Homework 5 Solutions

Problem 1: Demanding that equations (65.2)-(65.4) are of the form

$$D^\mu \phi D_\mu \phi \tag{1}$$

with $D_\mu = \partial_\mu - ie' A_\mu$, for some constant e', we must have

$$Z_2 = Z_1 \frac{e}{e'}, \quad Z_2 = Z_4 \left(\frac{e}{e'}\right)^2. \tag{2}$$

Eliminating the e's we get

$$Z_4 = \frac{Z_1^2}{Z_2}. \tag{3}$$

Problem 2: a) From equations (65.1)-(65.4) we can easily derive

$$j^\mu = ieZ_2(\phi \partial^\mu \phi^\dagger - \phi^\dagger \partial^\mu \phi) + 2iZ_1 e^2 \phi^\dagger \phi A^\mu, \tag{4}$$

and

$$\frac{\partial \mathcal{L}_1}{\partial A_\mu} = -iZ_1 e(\phi \partial^\mu \phi^\dagger - \phi^\dagger \partial^\mu \phi) - 2iZ_4 e^2 \phi^\dagger \phi A^\mu. \tag{5}$$

Hence, the classical equations of motion in Lorenz gauge is

$$-Z_3 \partial^2 A^\mu = \frac{\partial \mathcal{L}_1}{\partial A_\mu} = Z_1 Z_2^{-1} j^\mu, \tag{6}$$

where we used that $Z_4 = Z_1^2/Z_2$.

As in section (65), the LSZ formula implies that

$$iZ_3 \int d^4x d^4y d^4z e^{ikx - ip'y + ipz} (-\partial_x^2) \left\langle T A^\mu(x) \phi(y) \phi^\dagger(z) \right\rangle, \tag{7}$$

is the photon-scalar-scalar vertex with the photon propagator stripped off. Therefore, the quantity

$$C^\mu(k, p, p') = iZ_1 Z_2^{-1} \int d^4x d^4y d^4z e^{ikx - ip'y + ipz} \left\langle T j^\mu(x) \phi(y) \phi^\dagger(z) \right\rangle, \tag{8}$$

is equal to

$$C^\mu(k, p, p') = (2\pi)^2 \delta^4(k + p - p') \left[\frac{1}{i} \tilde{\Delta}(p') i V_3^\mu(k, p, p') \frac{1}{i} \tilde{\Delta}(p)\right]. \tag{9}$$

Having established these facts, it is straightforward to show (as in spinor QED on page 413) that

$$(p'-p)_\mu V_3^\mu(k,p,p') = Z_1 Z_2^{-1} e\left[\tilde{\Delta}(p')^{-1} - \tilde{\Delta}(p)^{-1}\right]. \tag{10}$$

b) Since both $V_3^\mu(k,p,p')$ and $\tilde{\Delta}(p)$ are finite, but the Z_i's diverge we must have

$$Z_1 = Z_2. \tag{11}$$

c) Similarly we can define the quantity

$$C^{\mu\nu}(k,k',p,p') = iZ_1^2 Z_2^{-2} \int d^4x d^4y d^4z e^{ikx+ik'w-ip'y+ipz} \left\langle T j^\mu(x) j^\nu(w) \phi(y) \phi^\dagger(z) \right\rangle. \tag{12}$$

This gets contribution from all the three- and four-point vertices

$$\begin{aligned} C^{\mu\nu}(k,k',p,p') =& (2\pi)^2 \delta^4(k+p-p') \frac{1}{i}\tilde{\Delta}(p')\Big[iV_4^{\mu\nu}(k,k',p,p') && (13)\\ &+ iV_3^\mu(p',p+k')\frac{1}{i}\tilde{\Delta}(p+k') iV_3^\nu(p+k',p) && (14)\\ &+ iV_3^\nu(p',p+k')\frac{1}{i}\tilde{\Delta}(p+k') iV_3^\mu(p+k',p)\Big]\frac{1}{i}\tilde{\Delta}(p) && (15)\end{aligned}$$

Proceeding as before we can show

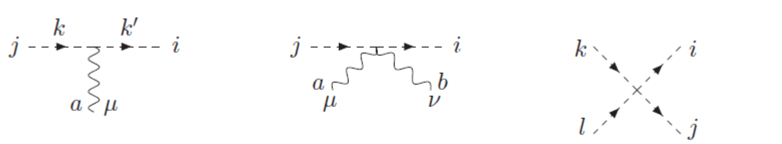
$$k_\mu C^{\mu\nu}(k,k',p,p') = Z_1 Z_2^{-1} e\left(C^\nu(k',p'-k,p) - C^\mu(k',p',p+k)\right), \tag{16}$$

which furthermore leads to

$$k_\mu V^{\mu\nu}(k,k',p,p') = Z_1 Z_2^{-1} e\left(V^\nu(p+k',p) - V^\mu(p',p'-k)\right). \tag{17}$$

