FieldsI-Notes

Quantum Field Theory I

Spring 2019

January 2019 - March 2019

1 Lecture 1 (January 29)

The subject of Field Theory is the dynamical systems (classical or quantum) with continuously many degrees of freedom. Typically, there is some number of degrees of freedom associated with each point of space $\mathbf{x} \in \mathbf{R}^3$. Examples are:

i) Electromagnetic Theory

$$\text{Degrees of Freedom}: \qquad \mathbf{E}(\mathbf{x},t),\ \mathbf{H}(\mathbf{x},t)\,.$$

ii) Gravitation (general Relativity)

$$\text{Degrees of Freedom} = \text{Spacetime metric} \quad g_{\mu\nu}(\mathbf{x},t)\,.$$

iii) More exotic example: Dirac (electron-positron) field

$$\text{Degrees of Freedom} = \text{Four} - \text{component (complex) Lorentz spinor} \quad \psi_a(\mathbf{x},t)\,.$$

Main subject of this course is Quantum Theory of Fields. The states of quantum fields with local interaction admit interpretation in terms of propagating and interacting particles. Thus the theory of quantum fields is the basis of particle theory.

Classical Field Theory

We start with a brief review of basic aspects of classical field theory. Fundamental object in classical mechanics is the **action**,

$$S = \int L\big(q(t), \dot{q}(t)\big)\, dt\,;$$

the equations of motion are derived from the extremal action principle $\delta S = 0$. In field theory the coordinates are replaced by the field degrees of freedom,

$$q(t) \to \Phi(\mathbf{x},t)\,,$$

and the action is written as a space-time integral

$$S = \int \mathcal{L}\big(\Phi(\mathbf{x},t), \dot{\Phi}(\mathbf{x},t), \nabla\Phi(\mathbf{x},t)\big)\, d^3\mathbf{x}\, dt\,.$$

Here $\Phi = \Phi(\mathbf{x}, t)$ stands for a collection of fields (functions of $\mathbf{x}, t$, representing local degrees of freedom), which may include scalar fields, components of vectors, tensors, spinors, etc. It is conventionally assumed that Φ has finitely many individual components. The function $\mathcal{L}$ is called the **Lagrangian density**, it depends on $\Phi(\mathbf{x}, t)$ and its derivatives over $\mathbf{x}$ and t, taken at the same space-time point $\big(\mathbf{x}, t\big)$. Two remarks are in order.

1. The above expression is suitable for imposing conditions of the special relativity. The integration is over the 4-dimensional Minkowskian space-time,

$$d^3\mathbf{x}\, dt = d^4x\,.$$

Recall that in relativistic theory one deals with the 4-dimensional space-time with the coordinates

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \mathbf{x})\,,$$

equipped with the "pseudo-metric"

$$d\tau^2 = c^2\, dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\,.$$

The Lorentz transformations are those linear transformations of the coordinates x^μ which preserve $d\tau^2$. In what follows I will use the units in which[1]

$$c = 1\,, \qquad \hbar = 1\,.$$

Then

$$d\tau^2 = dx^\mu\, dx_\mu\,, \qquad \text{where} \qquad x^\mu = (t, \mathbf{x})\,, \quad \text{and} \quad x_\mu = (t, -\mathbf{x})\,.$$

To ensure the relativistic invariance of the classical field theory it suffices to choose $\mathcal{L}$ to be a Lorentz scalar.

2. In writing the above action I already accepted severe restrictions, namely,

$$S = \int dt\, L\,, \qquad \text{with}$$

[1]These conditions are simply the convention to measure spatial distances and energy in the units

$$[\text{unit of length}] = c \times [\text{unit of time}]\,, \qquad [\text{unit of energy}] = \hbar/\,[\text{unit of time}]\,.$$

$$L = \int d^3\mathbf{x}\, \mathcal{L}\big(\Phi(\mathbf{x},t), \dot\Phi(\mathbf{x},t), \nabla\Phi(\mathbf{x},t)\big)\,,$$

i.e. L is a $d^3\mathbf{x}$ integral of a *spatial density* $\mathcal{L}$, which depends on Φ, $\dot\Phi$, and $\nabla\Phi$, all taken *at the same point* $\mathbf{x}$. This means that the field degrees of freedom $\Phi(\mathbf{x})$ at some point $\mathbf{x}$ do not have a direct interaction with $\Phi(\mathbf{x}')$ located at some other point $\mathbf{x}'$, with $|\mathbf{x}-\mathbf{x}'|$ greater then zero. Allowing such interactions would lead to possible terms in L like

$$L_{\text{nonlocal}} = \int F\big(\Phi(\mathbf{x},t), \Phi(\mathbf{x}',t)\big)\, d^3\mathbf{x}\, d^3\mathbf{x}'$$

("nonlocal interactions"). Nonlocal interactions do not go well along with the relativistic invariance. For if S includes, say, double integrals over $\mathbf{x}$ (as above), relativistic invariance of S forces one to include the terms which are nonlocal in time as well, like

$$\int F'\big(\Phi(t), \Phi(t')\big)\, dt\, dt'\,,$$

which evidently violate *causality*: The state in the future affects the dynamics now. Locality and causality are deeply connected.

The above local form, $L = \int \mathcal{L}(\Phi(\mathbf{x},t), \dot\Phi(\mathbf{x},t), \nabla\Phi(\mathbf{x},t))\, d^3\mathbf{x}$, describes the dynamics in which the interaction occurs only between the degrees of freedom associated with "infinitely close" spatial points (through the $\nabla\Phi$ in L).

