Homework 3 Solutions

Problem 1: A gauge field transforms as

$$A_\mu \to U A_\mu U^\dagger + U\partial_\mu U^\dagger. \tag{1}$$

For infinitesimal transformations, using the Baker-Campbell-Hausdorff formula we can easily evaluate both terms

$$U A_\mu U^\dagger = e^{-ig\theta^a T_R^a} A_\mu^b T_R^b e^{ig\theta^c T_R^c} = A_\mu^b \left(T_R^b - ig\theta^a [T_R^a, T_R^b]\right) \tag{2}$$

$$= A_\mu^b \left(T_R^b + g\theta^a f^{abc} T_R^c\right) \tag{3}$$

and similarly for the second term

$$U\partial_\mu U^\dagger = -\partial_\mu \theta^b T_R^b. \tag{4}$$

Hence we see that the components of the gauge field transforms as

$$A_\mu^c \to A^c + gA^b\theta^a f^{abc} - \partial_\mu\theta^c, \tag{5}$$

which is independent of the representation of T_R.

Problem 2: From the free field expansion of the gauge field we have that

$$-i\int d^3x e^{ikx} \overleftrightarrow{\partial_0} A_\mu = \sum_\lambda \epsilon_\lambda^\mu(k) a_\lambda^\dagger(k). \tag{6}$$

Contracting this equation with ck_μ we get

$$-c\sqrt{2}\omega a_<^\dagger(k) = -c\xi\{Q, b^\dagger(k)\}, \tag{7}$$

which is the BRST transformation of

$$|\chi\rangle = -c\xi b^\dagger(k)\,|\psi\rangle\,. \tag{8}$$

Problem 3: In this problem, instead of the usual gauge condition we have to use

$$f^a = \partial^i A_i^a. \tag{9}$$

However, the calculation is exactly the same with the only difference that the Faddev-Popov determinant will be

$$det[-\partial^i D_i^{ab}\delta^4(x-y)]. \tag{10}$$

Therefore the ghost action is

$$L_{ghost} = \bar{c^a}\left(\delta^{ab}\partial^i\partial_i - gf^{abc}\partial^i A_i^b\right)c^b. \tag{11}$$

From this expression we can easily read the ghost propagator which is

$$\frac{i}{k^2}\delta^{ab}. \tag{12}$$

Problem 4: The scaling dimensions of the available fields are

$$[A] = [c] = [\bar{c}] = 1, \quad [h^a] = 2. \tag{13}$$

The most general Lorentz-invariant Lagrangian of scaling-dimension four and ghost number zero is schematicaly

$$L = -\frac{1}{4}F^2 + a_1(\partial A)^2 + a_2\ \bar{c}c\partial A + a_3\ \bar{c}\partial cA + a_4\ \bar{c}\partial^2 c + a_5\ h^2 + a_6\ h\partial A, \tag{14}$$

where F is the usual field strength and g the coupling constant. First we note that by normalizing the fields h and c we can set $\alpha_4 = a_6 = 1$. Next we see that the BRST transformation of the second term cannot be canceled since

$$\delta_B\left((\partial A)^2\right) \sim (\partial^2 c)\partial A, \tag{15}$$

and therefore $a_1 = 0$. Finally the remaining coefficients a_2 and a_3 can be determined by requiring

$$\delta_B(h\partial A + a_2\ \bar{c}c\partial A + a_3\ \bar{c}\partial cA) = 0. \tag{16}$$

Looking at the resulting terms containing h we get

$$h\partial^\mu D_\mu c + a_2\ h\partial^\mu A_\mu c + a_3\ hA_\mu\partial^\mu c = 0 \tag{17}$$

and therefore $a_3 = 1$ and $a_2 = ig$.

Problem 5:

Considering the R_ξ gauge the only thing that changes is the extra term in the photos propagator. First, we note that the self-energy diagram of the photon does not include a photos propagator and therefore Z_3 doesn't change. For the fermion self-energy diagram the contribution of the ξ-term in the photon propagator is

$$i\Delta\Sigma = (\xi - 1)e^2 \int \frac{d^l}{(2\pi)^4}\frac{\not{l}(\not{p} - \not{l} + m)\not{l}}{((p+l)^2 + m^2)\, l^4} - i(\Delta Z_2)\not{p} - i(\Delta Z_m)m. \tag{18}$$

This can be evaluated in the usual way and one can get

$$Z_2 = 1 - \xi \frac{e^2}{8\pi} \frac{1}{\epsilon}, \tag{19}$$

and

$$Z_2 = 1 - (\xi + 3) \frac{e^2}{8\pi} \frac{1}{\epsilon}. \tag{20}$$

Similarly for the vertex correction we get an extra contribution

$$i\Delta V_\mu = (\xi - 1) e^3 \int \frac{d^l}{(2\pi)^4} \frac{\not{l}(-\not{l} + m)\gamma_\mu(-\not{l} + m)\not{l}}{(l^2 + m^2)^2 \, l^4} + ie(\Delta Z_1)\gamma_\mu, \tag{21}$$

and again one can get

$$Z_1 = 1 - \xi \frac{e^2}{8\pi} \frac{1}{\epsilon}. \tag{22}$$

Setting $\xi = 0$ we recover the Z-factors in the Lorenz gauge.

Problem 6: Consider the one loop diagram with four external photos ordered as 1234 clockwise. The divergent part of the diagram is

$$I = e^4 \int \frac{d^4 l}{(2\pi)^4} \frac{Tr(\not{\epsilon}_1 \not{l} \not{\epsilon}_2 \not{l} \not{\epsilon}_3 \not{l} \not{\epsilon}_4 \not{l})}{(l^2)^4} = e^4 Tr(\not{\epsilon}_1 \gamma_\mu \not{\epsilon}_2 \gamma_\nu \not{\epsilon}_3 \gamma_\rho \not{\epsilon}_4 \gamma_\sigma) \int \frac{d^4 l}{(2\pi)^4} \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2)^4}. \tag{23}$$

Apart from a divergent part that needs to be regularized, the integral over the loop momentum will be proportional to

$$g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\sigma\nu} + g^{\mu\sigma} g^{\nu\rho}. \tag{24}$$

This has to be contracted with the trace in the above equation. Using the properties of the gamma matrices

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu, \tag{25}$$

and

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4 g^{\nu\rho}, \tag{26}$$

we get that

$$I = C \left((\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4) + (\epsilon_1 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3) - (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4) \right), \tag{27}$$

for some constant C. We see now that if we add all the six permutations of the external photos (taking into account the extra minus sign when we swap a fermion propagator) the net results vanishes.