Homework 2 Solutions

Problem 1:

a) The field strength renormalization for the composite operator ϕ^2 can be calculated from the divergent part of $G(p_1, p_2, x)$. Using the Feynman rules is easy write down the expression for this Green's function

$$\left\langle \phi(p_1)\phi(p_2)\phi^2(x)\right\rangle = \frac{i\lambda}{2}\frac{1}{p_1^2 p_2^2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k+p_1+p_2)^2} \tag{1}$$

Following the standard procedure one finds that the divergent part is equal to

$$-\frac{\lambda}{2}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}}\int_0^1 dx \frac{1}{\left(x(1-x)(p_1+p_2)^2\right)^{2-d/2}}. \tag{2}$$

Subtracting the divergence for $p_1^2 = p_2^2 = \mu^2$ we get

$$\delta_{\phi^2} = \frac{\lambda}{2}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}(\mu^2)^{2-d/2}}. \tag{3}$$

Hence, the anomalous dimension of ϕ^2 is

$$\gamma_{\phi^2} = \frac{\lambda}{16\pi^2} \tag{4}$$

The Callan-Symanzik equation at the fixed point

$$\left(\mu\frac{\partial}{\partial\mu} + 2\gamma_{\phi^2}\right)\left\langle \phi^2(x)\phi^2(0)\right\rangle = 0 \tag{5}$$

has the following solution (using the method of characteristics)

$$\left\langle \phi^2(x)\phi^2(0)\right\rangle = \frac{1}{|x|^{2(d-2)}} C(\lambda(x)) e^{2\int_{1/\mu}^{|x|} d\log|x'|\gamma(\lambda(x'))} \tag{6}$$

The theory will be driven to the WF fixed point for large $|x|$. In this case the integral in the exponent will be dominated by the large values of x'. Therefore

$$\left\langle \phi^2(x)\phi^2(0)\right\rangle \sim \frac{1}{|x|^{2(d-2+\gamma_{\phi^2})}} = \frac{1}{|x|^{2\Delta_{\phi^2}}} \tag{7}$$

b) Consider the mass term in the lagrangian as an interaction. Any Green's function will be a power series in m

$$\begin{aligned} G_m^{(n)}(x_1,\ldots,x_n) &= \langle \phi(x_1)\ldots\phi(x_n)\rangle_m && (8)\\ &= \sum(-im^2)^l \left\langle \phi(x_1)\ldots\phi(x_n)\left(\int \frac{1}{2}\phi^2\right)^l\right\rangle_0 && (9)\\ &= \sum(-im^2)^l G_0^{(n,l)}(x_1,\ldots,x_n) && (10)\end{aligned}$$

The above Green's functions obey the following Callan-Symanzik equations

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+2\gamma_m m^2\frac{\partial}{\partial m^2}+n\gamma_\phi\right)G_m^{(n)}=0 \tag{11}$$

and

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+l\gamma_{\phi^2}+n\gamma_\phi\right)G_0^{(n,l)}=0 \tag{12}$$

Now let's act on equation (10) with the following operator

$$\begin{aligned}\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+n\gamma_\phi\right)G_m^{(n)} &= \sum(-im^2)^l\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+n\gamma_\phi\right)G_0^{(n,l)}\\ &= \sum(-im^2)^l\left(-l\gamma_{\phi^2}\right)G_0^{(n,l)}\\ &= \left(-\gamma_{\phi^2}\frac{\partial}{\partial m^2}\right)\sum(-im^2)^l\ G_0^{(n,l)}\\ &= \left(-\gamma_{\phi^2}\frac{\partial}{\partial m^2}\right)G_m^{(n)}\end{aligned} \tag{13}$$