Also, I did not include higher spatial derivatives of Φ as the arguments in $\mathcal{L}$. Although such dependence would not violate locality (on a superficial level), including such dependencies in a relativistic theory would require to add higher time derivatives, like $\ddot\Phi$, as well. Such dependence is not allowed in conventional Lagrangian dynamics, which requires that the time evolution $\varphi(\mathbf{x},t)$ is completely determined by the initial values of $\Phi(\mathbf{x})$ and $\dot\Phi(\mathbf{x})$ at $t = t_0$.

Anyhow, we start with the above local form of the action. In classical field theory the field equations of motion follow from the action principle

$$\delta S = 0\,.$$

Let us combine the spatial and time variable into the 4-coordinate $x = (x^\mu) = (x^0, \mathbf{x})$, with $x^0 = t$, and write $\Phi(x)$ instead of $\Phi(\mathbf{x},t)$; the Lagrangian density then is written as

$$\mathcal{L}\big(\Phi(x), \partial_\mu\Phi(x)\big)\,,$$

where

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right).$$

A simple exercise in functional analysis (look up the textbook, page 15 in PS) yields the classical field equations of motion

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \Phi(x)\big)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi(x)} = 0\,. \tag{1.1}$$

Remark: Essential step in deriving (1.1) is integrating by parts. The total derivative term

$$\int d^4x \, \frac{\partial}{\partial x^\mu} \left(\delta\Phi(x) \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi(x))} \right)$$

can be reduced to the integral over the boundary (Gauss theorem), which depends only on the variations $\delta\Phi(x)$ at the boundary; this term does not affect the "bulk" equations of motion (1.1), which hold at each point x of the space-time. But if the field dynamics is considered in a space-time domain D, with the boundary ∂D, the above term contributes to the variation of S via the variations of the boundary values $\delta\Phi(x)\,|_{x\in\partial D}$ of the fields,

$$\int_D d^4x \, \partial_\mu \left(...\right) = \int_{\partial D} \delta\Phi(x) \, \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi(x)} \, d\Sigma_\mu(x)$$

where $d\Sigma_\mu(x)$ is the normal[2] element of the boundary hypersurface ∂D. In this situation the total derivative term contributes to the boundary conditions (see the homework Problem 1). The default setup is D = full infinite space-time, with the condition that $\Phi(x)$ decays sufficiently fast at both space and time infinities ("scattering asymptotic conditions"); in this setup the above derivative term can be ignored.

The task of the classical field theory is to find solutions of these equations with such and such initial or boundary conditions.

One of the most important general results about classical field theory is the relation between symmetries and conservation laws known as the

[2] "Normal" means that $d\Sigma_\mu(x) T^\mu(x) = 0$ for any $T^\mu(x)$ tangent to the boundary ∂D, at any point $x \in \partial D$.

Noether's theorem (pp 17-18 of PS)

What is a symmetry? Suppose we have a family of continuous transformations of the field variables

$$\Phi(x) \to \Phi'(x) = F\big(x, \Phi(x)\big)$$

such that the action S does not change

$$S[\Phi] = S[\Phi'] \,.$$

The term continuous transformations means here that they depend on some continuous parameter(s), $F = F_s$, in such a way that

$$F_0(x, \Phi(x)) = \Phi(x) \qquad - \text{ no transformation} \,.$$

(discrete symmetries is a separate story). Then we can take an infinitesimal transformation

$$\Phi(x) \to \Phi'(x) = \Phi(x) + \epsilon\, E\big(x, \Phi(x)\big) \,,$$

where

$$E = \frac{d}{ds}\, F\Big|_{s=0} \,,$$

with an infinitesimal ϵ. Substituting in the Lagrangian density, one finds

$$\mathcal{L}\big(\Phi', \partial_\mu \Phi'\big) = \mathcal{L}\big(\Phi, \partial_\mu \Phi\big) + \epsilon \left[\frac{\partial \mathcal{L}}{\partial \Phi}\, E\big(x, \Phi\big) + \frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \Phi\big)}\, \partial_\mu E\big(x, \Phi\big) \right] + O(\epsilon^2) \,.$$

The invariance (i.e. the requirement that the action $S = \int \mathcal{L}\, d^4x$ does not change) implies

$$\big[\cdots \big] = \partial_\mu \tilde{J}^\mu\big(\Phi, \partial_\mu \Phi, x\big) \,,$$

with some $\tilde{J}^\mu$ which depend on our field variables in a local way (it may also have separate dependence on x if E does). Transforming the left-hand side by parts one finds

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \Phi\big)}\, E\big(x, \Phi\big) \right) + \left[\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \Phi\big)} \right) \right] E\big(x, \Phi\big) = \partial_\mu \tilde{J}^\mu \,.$$

