B. \tag{2}$$

We see that for $B = \beta'(\lambda) > 0$ the theory approaches the fixed point for $p \to 0$ while for $B = \beta'(\lambda) < 0$ the theory approaches the fixed point for $p \to \infty$. For the ϕ^4 theory the the one-loop beta function is

$$\beta(\lambda) = -\epsilon\lambda + \frac{3\lambda^2}{16\pi^2}. \tag{3}$$

At the trivial fixed point $\beta'(\lambda_*) \approx -\epsilon < 0$ which implies that it is a UV fixed point, while close to the Wilson-Fisher point $\beta'(\lambda_*) \approx \epsilon > 0$ while implies that it is an IR fixed point.

In addition, the mass term is marginal and it drives the system away from either of the fixed points. From these facts we can determine the RG-diagram for this theory, which is shown in figure (1).

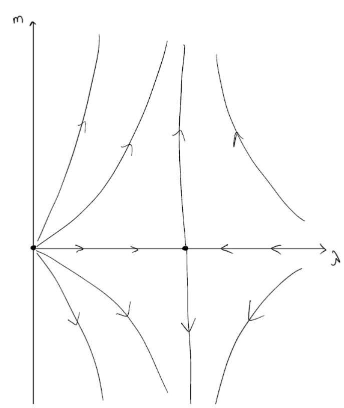


Figure 1: RG-diagmram for ϕ^4 theory in $4 - \epsilon$ dimensions.

Problem 2: From dimensional analysis, the general form of the two-point function is

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{|x|^{4(d-2)}} f\left(\mu|x|, \lambda(\mu)\right). \tag{4}$$

<u>$d = 4$</u>

- $|x| \to 0$: In the UV the theory becomes strongly coupled and we cannot really say anything about the above two-point function apart from the general from
$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{|x|^{8}} f\left(\mu|x|, \lambda(\mu)\right). \tag{5}$$

- $|x| \to \infty$: In the IR the theory will go to the Gaussian fixed point and the correlation function will be those of the free theory
$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{|x|^{8}}. \tag{6}$$

<u>$d = 4 - \epsilon$</u>

- $|x| \to 0$: In the UV the theory will go to the Gaussian fixed point and the correlation function will be those of the free theory
$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{|x|^{8-4\epsilon}}. \tag{7}$$

- $|x| \to \infty$: In the IR the theory will go to the WF fixed point and the correlation function will be
$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{|x|^{8-4\epsilon+2\gamma_{\phi^4}}}. \tag{8}$$

  In fact $\gamma_{\phi^4} = 2\epsilon$ and therefore
$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{|x|^{8}}. \tag{9}$$

Problem 3:
a)The QED Lagrangian is

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\Psi}i\not{D}\Psi - m\bar{\Psi}\Psi. \tag{10}$$

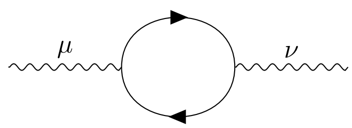
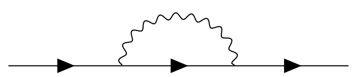
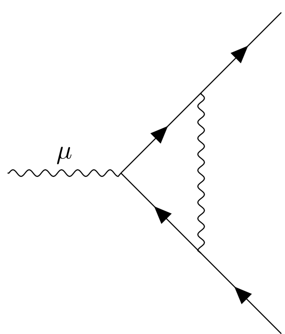
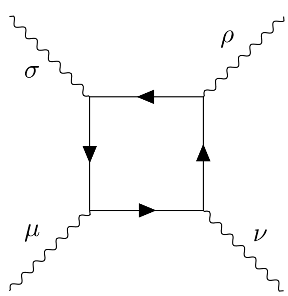
b) After gauge fixing it becomes

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\Psi}i\not{D}\Psi - m\bar{\Psi}\Psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \partial^\mu\bar{c}\partial_\mu c. \tag{11}$$

c) As usual the gauge fixing term is not renormalized and since the ghosts are decoupled $Z_\xi = Z_g = 1$. Therefore

$$L = -\frac{Z_3}{4}F^{\mu\nu}F_{\mu\nu} + Z_2\bar{\Psi}i\partial\!\!\!/\Psi + Z_1\bar{\Psi}eA\!\!\!/\Psi - Z_m m\bar{\Psi}\Psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \partial^\mu\bar{c}\partial_\mu c. \quad (12)$$

d) Using the formula for the superficial degree of diverge from a previous homework we determine which diagrams are potential divergent.

		D_{sup}	D_{real}
(i)		2	0
(ii)		1	0
(iii)		0	0
(iv)		0	−4

Gauge invariance requires that diagram (i) is of the form

$$\left(q^\mu q^\nu - \eta^{\mu\nu}q^2\right)\Pi_2(q^2). \quad (13)$$

Since the factor in front has dimension two, dimensional analysis dictates that $D_{real} = D_{sup} - 2 = 0$. Analogously for diagram (iv) gauge invariance requires that the amplitude will be schematically of the form

$$M_{\mu\nu\rho\sigma} = \left(q^{(\mu}q^\nu q^\rho q^{\sigma)} + c\,\eta^{(\mu\nu}q^\rho q^{\sigma)}q^2\right)\Pi_4(q^2) \quad (14)$$

where c is such that

$$q^\mu M_{\mu\nu\rho\sigma} = 0. \tag{15}$$

Hence, for this diagram $D_{real} = D_{sup} - 4 = -4$. Finally for diagram (ii) as it was explain in the class (see also *Peskin & Shroeder* p.319) Chiral symmetry requires that the result vanishes for zero mass, or in other words that it is proportional to the mass (which has dimension one) and therefore $D_{real} = D_{sup} - 1 = 0$.

e) To complete the proof for the renormalizability of QED we note that we have four divergent quantities, one from each of the diagrams (i) and (iii) and two from diagram (ii) which is of the form $mA_0 + A_1 \not{p}$. These four divergences can be absorbed into the four counterterms by appropriately defining Z_1, Z_2, Z_3, Z_m.

Problem 4:

a) The general classical action of a non abelian gauge theory is

$$S = \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} \right) \tag{16}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{17}$$

The gauge trnasformation of the gauge field is

$$A^a_\mu \to U^\dagger A^a_\mu U + \frac{i}{g} U^\dagger \partial_\mu U, \tag{18}$$

where $U = e^{i\alpha^a(x) T^\alpha}$ are the elements of the gauge group.

b) Imposing the constraint

$$G(A) = n^\mu A_\mu - \omega(x) = 0, \tag{19}$$

the Faddeev-Popov determinant is

$$\det\left(\frac{\delta G(A^\alpha)}{d\alpha} \right) = \det\left(\frac{1}{g} \partial^\mu D_\mu \right). \tag{20}$$

Hence, the generating functional is

$$\begin{aligned} Z &= \left(\int D\alpha \right) \int D\omega e^{-i\int d^4x \frac{\omega^2}{2\xi}} \int DADcD\bar{c} e^{i\int\left(-\frac{1}{4}F^2 + \bar{c} n^\mu D_\mu c\right)} \delta(n \cdot A - \omega) \\ &= \left(\int D\alpha \right) \int DADcD\bar{c} e^{i\int\left(-\frac{1}{4}F^2 + \bar{c} n^\mu D_\mu c - \frac{1}{2\xi}(n\cdot A)^2\right)} \end{aligned} \tag{21}$$

We see that the ghost propagator is

$$\frac{\delta^{\alpha\beta}}{n \cdot k}. \tag{22}$$

while the ghost vertex is

$$g f^{abc} n^\mu \tag{23}$$