Problem 3:

The Feynman rules for scalar electrodynamics can be easily read of from equations (65.1)-(65.4) to be



$ig(T^a)_{ij}(k+k')^\mu$ $\qquad -ig^2(T^aT^b+T^bT^a)_{ij}g^{\mu\nu}$ $\qquad -i\frac{1}{2}\lambda(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl})$

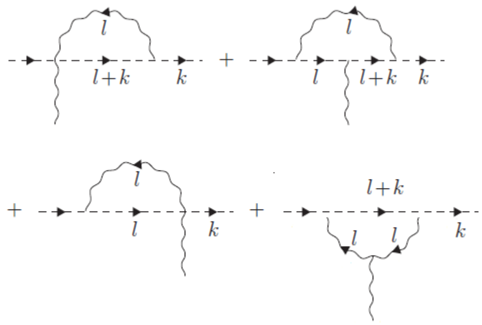


Figure 1

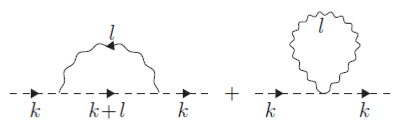


Figure 2

Problem 4:

The calculations of the Z factors in this case are the same as for one abelian scalar, except from some additional factors that come from group structure of the vertices.

For Z_1 we need to evaluate the diagrams in figure (1) and the result is

$$Z_1 = 1 + (3C(R) - T(A)) \frac{g^2}{8\pi^2} \frac{1}{\epsilon} \tag{18}$$

For Z_2 we need to evaluate the diagrams in figure (2) and the result is

$$Z_2 = 1 + C(R) \frac{3g^2}{8\pi^2} \frac{1}{\epsilon} \tag{19}$$

For Z_3 we need to evaluate diagrams shown in figures (3) and (4)

The diagrams in figure (3) contribute

$$-\frac{1}{3} T(R) \frac{g^2}{8\pi^2}, \tag{20}$$

while the diagrams in figure (4) contribute

$$\frac{5}{3} T(A) \frac{g^2}{8\pi^2}. \tag{21}$$

Combining the contributions we get

$$Z_3 = 1 + \left(\frac{5}{3} T(A) - \frac{1}{3} T(R) \right) \frac{g^2}{8\pi^2} \frac{1}{\epsilon}. \tag{22}$$

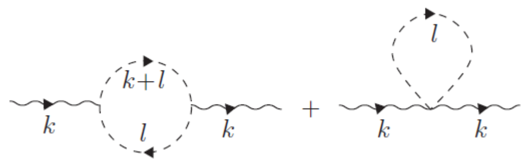


Figure 3

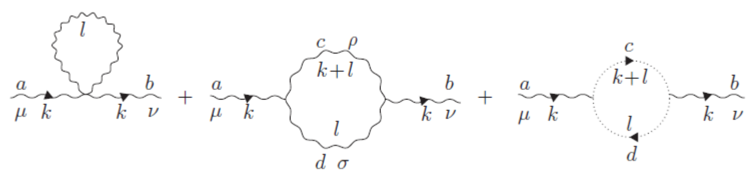


Figure 4

Following the general analysis of section (52)

$$\beta(g) = -\left(\frac{11}{3}C(R) - \frac{1}{3}T(R)\right)\frac{g^3}{(4\pi)^2}. \tag{23}$$

Problem 5: The general analysis of section (53) shows that when we integrate out a field with Lagrangian of the form

$$L = \phi K \phi, \tag{24}$$

for some operator K, we get

$$\det\{K\}^{\pm 1}, \tag{25}$$

where the minus sign is for bosons and the plus for fermions. Hence, all the possible one-loop contributions to the terms in the quantum action that do not depend on the ghost fields are

- $c^a(\bar{D}^2)^{ab}c^b \to \det \bar{D}^2 = \det \Box_{A,(1,1)}$
- $\Psi(i\bar{D})\Psi \to \det i\bar{\not{D}} = (\det\left(i\bar{\not{D}}^2\right)^{1/2} = \det \Box_{R_{DF},(2,1)\oplus(1,2)}$
- $\phi^a(\bar{D}^2)^{ab}\phi^b \to (\det \Box_{R_{CB},(1,1)})^{-1}$
- $\frac{1}{2}\mathcal{A}^a\left((\bar{D}^2)^{ab}c^b + g(T^a)^{bc}\bar{F}^a_{\mu\nu}S^{\mu\nu}_{(a,b)}\right)\mathcal{A}^b \to \det \Box_{A,(2,2)}$