Now, suppose that $\Phi = \Phi_c$, a solution of the classical field equations of motion. Then the second term vanishes, and we obtain the continuity equation

$$\partial_\mu J^\mu = 0 \,,$$

where

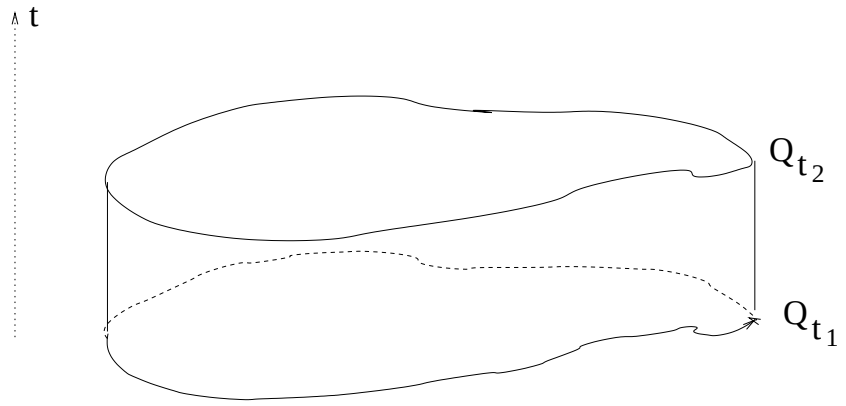
$$J^\mu\big(\Phi,\partial\Phi,x\big)=\frac{\partial\mathcal{L}}{\partial\big(\partial_\mu\Phi\big)}\,E\big(x,\Phi\big)-\tilde{J}^\mu\big(\Phi,\partial\Phi,x\big)\,,$$

which holds as long as Φ solves the classical field equations.

The continuity equation then implies a conservation law. Indeed, by the Gauss theorem

$$\oint_\Sigma J^\mu\,d\Sigma_\mu=\int_D\partial_\mu J^\mu\,d^4x=0\,.$$

Here $\Sigma=\partial D$ is any closed 3-dimensional hypersurface in the 4-dimensional space-time, and $d\Sigma_\mu$ is the normal element of $e_{\mu\nu\lambda\rho}dx^\nu dx^\lambda dx^\rho$ of this hypersurface. Taking the hypersurface to be a big slab between two equal-time hyperplanes and bringing the spatial boundary to to the spatial infinity,



and also assuming that all fields decay at $\mathbf{x}\to\infty$, one finds

$$Q_{t_1}=Q_{t_2}\,,$$

where

$$Q_t=\int J^0\,d^3\mathbf{x}\,,$$

that is Q is an integral of motion for the classical field equations. Let me stress again that both the continuity equation $\partial_\mu J^\mu=0$ and the conservation law $\frac{d}{dt}Q=0$ are satisfied only if we take $\Phi=\Phi_c$ (in terminology of QFT they are said to hold "on shell").

Energy-Momentum

The most common symmetry we encounter in field theory is the translational symmetry. Assume that the Lagrangian density $\mathcal{L}(\Phi, \partial\Phi)$ has no x dependence except for that coming through the field variable $\varphi(x)$. It means that the dynamics looks the same if we shift the space-time coordinates

$$x \to x' = x + a$$

by a constant 4-vector a. This can be viewed as the symmetry with respect to the following transformation of the field variables

$$\Phi(x) \to \Phi'(x) = \Phi(x + a)\,,$$

which leaves the action invariant. This field transformation has the infinitesimal form

$$\Phi'(x) = \Phi(x) + da^\mu\, \partial_\mu \Phi(x)\,,$$

i.e. in this example $E(\Phi, \partial\Phi) = \partial_\mu\Phi$. When we do this transformation the action is going to stay invariant, but the Lagrangian density changes by the total derivative

$$\mathcal{L}' = \mathcal{L} + da^\nu\, \partial_\nu \mathcal{L}\,,$$

where the derivative in the last term can be written as $\partial_\mu\big(\delta^\mu_\nu\, \mathcal{L}\big)$, and has the same meaning as $\partial_\mu \tilde{J}^\mu$ in the general argument above.

Note that there are four independent symmetries associated with four components of a^μ. Correspondingly, there are four Noether's currents

$$T^\mu_{\ \nu} = \frac{\partial\mathcal{L}}{\partial\big(\partial_\mu\Phi\big)}\, \partial_\nu\Phi - \delta^\mu_\nu\, \mathcal{L}\,,$$

which therefore satisfies the continuity equations

$$\partial_\mu T^\mu_{\ \nu} = 0 \qquad \text{on shell}\,.$$

This object is known as the **energy-momentum tensor**. It leads to four conserved charges

$$E = \int T^0_0\, d^3\mathbf{x}\,,$$

and

$$P^i = -\int T^0_i\, d^3\mathbf{x}\,, \quad \text{with} \quad i = 1, 2, 3\,.$$

Of these, E is interpreted as the energy of the field configuration $\Phi(\mathbf{x}, t)$, and $\mathbf{P}$ is interpreted as its momentum. Therefore one may take

$$T^{0\,0} = T^0_0 = \mathcal{E} \quad \text{for the energy density}\,,$$

and

$$T^{0\,i} = -T^0_i = \mathcal{P}^i \quad \text{for the momentum density}\,.$$

The spatial components $T^{i\,j}$ constitute the (spatial) *stress tensor*. It is interpreted as usual: Consider any domain $D = D^3$ in the 3-dimensional space (at a given time). The momentum of the field inside this domain is

$$P^i_D = \int_D \mathcal{P}^i(\mathbf{x})\, d^3\mathbf{x}\,.$$

According to the pulse-momentum theorem,

$$\frac{d}{dt}\, P^i_D = F^i_D\,, \quad \text{the net force applied to the field inside} \quad D\,.$$

Since the interaction is local, this force must come from the surface forces,

$$F^i_D = -\int_{\partial D} T^{i\,j}(\mathbf{x})\, d\sigma_j = -\int_D \partial_j T^{i\,j}(\mathbf{x})\, d^3\mathbf{x}\,.$$

Thus, $dF^i = T^{ij}(\mathbf{x}) d\sigma_j$ is the force exerted by the field on the surface element $d\sigma_j$.

