Homework 6 Solutions

Problem 1: Both equation (19.5.18) of Weinberg and equation (83.12) of Srednicki are the most general effective lagrangians for pions with exactly two derivatives. Higher order corrections, either of the form $(D_\mu D^\mu)^2, \ldots$ in Weinberg's formalism or $(\partial_\mu U^\dagger \partial^\mu U)^2, \ldots$ in Srednicki's formalism will contain more derivatives.

Hence, equation (19.5.18) of Weinberg and equation (83.12) of Srednicki have to be related by a field redefinition. Using the well-known formula

$$e^{i\pi^a \sigma^a / f_\pi} = \cos|\pi| + i\hat{\pi} \cdot \vec{\sigma} \sin|\pi|, \tag{1}$$

where we have rescaled $\vec{\pi}$ by $f_\pi/2$. Equation (83.12) can now be written as

$$\mathrm{tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) = 2\partial_\mu|\vec{\pi}|\partial^\mu|\vec{\pi}| + 2\partial_\mu\hat{\pi} \cdot \partial^\mu\hat{\pi}\ \sin^2|\vec{\pi}| \tag{2}$$

Starting from equation (19.5.18), the most general field redefinition is

$$\vec{\zeta} = \hat{\pi} f(|\vec{\pi}|). \tag{3}$$

Demanding that the two effective lagrangians are the same, i.e

$$\frac{\partial_\mu\vec{\zeta} \cdot \partial^\mu\vec{\zeta}}{(1+\zeta^2/F^2)^2} = 2\partial_\mu|\vec{\pi}|\partial^\mu|\vec{\pi}| + 2\partial_\mu\hat{\pi} \cdot \partial^\mu\hat{\pi}\ \sin^2|\vec{\pi}|, \tag{4}$$

one can get the the following algebraic equationfor the function f

$$\frac{f^2}{1+f^2/F^2} = 2\sin^2|\vec{\pi}|, \tag{5}$$

whose solution is

$$f^2(|\vec{\pi}|) = \frac{2\sin^2|\vec{\pi}|}{F^2 - 2\sin^2|\vec{\pi}|}. \tag{6}$$

Problem 2: The transformations of the fields ξ_a are given by equation (19.6.18)

$$g\gamma(\xi) = \gamma(\xi')h(\xi, g), \tag{7}$$

where

$$\begin{aligned} g &= e^{i(\theta_a^V \sigma^a + \theta_a^A \sigma^a \gamma_5)} \\ \gamma &= e^{-i\xi_a \sigma^a} \\ h &= e^{i\theta\sigma^a} \end{aligned} \tag{8}$$

for some $\theta(\xi, \theta^V, \theta^A)$. As in the case of $SU(3) \times SU(3)$ we get the analog of equation (19.7.8)

$$U'(x) = e^{iR_a\sigma^a} U(x) e^{-iL_a\sigma^a}, \tag{9}$$

where $L_a = \theta_a^V + \theta_a^A$ and $R_a = \theta_a^V - \theta_a^A$. One can now determine the transformation of ξ's using the formula

$$e^{ix^a\sigma^a} = \cos|x| + i\hat{x}\cdot\vec{\sigma}\sin|x|. \tag{10}$$

For example for $R = 0$ we get

$$i\xi'^a = \frac{i\hat{L}^a \cos\xi \sin L + i\hat{\xi}^a \sin\xi \cos L - i(\hat{\xi}\times\hat{L})^a \sin\xi \sin L}{\sqrt{1 - (\cos\xi\cos L - \hat{\xi}\cdot\hat{L}\sin\xi\sin L)^2}} \tag{11}$$

and similarly for $L = 0$.

