Homework 1 Solutions

Problem 1 : (Srednicki 27.1) Using the definition of a beta function we can easily get

$$\frac{d\alpha}{dm} = \frac{\frac{d\alpha}{\ln\mu}}{\frac{dm}{\ln\mu}} = \frac{b_1}{c_1}\frac{\alpha}{m}, \tag{1}$$

which can be written as

$$\frac{dm}{m} = \frac{c_1}{d_1}\frac{d\alpha}{a}. \tag{2}$$

Integrating both sides we find

$$\frac{m_1}{m_2} = \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{c_1}{d_1}}. \tag{3}$$

Problem 2 : (Srednicki 28.1) Starting with the analog of equations (28.3)-(28.6)

$$\phi_0 = \sqrt{Z_\phi}\phi, \tag{4}$$

$$m_0 = \sqrt{\frac{Z_m}{Z_\phi}}m, \tag{5}$$

$$\lambda_0 = Z_\phi^{-2} Z_\lambda \mu^\epsilon \lambda, \tag{6}$$

the strategy will be to repeat the same steps as in the ϕ^3-theory. We will also use the results from lat semester for the above Z factors

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2}\frac{1}{\epsilon}, \tag{7}$$

$$Z_m = 1 + \frac{\lambda}{16\pi^2}\frac{1}{\epsilon}, \tag{8}$$

$$Z_\phi = 1 + \mathcal{O}(\lambda^2). \tag{9}$$

As usual, all the bare quantities must be independent from the renormalization scale μ since it is an auxiliary parameter. Proceeding as in the ϕ^3 case we define the analog of equation (28.14)

$$G(\lambda, \epsilon) = \ln Z_\phi^{-2} Z_\lambda \approx \frac{3\lambda}{16\pi^2}\frac{1}{\epsilon}. \tag{10}$$

From (6) we get

$$\ln \lambda_0 = G + \epsilon \ln \mu + \ln \lambda, \tag{11}$$

and requiting that λ_0 is independent of μ, or in other words

$$\frac{d \ln \lambda_0}{d \ln \mu} = 0, \tag{12}$$

we obtain

$$\frac{d\lambda}{d \ln \mu} = -\frac{\epsilon \lambda}{1 + \frac{3\lambda}{16\pi^2}\frac{1}{\epsilon}} \overset{\epsilon \to 0}{=} -\frac{3\lambda^2}{16\pi^2}, \tag{13}$$

and therefore

$$\beta(\lambda) = -\frac{3\lambda^2}{16\pi^2}. \tag{14}$$

Similarly for the mass renormalization we define

$$M = \ln \sqrt{\frac{Z_m}{Z_\phi}} \approx \frac{\lambda}{32\pi^2}\frac{1}{\epsilon}, \tag{15}$$

and requiring that $\frac{d \ln m_0}{d \ln m} = 0$ we find

$$\gamma_m = \frac{\lambda}{32\pi^2}, \tag{16}$$

Lastly, since Z_ϕ doesn't have corrections at one loop order, equation (28.36) implies that

$$\gamma_\phi = \frac{1}{2}\frac{d \ln Z_\phi}{d \ln \mu} = \mathcal{O}(\lambda^2). \tag{17}$$

Figure 1: Relevant diagrams for β_λ. The single line correspond to ϕ_1 and the double line to ϕ_2.

Figure 2: Relevant diagrams for β_ρ. The single line correspond to ϕ_1 and the double line to ϕ_2.