Klein-Gordon Field

Let us consider a simple field theory known as the *Klein-Gordon* theory (it will be our main example for some time). It involves a single-component scalar field $\Phi(x) = \varphi(\mathbf{x}, t)$, and the action is

$$S = \int d^4x \left[\frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\,\varphi^2\right] \equiv$$

$$\int dt\, d^3\mathbf{x} \left[\frac{1}{2}\,\dot\varphi^2 - \frac{1}{2}\,(\nabla\varphi)^2 - \frac{m^2}{2}\,\varphi^2\right].$$

This form of the action shows that the theory is Lorentz-invariant, and the field φ is a scalar (i.e. $\varphi(x)$ has definite value at each space-time point x, independent of the choice of the inertial frame). Variation yields the Klein-Gordon field equation

$$\partial_\mu\partial^\mu\varphi + m^2\,\varphi = \partial_t^2\varphi - \nabla^2\varphi + m^2\,\varphi = 0\,.$$

Applying the above equations one finds for the energy-momentum tensor

$$T_{\mu\,\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}\,g_{\mu\,\nu}\left(\partial_\lambda\varphi\,\partial^\lambda\varphi - m^2\,\varphi^2\right),$$

where

$$g_{\mu\,\nu} = \mathrm{diag}\big(1,-1,-1,-1\big)$$

is the usual Minkowski metric. Thus,

$$\mathcal{E} = \frac{1}{2}\left(\dot\varphi^2 + \big(\nabla\varphi\big)^2 + m^2\,\varphi^2\right),$$

and

$$\mathcal{P} = \dot\varphi\,\nabla\varphi\,.$$

We see in this example that the energy-momentum tensor $T_{\mu\,\nu}$ is symmetric,

$$T_{\mu\,\nu} = T_{\nu\,\mu}\,.$$

Exercise

Show that in any Lorentz-invariant theory involving only a scalar field $\varphi(x)$ the energy-momentum tensor always comes out symmetric, $T_{\mu\,\nu} = T_{\nu\,\mu}$. Check that in such theory all components of the antisymmetric tensor

$$M^{\mu\,\nu} = \int\,\left[x^\mu\,T^{\nu\,0} - x^\nu\,T^{\mu\,0}\right]d^3\mathbf{x}$$

are conserved on-shell.

The equations of motion of the Klein-Gordon theory

$$\partial_\mu\partial^\mu\varphi + m^2\,\varphi = \partial_t^2\varphi - \nabla^2\varphi + m^2\,\varphi = 0\,.$$

are *linear*. That is, any linear combination of solutions are again the solutions. (In general, this property defines the *free field theory*). Therefore, general solution is a superposition of "elementary solutions". In the infinite space-time, one can take the plane waves

$$e^{-ipx} = e^{-ip^0x^0+i\mathbf{p}\mathbf{x}} \quad \text{with} \quad p^0 = \omega_{\mathbf{p}} := \sqrt{\mathbf{p}^2 + m^2}$$

as well as the complex-conjugate waves[3]

$$\varphi(x) = \int \frac{d^3\mathbf{p}}{2\omega_{\mathbf{p}}\,(2\pi)^3}\,\left(A_{\mathbf{p}}\,e^{-ipx} + A^*_{\mathbf{p}}\,e^{ipx}\right)\,. \tag{1.2}$$

This form represents general solution of the Klein-Gordon equation, which is bounded at the spatial infinity.

[3]As usual

2 Lecture 2 (January 31)

Before going to quantum theory, let us recall the Hamiltonian formalism of classical mechanics and see how it looks for fields.

Hamiltonian Formalism

In Lagrangian dynamics with the generalized coordinates q^i one passes to the Hamiltonian formalism through introducing the conjugate momenta

$$p_i = \frac{\partial L(q,\dot{q})}{\dot{q}^i}\,,$$

and then excluding $\dot{q}$ in favor of p, by solving these equations. The Hamiltonian function H appears as the Legendre transform

$$H(p,q) = \sum_i p_i\,\dot{q}^i - L\,.$$

The equations of motion then take the canonical form

$$\dot{p}_i = \big\{H, p_i\big\} = -\frac{\partial H}{\partial q_i}\,,$$

$$\dot{q}^i = \big\{H, q^i\big\} = \frac{\partial H}{\partial p^i}\,,$$

in terms of the Poisson brackets

$$\big\{f, g\big\} := \sum_i \frac{\partial f}{\partial p_i}\,\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\,\frac{\partial g}{\partial p_i}\,.$$

In a field theory we have continuously many generalized coordinates $\varphi(\mathbf{x})$ (I assume here a scalar field, for simplicity), labeled by the spatial coordinates $\mathbf{x}$. In this case the conjugated momenta are defined as

$$\pi(\mathbf{x}) = \frac{\delta L}{\delta\dot{\varphi}(\mathbf{x})}\,,$$

with the basic Poisson brackets

$$\big\{\pi(\mathbf{x}), \varphi(\mathbf{x}')\big\} = \delta(\mathbf{x}-\mathbf{x}')\,.$$

In the Klein-Gordon theory

$$\pi(\mathbf{x}) = \dot{\varphi}(\mathbf{x})\,,$$

and the Hamiltonian is

$$H = \int d^3\mathbf{x} \left[\frac{1}{2}\,\pi^2(\mathbf{x}) + \frac{1}{2}\,\big(\nabla\varphi\big)^2(\mathbf{x}) + \frac{m^2}{2}\,\varphi^2(\mathbf{x})\right]. \tag{2.1}$$

Of course, it coincides with the above expression for the energy, $E = \int T^{00}\, d^3\mathbf{x}$, expressed through π rather then $\dot{\varphi}$.

