Homework 8 Solutions

Problem 1: The **5** and **10** representations of $SU(5)$ decompose into representations of $SU(3) \times SU(2) \times U(1)$ as follows

$$\mathbf{5} = (3,1)_{-1/3} \oplus (1,2)_{1/2}, \tag{1}$$

and

$$\mathbf{10} = (3,2)_{1/6} \oplus (\bar{3},1)_{-2/3} \oplus (1,1)_1. \tag{2}$$

So we see that $\bar{\mathbf{5}} \oplus \mathbf{10}$ gives rise to a single generation of left handed fermions in the Standard Model.

Problem 2: For an anomaly-free theory we demand that

$$\mathrm{tr}\{T^a, T^b\}T^c = 0. \tag{3}$$

In this case, the matter fields carry $U(1)$ charge Q_i and belong to representations of the non abelian group G. The above requirement then translates to following conditions:

- $G \times G \times G:$ $\quad \sum_i A(R_i) d^{abc} = 0.$
- $G \times G \times Q:$ $\quad \sum_i T(R_i) Q_i = 0.$
- $Q \times Q \times Q:$ $\quad \sum_i d(R_i) q_i^3 = 0.$

Problem 3: Using the free field expansion (55.11) for the vector field one can easily prove

$$\epsilon^{\mu\nu\rho\sigma} \langle p, q| F_{\mu\nu}(z) F_{\rho\sigma}(z) |0\rangle = +8 \epsilon^{\mu\nu\rho\sigma} \epsilon_\mu e'_\nu p_\rho q_\sigma e^{i(p+q)z}. \tag{4}$$

Multiplying both sides with $g_2/16\pi^2$ and using equation (76.14) yields equation (76.29).

Problem 4: We begin by noting the following useful identity

$$\mathrm{tr}\big(T^a T^b T^c\big) = \frac{1}{2} i T(R) f^{abc} + A(R) d^{abc}. \tag{5}$$

The first term in (77.35) becomes

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} tr(T^a \partial_\mu (A_\nu \partial_\rho A_\sigma)) &= \epsilon^{\mu\nu\rho\sigma} tr(T^a \partial_\mu A_\nu \partial_\rho A_\sigma) && (6)\\
&= \epsilon^{\mu\nu\rho\sigma} \partial_\mu A^b_\nu \partial_\rho A^c_\sigma \, \mathrm{tr}\big(T^a T^b T^c\big) && (7)\\
&= A(R) \epsilon^{\mu\nu\rho\sigma} \partial_\mu A^b_\nu \partial_\rho A^c_\sigma d^{abc}. && (8)
\end{aligned}$$

Here we used the previous identity and also the fact that the term with f^{abc} cancels because it is contracted with a symmetric quantity. Next, the second term in (77.35) is the derivative of

$$
\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} A^b_\nu A^c_\rho A^d_\sigma \operatorname{tr}\big(T^a T^b T^c T^d\big) =& \frac{1}{2} e^{\mu\nu\rho\sigma} A^b_\nu A^c_\rho A^d_\sigma \operatorname{tr}\big(T^a T^b [T^c, T^d]\big) \\
=& \frac{i}{2} e^{\mu\nu\rho\sigma} A^b_\nu A^c_\rho A^d_\sigma f^{cde} \operatorname{tr}\big(T^a T^b T^e\big) \\
=& -\frac{1}{4} T(R) e^{\mu\nu\rho\sigma} A^b_\nu A^c_\rho A^d_\sigma f^{abe} f^{cde} \\
&+ \frac{i}{2} A(R) e^{\mu\nu\rho\sigma} A^b_\nu A^c_\rho A^d_\sigma d^{abe} f^{cde}
\end{aligned}
$$

We observe that the first term can be completely antisymmetrized and therefore vanishes by the Jacobi identity. In conclusion, we see that both terms in (77.35) are proportional to $A(R)$.

Problem 5: Starting from equation (77.7) we can arrive at equation (77.36) by noting the following two identities

$$
\epsilon^{\mu\nu\rho\sigma} \partial_\mu \operatorname{tr}(A_\nu A_\rho A_\sigma) = 3\epsilon^{\mu\nu\rho\sigma} \operatorname{tr}(\partial_\mu A_\nu A_\rho A_\sigma), \tag{9}
$$

and

$$
\begin{aligned}
\operatorname{tr}([A_\mu, A_\nu][A_\rho, A_\sigma]) &= A^a_\mu A^b_\nu A^c_\rho A^d_\sigma \operatorname{tr}\big([T^a, T^b][T^c, T^d]\big) && (10) \\
&= -A^a_\mu A^b_\nu A^c_\rho A^d_\sigma f^{abe} f^{cdg} \operatorname{tr}\{T^e T^g\} && (11) \\
&= -T(R) A^a_\mu A^b_\nu A^c_\rho A^d_\sigma f^{abe} f^{cde}. && (12)
\end{aligned}
$$

Using that the structure constant are completely antisymmetric, the second identity implies that

$$
\epsilon^{\mu\nu\rho\sigma} \operatorname{tr}([A_\mu, A_\nu][A_\rho, A_\sigma]) = 0. \tag{13}
$$

Hence we see that

$$
\begin{aligned}
\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \operatorname{tr}(F_{\mu\nu} F_{\rho\sigma}) &= \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \operatorname{tr}\big(\partial_{(\mu} A_{\nu)} - ig[A_\mu, A_\nu]\big)\big(\partial_{(\rho} A_{\sigma)} - ig[A_\rho, A_\sigma]\big) \\
&= \epsilon^{\mu\nu\rho\sigma} \operatorname{tr}\left(\partial_\mu A_\nu \partial_\rho A_\sigma - \frac{2}{3} ig \partial_\mu (A_\nu A_\rho A_\sigma)\right) \\
&= \epsilon^{\mu\nu\rho\sigma} \partial_\mu \operatorname{tr}\left(A_\nu \partial_\rho A_\sigma - \frac{2}{3} ig A_\nu A_\rho A_\sigma\right).
\end{aligned}
$$

Problem 6: As we mentioned in the first problem, the left handed Weyl fermions of the Standard Model have the following quantum numbers

$$
(1, 2)_{-1/2} \oplus (1, 1)_1 \oplus (3, 2)_{1/6} \oplus (\bar{3}, 1)_{-2/3} \oplus (\bar{3}, 1)_{1/3} \tag{14}
$$

We see that all the anomalies cancel

- 3-3-3:
$$0 + 0 + 2A(\mathbf{3}) + A(\bar{\mathbf{3}}) + A(\bar{\mathbf{3}}) = 0 \tag{15}$$
- 3-3-1:
$$0 + 0 + 2\frac{1}{6}T(\mathbf{3}) - \frac{2}{3}T(\bar{\mathbf{3}}) + \frac{1}{3}T(\bar{\mathbf{3}}) = 0 \tag{16}$$
- 2-2-1:
$$-\frac{1}{2}T(\mathbf{2}) + 0 + 3\frac{1}{6}T(\mathbf{2}) + 0 + 0 = 0 \tag{17}$$
- 1-1-1:
$$2\left(-\frac{1}{2}\right)^3 + 1 + 3 \times 2\left(\frac{1}{6}\right)^3 + 3\left(-\frac{2}{3}\right)^3 + 3\left(\frac{1}{3}\right)^3 = 0 \tag{18}$$
- 2-2-2:

  This anomaly is zero because the **2** is pseudoreal.