Comparing this equation with (11) we see that

$$\gamma_{\phi^2}=2\gamma_m. \tag{14}$$

c) An is part (a) one can similarly evaluate

$$\left\langle\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)\phi^4(x)\right\rangle, \tag{15}$$

and find

$$\gamma_{\phi^4}=\frac{3\lambda^2}{16\pi^2} \tag{16}$$

d) In the previous homework (Problem 5) we argued that

$$\Delta_{\phi^4}=d+\beta'(\lambda_*) \tag{17}$$

e) The current associated to the global $U(1)$ symmetry is

$$J_\mu=i(\phi^*\partial_\mu\phi-\phi\partial_\mu\phi^*). \tag{18}$$

The one loop contribution to the three-point function

$$\langle\phi(x_1)\phi(x_2)J_\mu(x)\rangle \tag{19}$$

is given by

$$\frac{\lambda}{4}\int d^4y\left\langle\phi(x_1)\phi(x_2)J_\mu(x)\phi^4(y)\right\rangle. \tag{20}$$

It is straightforward to show that both terms in $J_\mu(x)$ give

$$\lambda\int d^4y\ D(x_1-y)D(x_2-y)D(x-y)\partial_\mu D(x-y), \tag{21}$$

and therefore the above three point function vanishes.

We can prove that $\gamma_J = 0$ to all orders in perturbation theory s follows. Consider the Ward identity for this symmetry

$$\begin{aligned}
\langle \phi(x_1)\phi^*(x_2)\partial_\mu J^\mu(x)\rangle &= -i\delta^{(4)}(x-x_1)\langle \delta\phi(x_1)\phi(x_2)\rangle - i\delta^{(4)}(x-x_2)\langle \phi(x_1)\delta\phi(x_2)\rangle \\
&= \delta^{(4)}(x-x_1)\langle \phi(x_1)\phi(x_2)\rangle - \delta^{(4)}(x-x_2)\langle \phi(x_1)\phi(x_2)\rangle \\
&= \frac{1}{\pi^2}\partial_\mu \left(\frac{(x-x_1)^\mu}{(x-x_1)^4} - \frac{(x-x_2)^\mu}{(x-x_2)^4} \right) \langle \phi(x_1)\phi(x_2).\rangle \qquad (22)
\end{aligned}$$

We see that

$$\langle \phi(x_1)\phi^*(x_2)J^\mu(x)\rangle = \frac{1}{\pi^2}\left(\frac{(x-x_1)^\mu}{(x-x_1)^4} - \frac{(x-x_2)^\mu}{(x-x_2)^4} \right) \langle \phi(x_1)\phi(x_2)\rangle \qquad (23)$$

Note that there is no integration "constant" since for $x \to \infty$ the three point function must vanish. Hence, all the divergences are coming from the propagator and therefore can be taken care of by the the renormalization of ϕ.

f) From dimensional analysis any two-point function in a generic massless theory will have the form

$$\begin{aligned}
G^{(2)}(p) &= \frac{i}{p^2} + \frac{i}{p^2}(A\log\frac{\Lambda^2}{p^2} + \dots) + \frac{i}{p^2}(ip^2\delta_Z)\frac{i}{p^2} \qquad (24) \\
&= \frac{i}{p^2} + \frac{i}{p^2} f\left(\frac{\Lambda}{\mu}\right) - \frac{i\delta_Z}{p^2} \qquad (25)
\end{aligned}$$

for some function f. So we see that all the divergent terms (terms that diverge for $\Lambda \to \infty$) are functions of $\frac{\Lambda}{\mu}$, and therefore

$$\mu\frac{d\delta_Z}{d\mu} = -\Lambda\frac{d\delta_Z}{d\Lambda}. \qquad (26)$$

from which it follows that

$$\gamma_O = -\Lambda\frac{dZ_O}{d\Lambda} \qquad (27)$$

Consider now the integral in (1). After using Feynman parameters, shifting the loop momentum (assuming $\Lambda \to \infty$) and Wick rotate, we have

$$\begin{aligned}
\left\langle\phi(p_1)\phi(p_2)\phi^2(x)\right\rangle &= \frac{i\lambda}{2}\frac{1}{p_1^2p_2^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k+p_1+p_2)^2} \\
&= -\frac{\lambda}{2}\frac{1}{p_1^2p_2^2}\int_0^1 dx\int_0^\Lambda\frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2-x(1-x)(p_1+p_2)^2)^2} \\
&= -\frac{\lambda}{(4\pi)^2}\frac{1}{p_1^2p_2^2}\int_0^1 dx\int_0^\Lambda dk_E\frac{1}{(k_E^2-x(1-x)(p_1+p_2)^2)^2} \\
&\overset{\Lambda\to\infty}{=} -\frac{\lambda}{(4\pi)^2}\frac{1}{p_1^2p_2^2}\int^\Lambda dk\frac{1}{k} \\
&= -\frac{\lambda}{(4\pi)^2}\frac{1}{p_1^2p_2^2}\log\Lambda
\end{aligned}$$