Quantum Klein-Gordon field

We can try now to develop quantum theory of the Klein-Gordon field, following the usual rules of canonical quantization. One starts with the correspondence

Classical Theory	Quantum Theory
Phase space	Hilbert space $\mathcal{H}$
p_i, q^i	Hermitian operators $\hat{p}_i, \hat{q}^i$: $\mathcal{H} \to \mathcal{H}$
$H(p,q)$	$\hat{H}(\hat{p},\hat{q})$: $\mathcal{H} \to \mathcal{H}$

where the Hermitian operators $\hat{q}, \hat{p}$ must obey the canonical commutation relations

$$\big[\hat{q}^i, \hat{p}_j\big] = i\,\delta^i_j\,.$$

Remark: In QM with finitely many degrees of freedom, in the Schroedinger's quantization one can take the space of square-integrable functions $\Psi(q)$ for $\mathcal{H}$, and define the corresponding representation

$$\hat{q}^i\,\Psi(q) = q^i\,\Psi(q)\,,$$

$$\hat{p}_i\,\Psi(q) = -i\,\frac{\partial}{\partial q^i}\,\Psi(q)\,,$$

and $\hat{H} = H(\hat{p},\hat{q})$. The time evolution of the state $\Psi(q)$ is governed by the Schroedinger equation

$$i\frac{\partial}{\partial t}\Psi(q) = \hat{H}\,\Psi(q)\,.$$

In principle, one can follow this prescription and introduce the operators

$$\hat{\varphi}(\mathbf{x})\,, \qquad \hat{\pi}(\mathbf{x})$$

and demand that they satisfy the canonical commutators in the form

$$\big[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{x}')\big] = i\,\delta(\mathbf{x}-\mathbf{x}')\,.$$

Furthermore, one can take the space of functionals $\Psi[\varphi(\mathbf{x})]$ as the Hilbert space $\mathcal{H}$; it provides a formal representation of the above canonical commutation relations through the identifications

$$\hat{\varphi}(\mathbf{x})\,\Psi[\varphi] = \varphi(\mathbf{x})\,\Psi[\varphi]\,,$$

$$\hat{\pi}(\mathbf{x})\,\Psi[\varphi] = -i\,\frac{\delta}{\delta\varphi(\mathbf{x})}\,\Psi[\varphi]\,.$$

The Hamiltonian then is

$$\hat{H} = \int d^3\mathbf{x}\left[-\frac{1}{2}\,\frac{\delta^2}{\delta\varphi(\mathbf{x})^2} + \frac{1}{2}\,\big(\nabla\varphi\big)^2 + \frac{m^2}{2}\,\varphi^2\right].$$

and it remains to find all solutions of the stationary Schroedinger equation

$$\hat{H}\Psi[\varphi] = E\,\Psi[\varphi]\,.$$

Exercise: Show that this Schroedinger equation admits formal solution of the form

$$\Psi_0[\varphi] = \text{Const}\,\exp\Big\{\int \varphi(\mathbf{x})G(\mathbf{x}-\mathbf{x}')\varphi(\mathbf{x}')\,d^3\mathbf{x}\,d^3\mathbf{x}'\Big\}$$

Find the kernel $G(\mathbf{x}-\mathbf{x}')$. (At this point, do not worry when you discover divergent integrals). Try to look for further solutions in the form

$$\Psi_n[\varphi] \sim \int F(\mathbf{x}_1,...,\mathbf{x}_n)\,\varphi(\mathbf{x}_1)...\varphi(\mathbf{x}_n)\,\exp\Big\{\int \varphi(\mathbf{x})G(\mathbf{x}-\mathbf{x}')\varphi(\mathbf{x}')\,d^3\mathbf{x}\,d^3\mathbf{x}'\Big\}\,.$$

Although this straightforward approach, with certain refinements, would work in the Klein-Gordon theory, it has a number of unpleasant features. One is that the scalar product in $\mathcal{H}$ has to be defined through the functional integral

$$\big(\Psi_1,\Psi_2\big) \sim \int \Psi_1^*[\varphi]\,\Psi_2[\varphi]\,D[\varphi]\,,$$

and certain care must be taken to give it a satisfactory definition (We are going to use functional integrals, in somewhat different context, though). This and other features of the above representation in terms of the variational derivatives makes this approach somewhat cumbersome. Much shorter route is based on explicit separation of variables, as we discuss below.

One observes that the Hamiltonian functional $H[\varphi]$ of the Klein-Gordon theory is quadratic in the field variables $\varphi(\mathbf{x})$ and their conjugate momenta $\pi(\mathbf{x})$; this means that the theory can be understood as the *collection of harmonic oscillators* (this observation actually suggests the Gaussian form of $\Psi_0[\varphi]$ in the above Exercise). It is then straightforward to make linear transformation of the canonical variables $\varphi(\mathbf{x}), \pi(\mathbf{x})$ which separates the variables into a system of non-interacting harmonic

oscillators, and then quantize the individual oscillators. This transformations are described in details in PS, and I refer to **Sect 2.3** there for the detailed calculation.

