Homework 4 Solutions

Problem 1: For QED the superficial degree of divergence is

$$D = dL - P_e - 2P_\gamma, \tag{1}$$

where L the number of loop momenta and P_e and P_γ the number of electron and photon propagators respectively. Moreover the number of loop momenta is equal to

$$L = P_e + P_\gamma - V + 1, \tag{2}$$

since every propagator has a momentum integral and every vertex a delta function that imposes momentum conservation. The extra plus one comes from the delta function that imposes the overall momentum conservation. Also the number of vertices is

$$V = 2P_\gamma + N_\gamma = P_e + \frac{1}{2}N_e, \tag{3}$$

since for every internal photon we have two vertices and for every external photon an additional one. Combining these equation we can write (1) as

$$D_{QED} = d + \frac{d-4}{2}V - \frac{d-2}{2}N_\gamma - \frac{d-2}{2}N_e. \tag{4}$$

For scalar QED the superficial degree of divergence is

$$D = dL - 2P_\phi - 2P_\gamma + V_3, \tag{5}$$

where V_3 is the number of cubic vertices $\phi\partial\phi A$. We added this term because for each such vertex we get an additional momentum coming from the derivative acting on ϕ. Similarly with QED we have

$$L = P_\phi + P_\gamma - V_3 - V_4 + 1, \tag{6}$$

and

$$2P + N_\phi + N_\gamma = 3V_3 + 4V_4. \tag{7}$$

Again combining these equations we get

$$D_{sQED} = d + (d-4)V_4 + \frac{d-4}{2}V_3 - \frac{d-2}{2}N. \tag{8}$$

Problem 2: a)Gauge invariance implies that the $q^\mu q^\nu$ in the photon propagator should not contribute to the scattering amplitude of two fermions and one photon. Hence we should have

$$q_\mu \bar{u}(p')V^\mu u(p) = 0, \quad q = p' - p. \tag{9}$$

Acting with q^μ in (63.23) we get

$$0 = q_\mu \bar{u}(p') V^\mu u(p) = e(p'-p)_\mu \bar{u}'[A\gamma^\mu + B(p'+p)^\mu + C(p'-p)]u \quad (10)$$
$$= e\bar{u}'[A(\not{p}' + \not{p}) + B(p'^2+p^2) + Cq^2]u \quad (11)$$
$$= eCq^2\bar{u}'u. \quad (12)$$

To go from the second to the third line we used $\bar{u}'\not{p}' = -m\bar{u}'$ and $\not{p}u = -mu$ as well as $p'^2 = p^2 = m^2$. Hence we see that gauge invariance require $C = 0$.
b)Using equation (63.16) we make the replacement

$$A\gamma_\mu + B(p'+p)_\mu \to (A+2mB)\gamma_\mu + 2iBS^{\mu\nu}q_\nu. \quad (13)$$

Comparing this equation with (63.23) we see that

$$F_1 = A + 2mB, \quad F_2 = -2mB. \quad (14)$$

Problem 3: For the purpose of this problem it's more convenient to chose a gauge such that

$$A = \frac{B}{2}(-y, x, 0). \quad (15)$$

In this gauge the in stead of $i\gamma^2\partial_1$ in equation (64.10), we have $\frac{1}{2}i(\gamma^2\partial_1 - \gamma^1\partial_2)$. Using that equations (64.12) and (64.13) and then the properties $\bar{u}\gamma^i u = 2p^i\bar{u}u$ and $\bar{u}u = 2m$ the above term in the Hamiltonial will become

$$-\frac{e}{2m}i(p_1\partial_2 - p_2\partial_1), \quad (16)$$

which is just $\frac{e}{2m}L_z$. Hence the elector magnetic moment is

$$\mu = \frac{e}{m}\left(\frac{1}{2}\left(1+\frac{\alpha}{2\pi}\right) + \frac{m_l}{2}\right), \quad (17)$$

so the new part due to the orbital angular momemtum is

$$\mu_l = \frac{em_l}{2m}. \quad (18)$$

Problem 4:
a) The field equation that follows from (64.3) is

$$(i\not{D} - m)\Psi + \frac{e}{2m}F_2(0)F_{\mu\nu}S^{\mu\nu}\Psi = 0, \quad (19)$$

where

$$\not{D} = \not{\partial} - ieF_1(0)\not{A}. \quad (20)$$

Working in the Dirac representation of the gamma matrices

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \tag{21}$$

the matrices $S^{\mu\nu}$ are

$$S^{0i} = \frac{i}{2}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \tag{22}$$

In this representation the Dirac equation (19) becomes

$$\begin{pmatrix} i\partial_0 - eF_1\varphi - m & i\vec{\sigma}\cdot\vec{\partial} + eF_1\vec{\sigma}\cdot\vec{A} \\ -i\vec{\sigma}\cdot\vec{\partial} - eF_1\vec{\sigma}\cdot\vec{A} & -i\partial_0 + eF_1\varphi - m \end{pmatrix}\Psi + \frac{e}{2m}F_2\begin{pmatrix} \vec{\sigma}\cdot\vec{B} & \vec{\sigma}\cdot\vec{E} \\ \vec{\sigma}\cdot\vec{E} & \vec{\sigma}\cdot\vec{B} \end{pmatrix}\Psi = 0, \tag{23}$$