Problem 3 :

a) The relevant Feynman diagrams for β_λ are shown in figure (1). Apart from the symmetry factors, the loop integral is the same as in the usual ϕ^4 theory. The calculation is done in detail in the book by *Peskin & Shroeder* (see p. 326-327). Using the notation of this book the above diagrams are equal to

$$-\left(\lambda^2+\frac{1}{9}\rho^2\right)[iV(s)+iV(t)+iV(u)]-i\delta_\lambda, \tag{18}$$

where the factor in front of ρ^2 comes from the symmetry factor of the diagrams with two ρ-vertices. Following the same step as in the previous problem, the beta function can be easily evaluated

$$\beta_\lambda=\frac{9\lambda^2+\rho^2}{3(4\pi)^2} \tag{19}$$

For the other beta function, the relevant diagrams are shown in figure (2). They are equal to

$$\left(\frac{1}{3}\lambda\rho\right)2iV(s)-\frac{1}{9}\rho^2[iV(t)+iV(u)]-i\delta_\rho. \tag{20}$$

As before the coefficients $\frac{1}{3}$ and $\frac{1}{9}$ are coming from the symmetry factors. In this case the beta function is

$$\beta_\rho=\frac{6\lambda\rho+4\rho^2}{3(4\pi)^2} \tag{21}$$

b)Using the above beta functions it is easy to calculate the beta function for

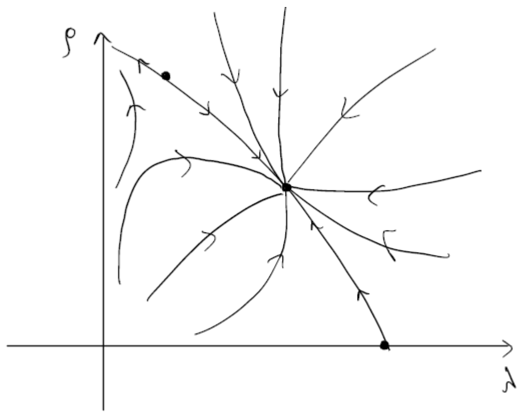

Figure 3: RG graph.

the ration ρ/λ as

$$\mu\frac{d(\rho/\lambda)}{d\mu} = \mu\frac{d\rho}{d\mu}\frac{1}{\lambda} - \mu\frac{d\lambda}{d\mu}\frac{\rho}{\lambda^2} = \frac{\rho}{3(4\pi^2}\left(-\left(\frac{\rho}{\lambda}\right)^2 + 4\frac{\rho}{\lambda} - 3\right) \tag{22}$$

We see that there are two fixed point for $\rho/\lambda = 0, 1, 3$. If we start with $\rho/\lambda < 3$ then the ration will flow toward the other fixed point which satisfies $\rho = \lambda$.

c) The beta functions in $d = 4 - \epsilon$ dimensions are

$$\beta_\lambda = -\epsilon\lambda + \frac{9\lambda^2 + \rho^2}{3(4\pi)^2} \tag{23}$$

$$\beta_\rho = -\epsilon\rho + \frac{6\lambda\rho + 4\rho^2}{3(4\pi)^2} \tag{24}$$

However, the terms contacting ϵ cancel out in the beta function for the ratio and the result is the same as in part (b). The reason is that the ratio is still a dimensionless parameter. As before, the fixed points happen for $\rho/\lambda = 0, 1, 3$ and the RG diagram is shown in figure (3).

Problem 4 :

The beta functions as given in Cardy's book are

$$\begin{aligned} \frac{du}{dl} &= \epsilon u - bu^2, \\ \frac{dh}{dl} &= h(y + b'u), \end{aligned} \tag{25}$$

where the relation of the variable l to the usual renormalization scale μ is

$$\mu = e^{-l}, \tag{26}$$

and in terms of μ the above set of equations take the usual form

$$\mu \frac{du}{d\mu} = -\epsilon u + b u^2, \tag{27}$$

$$\mu \frac{dh}{d\mu} = -h(y + b' u). \tag{28}$$

We see that there are two fixed points with

$$u = 0, \quad h = 0, \quad y_h = y \tag{29}$$

$$u = \frac{\epsilon}{b}, \quad h = 0, \quad y_h = y + \epsilon \frac{b'}{b} \tag{30}$$