Instead, here I will follow equivalent routine based on special symmetries of the KG theory. Recall that the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\,\varphi^2\,,$$

and observe that the transformation of the field

$$\varphi(x) \to \varphi'(x) = \varphi(x) + f(x)\,,$$

where $f(x)$ is an arbitrary solution of the KG equation

$$\partial_\mu\partial^\mu\, f(x) + m^2\, f(x) = 0$$

changes the Lagrangian density changes by a total derivative term

$$\mathcal{L}(\varphi', \partial\varphi') = \mathcal{L}(\varphi, \partial\varphi) + \partial_\mu \tilde{J}^\mu_f\,,$$

where

$$\tilde{J}^\mu_f(x) = \varphi(x)\partial^\mu f(x)$$

(In writing this equation I have assumed that f is infinitesimal and neglected the terms $\sim f^2$). The situation meets the conditions of the Noether's theorem. It follows that every solution f of the KG equation generates a conserved current J^μ_f, such that

$$\partial_\mu J^\mu_f = 0 \qquad \text{on shell.}$$

Explicitly

$$J^\mu_f = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\, f - \tilde{J}^\mu_f = \partial^\mu\varphi\ f - \varphi\ \partial^\mu f\,.$$

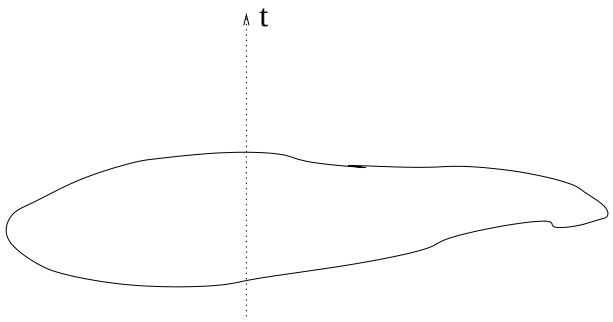
The corresponding Integrals of Motion are (at a time t)

$$A_f = \int d^3\mathbf{x}\left(\dot{\varphi}(\mathbf{x})\, f(\mathbf{x}) - \varphi(\mathbf{x})\,\dot{f}(\mathbf{x})\right) = \int d^3\mathbf{x}\left(\pi(\mathbf{x})\, f(\mathbf{x}) - \varphi(\mathbf{x})\,\dot{f}(\mathbf{x})\right), \quad (2.1)$$

where $\pi(\mathbf{x}) = \dot{\varphi}(\mathbf{x})$ is the canonical momentum conjugate to $\varphi(\mathbf{x})$. The integrals A_f are Lorentz scalars. This becomes explicit if one rewrites their definition in covariant form

$$A_f = \int_\Sigma d\Sigma_\mu \left(\partial^\mu\varphi\ f - \varphi\ \partial^\mu f\right)\,,$$

where Σ is any space-like 3-surface; because of the continuity equation the currents J^μ_f obey, the above integrals do not depend on specific choice of Σ.



In classical theory one straightforwardly evaluates the Poisson brackets

$$\{A_f, A_g\} = \int d^3\mathbf{x}\Big(\dot{f}(\mathbf{x})\, g(\mathbf{x}) - f(\mathbf{x})\, \dot{g}(\mathbf{x})\Big)$$

(This expression also can be written in the covariant form, in terms of the integral over arbitrary Σ).

In quantum theory we should replace, as usual,

$$\{A, B\} \to i\,[\hat{A}, \hat{B}]\,,$$

so that

$$[\hat{A}_f, \hat{A}_g] = -\,i\int d^3\mathbf{x}\left(\dot{f}(\mathbf{x})\, g(\mathbf{x}) - f(\mathbf{x})\, \dot{g}(\mathbf{x})\right).$$

It is convenient to use standard plane-wave solutions of the Klein-Gordon equation

$$f_{\mathbf{p}} = e^{i\,\omega_{\mathbf{p}}\,t - i\,\mathbf{p}\,\mathbf{x}}\ , \qquad f^*_{\mathbf{p}} = e^{-i\,\omega_{\mathbf{p}}\,t + i\,\mathbf{p}\,\mathbf{x}}\ ,$$

where

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}\ .$$

Denoting

$$\hat{A}_{\mathbf{p}} = \hat{A}_{f_{\mathbf{p}}}\ , \qquad \hat{A}^\dagger_{\mathbf{p}} = \hat{A}_{f^*_{\mathbf{p}}}\ ,$$

we find the commutation relations

$$[\hat{A}_{\mathbf{p}}, \hat{A}^\dagger_{\mathbf{p}'}] = (2\pi)^3\ 2\omega_{\mathbf{p}}\ \delta^{(3)}(\mathbf{p} - \mathbf{p}')\ ;$$

$$[\hat{A}_{\mathbf{p}}, \hat{A}_{\mathbf{p}'}] = 0 \; ; \qquad [\hat{A}^\dagger_{\mathbf{p}}, \hat{A}^\dagger_{\mathbf{p}'}] = 0 \; .$$

These are the *creation and annihilation operators*. The above operators $\hat{A}_{\mathbf{p}}$ and $\hat{A}^\dagger_{\mathbf{p}}$ differ in normalization from $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ used in PS (see **Sect.2.3**)

$$\hat{A}_{\mathbf{p}} = \sqrt{2\omega_{\mathbf{p}}}\, a_{\mathbf{p}} \; ;$$

$$\hat{A}^\dagger_{\mathbf{p}} = \sqrt{2\omega_{\mathbf{p}}}\, a^\dagger_{\mathbf{p}} \; .$$

Advantage of $A, A^\dagger$ is their Lorentz invariance.