Problem 3:

a) Each Dirac field equals two left-handed Weyl fields and therefore the flavor symmetry is $U(2n_F)$. However, the $U(1)$ is anomalous, leaving only a $SU(2n_F)$ flavor symmetry.

b)The condensate is of the form

$$\langle \chi_{\alpha i}\chi_{\alpha j}\rangle = -u^3\delta_{ij}. \tag{12}$$

A general $SU(2n_F)$ transformation $\chi_{\alpha i} \to L_{ij}\chi_{\alpha j}$ leaves the above condensate invariant only if L belongs to the $SO(2n_F)$ subgroup. Hence, the $SU(2n_F)$ flavor symmetry is broken down to $SO(2n_F)$.

c) The number of Goldstone bosons is

$$dim SU(4) - dim SO(4) = 15 - 6 = 9. \tag{13}$$

d) The non anomalous symmetry if again $SU(2n_F)$. In this case the condensate is

$$\langle \epsilon^{\alpha\beta}\chi_{\alpha i}\chi_{\beta j}\rangle = -u^3\eta_{ij}, \tag{14}$$

with $\eta_{ij} = -\eta_{ji}$. In this case the $SU(2n_F)$ flavor symmetry is broken down to $Sp(2n_F)$ and the number of Goldstone bosons is

$$dim SU(4) - dim Sp(4) = 15 - 10 = 5. \tag{15}$$

Problem 4:

For equations (83.13) and (83.37) the relevant interaction terms are

$$L_{int} = \frac{1}{6} f_\pi^{-2}(\pi^a\pi^a\partial_\mu\pi^b\partial^\mu\pi^b - \pi^a\pi^b\partial_\mu\pi^a\partial^\mu\pi^b + \frac{1}{4}m_\pi^2\pi^a\pi^a\pi^b\pi^b). \tag{16}$$

The scattering amplitude with four outgoing momenta is

$$V^{abcd} = f^{-2}\left(\delta^{ab}\delta^{cd}(s - m_\pi^2) + \delta^{ac}\delta^{bd}(t - m_\pi^2) + \delta^{ad}\delta^{bc}(u - m_\pi^2)\right). \qquad (17)$$

Problem 5: a) From (83.19) one can easily get

$$\begin{aligned}
m_{\pi^\pm}^2 &= 2v^3 f_\pi^{-2}(m_u + m_d)\\
m_{K^\pm}^2 &= 2v^3 f_\pi^{-2}(m_u + m_s)\\
m_{K^0}^2 &= 2v^3 f_\pi^{-2}(m_d + m_s)\\
m_{\pi^0,\eta}^2 &= \frac{4}{3}v^3 f_\pi^{-2}(m_u + m_d + m_s) \pm \sqrt{m_u^2 + m_d^2 + m_s^2 - m_u m_d - m_u m_s - m_d m_s})
\end{aligned}$$

b) Expanding for small $m_{u,d}/m_s$ we get

$$\begin{aligned}
\Delta m_{EM}^2 &= m_{\pi^\pm}^2 m_{\pi^0}^2 = 0.00138 GeV^2\\
m_u v^3 f_\pi^2 &= \frac{1}{4}(+m_{K^\pm}^2 - m_{K^0}^2 + m_{\pi^0}^2 - \Delta m_{EM}^2) = 0.00288 GeV^2\\
m_d v^3 f_\pi^2 &= \frac{1}{4}(-m_{K^\pm}^2 + m_{K^0}^2 + m_{\pi^0}^2 + \Delta m_{EM}^2) = 0.00624 GeV^2\\
m_s v^3 f_\pi^2 &= \frac{1}{4}(+m_{K^\pm}^2 + m_{K^0}^2 - m_{\pi^0}^2 - \Delta m_{EM}^2) = 0.00624 GeV^2
\end{aligned}$$

c) $m_u/m_d = 0.46$ and $m_s/m_d = 19$.

d) Using equations (83.50)-(83.52) in (83.48), we find $m_\eta = 0.566 GeV$ which is 3% larger than its observed value, $0.548 GeV$.

Problem 6:

a) Requiring the coefficient of $\partial_\mu \pi^9 \partial^\mu \pi^9$ to be $\frac{1}{2}$ yields

$$F^2 = \frac{1}{9}(2f_9^2 - 3f_\pi^2). \qquad (18)$$

b) Only the mass terms for π_0, η, and π^9 are different and one can easily get

$$m_\pi^2 = 4m\frac{v^3}{f_\pi^2},\ m_\eta^2 = \frac{8}{3}m_s\left(f_\pi^{-2} + \frac{3}{4}f_9^{-2}\right)v^3,\ m_{\pi^9}^2 = \frac{9f_\pi^2}{4f_9^2 + 3f_\pi^2}m_\pi^2. \qquad (19)$$