So we see that

$$Z_{\phi^2}(\Lambda) = -\frac{\lambda}{(4\pi)^2}\log\Lambda \tag{28}$$

which gives the expected value for γ_{ϕ^2}. One can similarly repeat the calculation for γ_{ϕ^4} and γ_J.

Problem 2: Recall that the connected Green's functions are given by

$$\langle\phi(x_1)\dots\phi(x_n)\rangle_{conn} = \frac{\delta^n W[J]}{\delta J(x_1)\dots\delta J(x_n)}\Big|_{J=0}. \tag{29}$$

The effective action $\Gamma[\phi_J]$ is defined as the Legendre transform of $W[J]$

$$\Gamma[\phi_J] = W[J] - \int d^dyJ(y)\phi_J(y). \tag{30}$$

where

$$\phi_J(x) = \frac{\delta W[J]}{\delta J(x)} \tag{31}$$

Taking the ϕ_J functional derivative of $\Gamma[\phi_J]$ we get

$$\frac{\delta\Gamma[\phi_J]}{\delta\phi_J(x)} = \int d^dy\frac{\delta J(y)}{\delta\phi_J(x)}\frac{\delta W[J]}{\delta J(y)} - J(x) - \int d^dy\frac{\delta J(y)}{\delta\phi_J(x)}\phi_J(y) = -J(x). \tag{32}$$

From this equation we conclude that

$$-\delta(x-y) = \frac{\delta}{\delta J(y)}\frac{\delta\Gamma[\phi_J]}{\delta\phi_J(x)} \tag{33}$$

$$= \int d^dz\frac{\delta\phi_J(z)}{\delta J(y)}\frac{\delta^2\Gamma[\phi_J]}{\delta\phi_J(x)\delta\phi_J(z)} \tag{34}$$

$$= \int d^dz\frac{\delta^2W[J]}{\delta J(y)\delta J(z)}\frac{\delta^2\Gamma[\phi_J]}{\delta\phi_J(x)\delta\phi_J(z)} \tag{35}$$

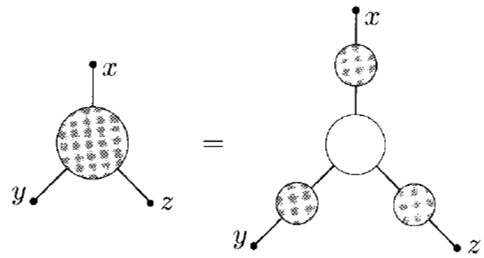

Figure 1: The black circles denote the connected graphs, while the white circle denotes the 1PI graphs.

Treating the arguments of a function as an index of a matrix and an integral as a sum over indices, this equation can be written as

$$\frac{\delta^2 W[J]}{\delta J_y \delta J_z} \frac{\delta^2 \Gamma[\phi_J]}{\delta \phi_{J,z} \delta \phi_{J,x}} = -1_{yx}. \tag{36}$$

IN this form we can see that

$$\frac{\delta^2 W[J]}{\delta J_y \delta J_z} = -\left(\frac{\delta^2 \Gamma[\phi_J]}{\delta \phi_{J,z} \delta \phi_{J,x}} \right)^{-1}, \tag{37}$$