The Hamiltonian can be expressed through the creation and annihilation operators as

$$\hat{H} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \frac{1}{4} \left(\hat{A}^\dagger_{\mathbf{p}}\, \hat{A}_{\mathbf{p}} + \hat{A}_{\mathbf{p}}\, \hat{A}^\dagger_{\mathbf{p}} \right), \tag{2.2}$$

which indeed is the Hamiltonian of a collection of harmonic oscillators, one oscillator per each wave-vector $\mathbf{p}$, the associated frequency being $\omega_{\mathbf{p}}$. It is straightforward to check that $\hat{A}^\dagger_{\mathbf{p}}$ rises (and $\hat{A}_{\mathbf{p}}$ lowers) the energy by the amount $\omega_{\mathbf{p}}$, namely

$$[\hat{H}, \hat{A}^\dagger_{\mathbf{p}}] = \omega_{\mathbf{p}}\, \hat{A}^\dagger_{\mathbf{p}} \; ,$$

$$[\hat{H}, \hat{A}_{\mathbf{p}}] = -\, \omega_{\mathbf{p}}\, \hat{A}_{\mathbf{p}} \; .$$

Similarly, for the momentum

$$\hat{\mathbf{P}} = -\int d^3\mathbf{x}\, \hat{\pi}(\mathbf{x})\, \nabla\hat{\varphi}(\mathbf{x})$$

we have

$$\hat{\mathbf{P}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \frac{\mathbf{p}}{4\,\omega_{\mathbf{p}}} \left(\hat{A}^\dagger_{\mathbf{p}}\, \hat{A}_{\mathbf{p}} + \hat{A}_{\mathbf{p}} \hat{A}^\dagger_{\mathbf{p}} \right).$$

Hence

$$[\hat{\mathbf{P}}, \hat{A}^\dagger_{\mathbf{p}}] = \mathbf{p}\, \hat{A}^\dagger_{\mathbf{p}} \; ,$$

$$[\hat{\mathbf{P}}, \hat{A}_{\mathbf{p}}] = -\, \mathbf{p}\, \hat{A}_{\mathbf{p}} \; ,$$

i.e. $\hat{A}^\dagger_{\mathbf{p}}$ adds $\mathbf{p}$ to the total momentum.

To find a physically acceptable solution of quantum KG theory it remains to find appropriate representation of the above commutation relations in a Hilbert space $\mathcal{H}$. Classically, the Hamiltonian (2.1) is non-negative defined. Correspondingly,

under physically meaningful quantization the quantum Hamiltonian (2.2) must be bounded from below. Recall from the theory of harmonic oscillator that in order to satisfy this property one has to assume that $\mathcal{H}$ contains the ground state $|0\rangle$ such that

$$\hat{A}_{\mathbf{p}} \mid 0\rangle = 0 \qquad \text{for all} \qquad \mathbf{p}\,.$$

Just as in the harmonic oscillator, by applying the operators $A^\dagger_{\mathbf{p}}$ to $\mid 0\rangle$ one generates the complete space of normalizable states. In QFT this is referred as the *Fock space*,

$$\mathcal{H}_{\rm KG} = \text{Fock space} = \text{Span}\left\{\hat{A}^\dagger_{\mathbf{p}_1}\,\hat{A}^\dagger_{\mathbf{p}_2}\,\cdots\,\hat{A}^\dagger_{\mathbf{p}_N}\;\mid 0\rangle\right\}\,.$$

The state $\mid 0\rangle$ is called *vacuum* state, or *Fock vacuum*. Note that the above basis already solves the diagonalization problem for $\hat{H}$: these vectors are the eigenstates of both $\hat{H}$ and $\hat{\mathbf{P}}$, with the eigenvalues

$$E = \omega_{\mathbf{p}_1} + \omega_{\mathbf{p}_2} + \ldots + \omega_{\mathbf{p}_N} + E_0\,,$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \ldots + \mathbf{p}_N + \mathbf{P}_0\,,$$

where E_0 and $\mathbf{P}_0$ are the eigenvalues associated with the vacuum state $\mid 0\rangle$,

$$\hat{H} \mid 0\rangle = E_0 \mid 0\rangle\,, \qquad \hat{\mathbf{P}} \mid 0\rangle = \mathbf{P}_0 \mid 0\rangle\,.$$

The basic vectors $\hat{A}^\dagger_{\mathbf{p}_1}\,\hat{A}^\dagger_{\mathbf{p}_2}\,\cdots\,\hat{A}^\dagger_{\mathbf{p}_N}\;\mid 0\rangle$ are interpreted as the states of N identical (Bose) free particles, with the momenta $\mathbf{p_1}, ..., \mathbf{p_N}$.

The above expressions for the Hamiltonian can be rewritten as

$$\hat{H} = E_0 + \int \frac{d^3\mathbf{p}}{(2\pi)^3}\;\frac{1}{2}\,\hat{A}^\dagger_{\mathbf{p}}\,\hat{A}_{\mathbf{p}}\,,$$

with

$$E_0 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{4}\,[\hat{A}_{\mathbf{p}}, \hat{A}^\dagger_{\mathbf{p}}]\,,$$

which of course is the sum of the zero-point energies of all the constituent oscillators.

There are two evident problems with the last expression:

(i) It is not zero, and moreover it contains seemingly meaningless factor $\delta^{(3)}(\mathbf{0})$ (recall that $[\hat{A}_{\mathbf{p}}, \hat{A}^\dagger_{\mathbf{p}'}] = 2\omega_{\mathbf{p}}\,(2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{p}')$).

(ii) The remaining integral over $\mathbf{p}$ diverges at large $\mathbf{p}$.