$$\tag{31}$$

Starting with equation (3.47) in Cardy's book we have

$$G(r/b, u', h') = b^{2d} \left(\frac{h'}{h} \right)^{-2} G(r, u, h). \tag{32}$$

This equation describes how the correlation functions transforms under scaling transformation $r \to r' = r/b$, or in other words by changing the mass scale $\mu \to b\mu$. So if we choose $r = r_0$ and equivalently

$$b = \frac{r}{r_0} = \frac{\mu_0}{\mu} = e^{-(l-l_0)}, \tag{33}$$

equation (32) can be written as

$$G(r, u(l), h(l)) = e^{-2d(l-l_0)} \left(\frac{h(l)}{h(l_0)} \right)^{-2} G(r_0, u(l_0), h(l_0)). \tag{34}$$

Hence, we see that by solving the differential equations for $u(l)$ and $h(l)$ we can plug them in the above equation and determine the correlation function. We can easily integrate equations (25) and obtain

$$u(l) = e^{(l-l_0)\epsilon} \frac{\epsilon}{b e^{(l-l_0)\epsilon} - 1} \tag{35}$$

$$h(l) = c\, e^{(l-l_0)y} (1 - b e^{(l-l_0)\epsilon})^{b'/b} \tag{36}$$

for some constant c. Plugging the latter in the transformation for $G(r, u, h)$ we get

$$G(r, u(l), h(l)) = G_0 e^{-2d(l-l_0)} \left(\frac{1 - b e^{(l-l_0)\epsilon}}{1 - b} \right)^{-2b'/b}, \tag{37}$$

and in terms of r

$$G(r, u(l), h(l)) \approx \left(\frac{r_0}{r} \right)^{2(d-y)} \left(1 - b \left(\frac{r_0}{r} \right)^{\epsilon} \right)^{-2b'/b}. \tag{38}$$

From this expression we can read of the asymptotic behavior for small and large r. In particular

$$r >> r_0 \quad G \approx \left(\frac{r_0}{r}\right)^{2(d-y)} \tag{39}$$

$$r << r_0 \quad G \approx \left(\frac{r_0}{r}\right)^{2(d-y-\epsilon\frac{b'}{b})} \tag{40}$$

Therefore, we see that for large r the behavior of the correlation function is the same as in the first fixed point in (31) and for small r is same as in the second fixed point.

Problem 5 :

a) A term that respects the $O(N)$ symmetry can be written as dot-products of the vector $\vec{\phi}$ and its derivatives. The only term that cannot be written in this form is the last one. The other terms can be written as

$$S = \int d^dx \left(\frac{1}{2}(\partial^\mu \vec{\phi}) \cdot (\partial_\mu \vec{\phi}) + t_0 \vec{\phi} \cdot \vec{\phi} + u_0 (\vec{\phi} \cdot \vec{\phi})^2 \right). \tag{41}$$

b)As usual

$$t_0 = Z_t Z_\phi^{-1} \mu^2 t \tag{42}$$

$$u_0 = Z_u Z_\phi^{-2} \mu^\epsilon u \tag{43}$$

$$v_0 = Z_t Z_\phi^{-2} \mu^\epsilon v \tag{44}$$

In the second problem we saw that the linear term in the beta functions comes from the powers of μ in the above equations. More specifically, the coefficient in front of ϵ is equal to d minus the dimension of the corresponding operator. Hence

$$c_1 = 2, \quad c_2 = c_3 = \epsilon \tag{45}$$

c) A fixed point is a solution of the set of equations

$$\beta_t = 0, \quad \beta_u = 0, \quad \beta_v = 0. \tag{46}$$

Using the expressions for the beta functions we find the following solutions

1. $t = 0, \quad u = 0, \quad v = 0,$
2. $t = 0, \quad u = 0, \quad v = \frac{\epsilon}{72},$
3. $t = 0, \quad u = \frac{\epsilon}{8(N+8)}, \quad v = 0,$
4. $t = 0, \quad u = \frac{\epsilon}{24N}, \quad v = \frac{(N-4)\epsilon}{72N}.$