Using the following rule for a derivative of an inverse of a matrix

$$\frac{dM^{-1}}{dx} = -M^{-1} \frac{dM}{dx} M^{-1}, \tag{38}$$

we can finally find

$$\begin{aligned} \frac{\delta^3 W[J]}{\delta J_y \delta J_z \delta J_x} &= -\frac{\delta \phi_{J,u}}{\delta J_x} \left(\frac{\delta^2 \Gamma[\phi_J]}{\delta \phi_{J,z} \delta \phi_{J,v}} \right)^{-1} \frac{\delta^3 \Gamma[\phi_J]}{\delta \phi_{J,u} \delta \phi_{J,v} \delta \phi_{J,r}} \left(\frac{\delta^2 \Gamma[\phi_J]}{\delta \phi_{J,r} \delta \phi_{J,x}} \right)^{-1} \\ &= -\int d^d u d^d v d^d r D(u-x) D(z-v) D(r-x) \frac{\delta^2 \Gamma[\phi_J]}{\delta \phi_J(u) \delta \phi_J(v) \delta \phi_J(r)}. \end{aligned}$$

This shows that the third derivative of the effective action is the connected graphs with the external propagators stripped off. Pictorially this is shown in figure (1)

Similarly one can find

$$\begin{aligned} \frac{\delta^3 W[J]}{\delta J_w \delta J_x \delta J_y \delta J_z} = -D_{sw} D_{xt} D_{yu} D_{zv} \Big(& \frac{\delta^4 \Gamma[J]}{\delta \phi_{J,s} \delta \phi_{J,t} \delta \phi_{J,u} \delta \phi_{J,v}} \\ & + \frac{\delta^3 \Gamma[\phi_J]}{\delta \phi_{J,s} \delta \phi_{J,t} \delta \phi_{J,r}} D_{rq} \frac{\delta^3 \Gamma[\phi_J]}{\delta \phi_{J,q} \delta \phi_{J,u} \delta \phi_{J,v}} + (t \leftrightarrow u) + (t \leftrightarrow v) \Big) \end{aligned}$$

Again pictorially this is shown in figure (2).

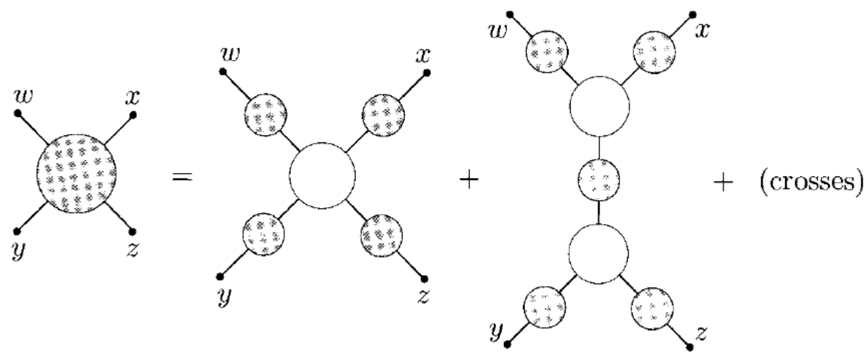

Figure 2: Four-point function.

Problem 3: Since the gauge field is in the adjoint representation of the gauge group, the covariant derivative acting on the field strength is equal to

$$D_\rho F^a_{\mu\nu} = \partial_\rho F^a_{\mu\nu} - g f^{abc} A^b_\rho F^c_{\mu\nu}. \tag{39}$$

Adding three cyclically permuted covariant derivatives we get

$$\begin{aligned} &D_\mu F^a_{\nu\rho} + D_\nu F^a_{\rho\mu} + D_\rho F^a_{\mu\nu} \\ =&\partial_\nu F^a_{\nu\rho} + \partial_\nu F^a_{\rho\mu} + \partial_\rho F^a_{\mu\nu} - g f^{abc} A^b_\mu F^c_{\nu\rho} - g f^{abc} A^b_\nu F^c_{\rho\mu} - g f^{abc} A^b_\rho F^c_{\mu\nu} \end{aligned} \tag{40}$$

Using the definition of the field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{41}$$

most of the terms cancel and we end up with

$$\begin{aligned} D_\mu F^a_{\nu\rho} + D_\nu F^a_{\rho\mu} + D_\rho F^a_{\mu\nu} &= -g A^c_\mu A^d_\nu A^e_\rho \left(f^{acd} f^{bde} + f^{adb} f^{bce} + f^{eab} f^{cdb} \right) \\ &= 0 \end{aligned} \tag{42}$$

which is zero by the Jacobi Identity (70.4).