where we also used $A_\mu = (-\varphi, \vec{A})$. Extracting the time dependence due to the rest energy we can write

$$\Psi = e^{imt}\begin{pmatrix} \Phi \\ X \end{pmatrix}. \tag{24}$$

In the non-relativistic limit the kinetic energy of the electron as well as the potential energy due to the electromagnetic field are much smaller than the rest energy. In other words

$$\dot{X} \ll mX, \quad eA \ll m. \tag{25}$$

In this limit, the above equation in components reads

$$(i\partial_0 - eF_1\varphi + \frac{e}{2m}F_2\vec{\sigma}\cdot\vec{B})\Phi + (i\vec{\sigma}\cdot\vec{\partial} + eF_1\vec{\sigma}\cdot\vec{A} + \frac{e}{2m}F_2\vec{\sigma}\cdot\vec{E})X = 0 \tag{26}$$

$$(-i\vec{\sigma}\cdot\vec{\partial} - eF_1\vec{\sigma}\cdot\vec{A} + \frac{e}{2m}F_2\vec{\sigma}\cdot\vec{E})\Phi - 2mX = 0. \tag{27}$$

We can solve the second equation for X

$$X = \frac{1}{2m}\left(-i\vec{\sigma}\cdot\vec{\partial} - eF_1\vec{\sigma}\cdot\vec{A} + \frac{e}{2m}F_2\vec{\sigma}\cdot\vec{E}\right), \tag{28}$$

and substituting in the above equation for Φ, neglecting higher order terms, we get

$$(i\partial_0 - eF_1\varphi + \frac{e}{2m}F_2\vec{\sigma}\cdot\vec{B})\Phi - (i\vec{\sigma}\cdot\vec{\partial} + eF_1\vec{\sigma}\cdot\vec{A})^2\Phi = 0, \tag{29}$$

which can be written as

$$i\frac{\partial\Phi}{\partial t} = H\Phi, \tag{30}$$

with

$$H = (i\vec{\sigma}\cdot\vec{\partial} + eF_1\vec{\sigma}\cdot\vec{A})^2 + eF_1\varphi - \frac{e}{2m}F_2\vec{\sigma}\cdot\vec{B}. \tag{31}$$

Moreover the first term can be simplified by using the properties of the Pauli matrices

$$(\vec{\sigma} \cdot (i\vec{\partial} + eF_1\vec{A})) = (i\vec{\partial} + eF_1\vec{A})^2 + i\vec{\sigma} \cdot (i\vec{\partial} + eF_1\vec{A}) \times (i\vec{\partial} + eF_1\vec{A}) \quad (32)$$

$$= (i\vec{\partial} + eF_1\vec{A})^2 - eF_1\vec{\sigma} \cdot \vec{\nabla} \times \vec{A} \quad (33)$$

$$= (i\vec{\partial} + eF_1\vec{A})^2 - eF_1\vec{\sigma} \cdot \vec{B}. \quad (34)$$

Hence, the Hamiltonian is

$$H = (i\vec{\partial} + eF_1\vec{A})^2 + eF_1\varphi - \frac{e}{2m}(F_1 + F_2)\vec{\sigma} \cdot \vec{B}. \quad (35)$$

Now we can easily read the magnetic moment of the electron to be

$$\vec{\mu} = \frac{e}{2m}(F_1 + F_2)\sigma \rightarrow \frac{e}{2mc}(1 + F_2)\hbar\sigma = g\frac{e}{2mc}\frac{\hbar\sigma}{2}. \quad (36)$$

Problem 5: Replacing ε_1' by k_1' and using momentum conservation the amplitude becomes

$$T = e^2\bar{v}_2 \left[\varepsilon_2' \left(\frac{-\not{p}_1 + \not{k}_1' + m}{m^2 - t} \right) \not{k}_1' + \not{k}_1' \left(\frac{-\not{p}_2 - \not{k}_1' + m}{m^2 - u} \right) \varepsilon_2' \right] u_1. \quad (37)$$

Using $\not{k}_1'\not{k}_1' = -k_1^2 = 0$ we have

$$T = e^2\bar{v}_2 \left[\varepsilon_2' \left(\frac{-\not{p}_1 + m}{m^2 - t} \right) \not{k}_1' + \not{k}_1' \left(\frac{-\not{p}_2 + m}{m^2 - u} \right) \varepsilon_2' \right] u_1. \quad (38)$$

Then we can commmute the momenta by using

$$(-\not{p}_1 + m)\not{k}_1' = \not{k}_1'(\not{p}_1 + m) + 2p_1 \cdot k_1, \quad (39)$$

and

$$\not{k}_1'(-\not{p}_2 + m) = (\not{p}_2 + m)\not{k}_1' - 2p_2 \cdot k_1, \quad (40)$$

and noting that $(\not{p}_1 + m)u_1 = 0$ and $\bar{v}_2(\not{p}_2 + m) = 0$ we arrive at

$$T = e^2\bar{v}_2 \left[\varepsilon_2' \left(\frac{2p_1 \cdot k_1}{m^2 - t} \right) \not{k}_1' - \not{k}_1' \left(\frac{2p_2 \cdot k_1}{m^2 - u} \right) \varepsilon_2' \right] u_1. \quad (41)$$

It's easy to see now that this vanishes since $2p_1 \cdot k_1 = t - m^2$ and $2p_1 2 \cdot k_1 = u - m^2$.