In principle there is mixing between operators with the sane scaling dimensions. For this reason, the Z factor will be a matrix defined as

$$O_0^i = Z^{ij} O^j. \tag{47}$$

Let M be the matrix that diagonlize Z^{ij}. Then the interaction terms in the Langranzian can be written as

$$L_{int} = g^i O_0^i = g_i Z^{ij} O_j = (g\Lambda^{-1})_i (\Lambda Z \Lambda^{-1})^{ij} (\Lambda O)_j = g'^i Z'^i O'^i \tag{48}$$

where the primes denote the diagonalized quantities. The scaling dimensions of the diagonalized operators minus the spacetime dimensions are then equal to the derivative of the beta function at the fixed point (see Peskin & Schroeder pages 428-435).

A priori we don't know which combinations of operators appearing in the Lagrangian diagonalize the Z matrix so instead we have to diagonalize the matrix

$$H^{ij} = \frac{\partial \beta_i}{\partial g^j}, \tag{49}$$

whose eigenvalues h^i satisfy

$$h^i = \Delta_i - d \tag{50}$$

The first fixed point corresponds to the free theory. There is no mixing between the operators and their scaling dimensions are

$$\Delta_m = c_1 = 2, \quad \Delta_u = \Delta_v = \epsilon. \tag{51}$$

For the other three fixed points there is mixing between the two quartic operators and the scaling dimensions are

2)

$$\Delta_m = d - 2 - \frac{\epsilon}{3}, \quad \Delta_1 = d - \epsilon, \quad \Delta_2 = d + \frac{\epsilon}{3}, \tag{52}$$

3)

$$\Delta_m = d - 2 - \frac{2}{3}\frac{N-1}{n}\epsilon, \quad \Delta_1 = d - \epsilon, \quad \Delta_2 = d - \frac{N-4}{3N}\epsilon, \tag{53}$$

4)

$$\Delta_m = d - 2 - \frac{N+2}{N+8}\epsilon, \quad \Delta_1 = d - \epsilon, \quad \Delta_2 = d + \frac{N-4}{N+8}\epsilon. \tag{54}$$

d) A fixed point is more stable is the there are no relevant operators at the fixed point. That is because we have to tune less bare parameters in order to end up at the fixed point at some lower energies. However, in problem we see that the stability of the fixed points change for $N = N_c = 4$. For $N < 4$ the most stable fixed point (apart from the trivial one) is the third one and for $N > 4$ the most stable fixed point is the fourth one.

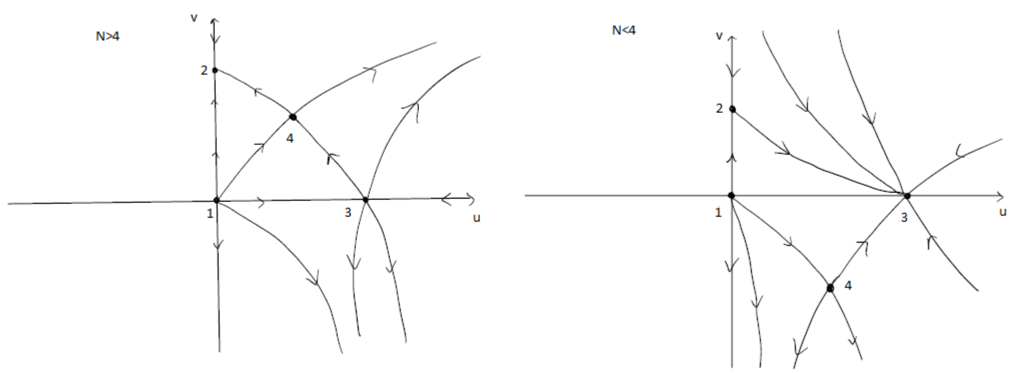

Figure 4: These are the two RG-graphs depending on the value of N.