Homework 7 Solutions

Problem 1:
a) The potential is

$$V = \frac{1}{2}m^2v^2\sum_i a_i^2 + \frac{1}{4}v^4\left(\lambda_1\sum_i a_i^4 + \lambda_2\left(\sum_i a_i^2\right)^2\right). \tag{1}$$

Solving the equation $\frac{\partial V}{\partial v} = 0$ for v and plugging it back into V one gets

$$V = \frac{-\frac{1}{4}m^4}{\lambda_1 A(\alpha) + \lambda_2 B(\alpha)}, \tag{2}$$

where $A(\alpha) = \sum_i a_i^4$ and $B(\alpha) = 1$.

b) From equation (1) we see that in order for V to be bounded from below

$$\lambda_1 A(\alpha) + \lambda_2 B(\alpha) \geq 0. \tag{3}$$

c) Equation (2) implies that V reaches it's minimum when the denominator is minimum (note the minus sign in the numerator).

d) Here we need to minimize the function

$$f(a_i,\alpha,\beta) = \sum_i\left(\frac{1}{4}a_i^4 + \frac{1}{2}a_i^2 + \beta a_i\right) \tag{4}$$

For the a_i'a we get the cubic equation

$$a_i^3 + \alpha a_i + \beta = 0. \tag{5}$$

Hence, there are three different solutions whose sum is equal to zero (since the coefficient of a_i^2 is zero).

e)Starting from the identity $\sum_i(x_i - \bar{x})^2 \geq 0$ which further implies $\sum_i x_i^2 \geq N\bar{x}^2$, when $x_i = a_i^2$ we have that

$$\sum_i \alpha^4 \geq \frac{1}{N}\left(\sum_i a_i^2\right)^2 = \frac{1}{N}. \tag{6}$$

For even N it is easy to see that $a_i = \pm\frac{1}{\sqrt{N}}$ with $N_+ = N_- = \frac{1}{2}N$. For odd N a little bit more complicated analysis shows that again $a_i = \pm\frac{1}{\sqrt{N}}$ with $N_\pm = \frac{1}{2}(N \pm 1)$.

Problem 2: The unbroken subgroup in this case is $H = SU(3) \times SU(3) \times U(1)$. Hence, the number of Goldstone bosons is

$$dimSU(6) - dimH = 35 - 17 = 18. \tag{7}$$

The fundamental and antifundamental representations of $SU(6)$ decompose into representations of H as follows

$$6 = (3,1)_{-1/3} \oplus (1,3)_{1/3}, \tag{8}$$

and

$$\bar{6} = (\bar{3},1)_{1/3} \oplus (1,\bar{3})_{-1/3}, \tag{9}$$

So we see that

$$35 \oplus 1 = 6 \otimes \bar{6} = (8,1)_0 \oplus (1,8)_0 \oplus (1,1)_0 \oplus (3,\bar{3})_{-2/3} \oplus (\bar{3},3)_{2/3}. \tag{10}$$

The last two factors correspond to the gauge bosons that remain massless while the first three factors to those that acquire mass. The mass of the latter can be calculated from

$$(M^2)^{ab} = \frac{1}{2} g^2 Tr\{[T^a, \langle\Phi\rangle], [T^b, \langle\Phi\rangle]\}, \tag{11}$$

using any generator, to be

$$M^2 = \frac{2}{3} g^2 v^2. \tag{12}$$

Problem 3: From equation (88.15) we see that

$$Q = T^3 + Y. \tag{13}$$

Problem 4: The Feynamn rules for the Lagrangian (87.27) are

$$
\begin{aligned}
iV^{\mu\nu\rho}_{WW\gamma}(p,q,r) &= -ie[(p-q)^{\rho}q^{\mu\nu} + (q-r)^{\mu}g^{\nu\rho} + (r-p)^{]}ng^{\rho\mu}],\\
iV^{\mu\nu\rho}_{WWZ}(p,q,r) &= -ie\cot\theta_W(p-q)^{\rho}q^{\mu\nu} + (q-r)^{\mu}g^{\nu\rho} + (r-p)^{]}ng^{\rho\mu}],\\
iV^{\mu\nu\rho\sigma}_{\gamma\gamma WW} &= ie^2(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}),\\
iV^{\mu\nu\rho\sigma}_{\gamma ZWW} &= ie^2\cot\theta_W(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}),\\
iV^{\mu\nu\rho\sigma}_{ZZWW} &= ie^2\cot^2\theta_W(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}),\\
iV^{\mu\nu\rho\sigma}_{WWWW} &= i\frac{e^2}{\sin^2\theta_W}(2g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}),\\
iV^{\mu\nu}_{HWW} &= -2i\frac{M_W^2}{v}g^{\mu\nu},\\
iV^{\mu\nu}_{HZZ} &= -2i\frac{M_Z^2}{v}g^{\mu\nu},\\
iV^{\mu\nu}_{HHWW} &= -2i\frac{M_W^2}{v^2}g^{\mu\nu},\\
iV^{\mu\nu}_{HHZZ} &= -2i\frac{M_Z^2}{v^2}g^{\mu\nu},\\
iV_{3H} &= -3i\frac{m_H^2}{v},\\
iV4H &= -3i\frac{m_H^2}{v^2}.
\end{aligned}
$$

Problem 5: Since in this scenario the Higgs is a real triplet, its hypercharge is zero. Hence, the $U(1)$ part of $SU(2) \times U(1)$ is unbroken and the B_μ gauge boson will remain massless. Moreover, similarly to the doublet case the photon will be massless while the W will get a mass.

Problem 6: From equation (75.8) we see that Neutrinos are created by $b^\dagger$ and therefore have helicity $-1/2$ while antineutrinos are created by $d^\dagger$ and therefore have helicity $+1/2$.

Problem 7: Fields in equation (88.33) have the symmetry $\ell_I \to e^{-i\alpha_I}\ell_I$ and $e_I \to e^{i\alpha_I}e_I$ with an independent α_I for each generation. Hence, the Dirac fields $\mathcal{E}_I$ and $\mathcal{N}_{LI}$ have charge $+1$ under the I transformation and zero under the other two.

Problem 8: a) The effective Lagrangian has the form (88.43) with

$$C_V = -\frac{1}{2} + 2s_W^2, \quad C_A = -\frac{1}{2}. \tag{14}$$

b) In this case, again the effective Lagrangian has the form (88.43) but now

$$C_V = \frac{1}{2} + 2s_W^2, \quad C_A = \frac{1}{2}. \tag{15}$$