Note that these problem don't seem to be as bad for $\mathbf{P}_0$, where we have

$$\mathbf{P}_0 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{\mathbf{p}}{4\,\omega_{\mathbf{p}}}\,[\hat{A}_{\mathbf{p}}, \hat{A}^\dagger_{\mathbf{p}}] = 0\,,$$

since the contributions of $\mathbf{p}$ and $-\mathbf{p}$ cancel pairwise. This is good, as we do not want the vacuum to have any net momentum. But this seems to create another problem, if one thinks that the energy and momentum should be the components of 4-vector $(E_0, \mathbf{P}_0)$. If $\mathbf{P}_0 = 0$ and $E_0 \neq 0$, it seems the Lorentz invariance is broken.

To sort this part of the problem out, let us recall that we are dealing with a system in which the degrees of freedom $\varphi(\mathbf{x})$ are attached to all points of the space. For such system one expects that its vacuum energy E_0, whatever it is, must be proportional to the volume of the space,

$$E_0 = \varepsilon\, V^{(3)}\,,$$

where ε is the vacuum energy density. This answers the question how to interpret the $\delta^{(3)}(\mathbf{0})$ factor: Recall that

$$(2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{p}') = \int d^3\mathbf{x}\, e^{i\,(\mathbf{p}-\mathbf{p}')\,\mathbf{x}}\,,$$

and so the above nonsense factor in fact is

$$(2\pi)^3\,\delta^{(3)}(\mathbf{0}) = \int d^3\mathbf{x} \;\to\; V^{(3)}\,.$$

This also solves the problem of Lorentz invariance. The energy density ε transforms as 00 component of 4-tensor (the energy-momentum tensor),

$$\varepsilon = \langle 0 \mid T^{00} \mid 0\rangle\,.$$

The Lorentz-invariance of the vacuum state $\mid 0\rangle$ only requires that

$$\langle 0 \mid T^{\mu\nu} \mid 0\rangle = \varepsilon_0\,\eta^{\mu\nu}\,, \qquad \eta^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)\,,$$

and we soon will have techniques to check that this equation indeed holds in the KG theory. It is OK for a Lorentz-invariant state to have nonzero energy density.

Still, something has to be done about the badly divergent integral for this energy density,

$$\varepsilon_0 = \frac{1}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\omega_{\mathbf{p}}\,.$$

The integral diverges at large momenta, i.e. the divergent contribution comes from small length scales. This is not too surprising, since our idea of the field theory was to associate a degree of freedom with every point of the space, and there are continuously many points in each finite part of it. One could assume that perhaps some microscopic physics takes care of the problem, replacing the above mathematically meaningless expression by the "cutoff" integral

$$\varepsilon_0 = \frac{1}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \omega_{\mathbf{p}}\, \Phi\left(\frac{\mathbf{p}^2}{\Lambda^2}\right).$$

where the "cutoff" factor $\Phi(\xi)$ is close to 1 when $\xi << 1$ and decays fast at large ξ, and Λ represents the energy scale beyond which the unknown microscopic physics becomes essential.

Whereas introducing the cutoff rids us of mathematically meaningless expression, needless to say it does not solve the problem of vacuum energy density, which now is finite but completely undetermined. In many practical cases one can simply ignore the problem. Large class of physical quantities are only sensitive to differences of energies, and in such problems the vacuum energy is invisible. But in fact the vacuum energy is measurable, in principle. Generally, if we want to be able to apply QFT to physics, we need to know what happens when we couple it to gravity, and gravity is very sensitive to the energy density. Even without gravity, suppose we can confine our system to finite part of the space, say inside a box. Then we want to know what forces it exerts upon the wall of that box. The last situation can actually be realized in laboratory, and it leads to a measurable effect - the so called Casimir effect. We are going to discuss it next, but let me make one more remark, in order to clarify the nature of the problem.

When a divergent integral appears in otherwise sensible calculations, it usually signals some ambiguity in the theory. Indeed, there is an ambiguity in the definition of the KG theory. One can add a constant term to the Lagrangian density,

$$\mathcal{L} \to \mathcal{L} + \text{Const}\,.$$

This extra term is fully consistent with both relativistic symmetry and locality. Of course this modification is completely irrelevant in the classical theory, since it does not affect the equations of motion. But the Lagrangian density will pass this term to the energy-momentum tensor, and finally to the Hamiltonian,

$$\hat{H} \to \hat{H} - \text{Const} \int d^3\mathrm{x}\,.$$

One can now "absorb" the vacuum energy divergence into this constant. We write

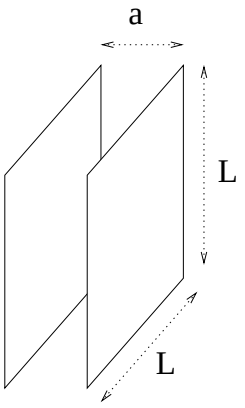
$$\text{Const} = \varepsilon_0 - \varepsilon \,,$$

where ε_0 is the above ugly integral, and ϵ is a finite constant (of the dimension [Mass]^4). The result is that the divergent integral disappears, but at the same time we learn that the quantum KG theory in fact has not one but two parameters, m^2 and ε. This extra term we added to the action is the simplest example of what we will call the *counterterms*.

3 Lecture 3

Casimir Effect

To understand possible manifestations of the vacuum energy, let us now briefly discuss the *Casimir Effect*. Originally, the Casimir effect refers to the situation where we have two parallel conducting plates, of large size $L \times L$, separated by a distance a.



Due to the quantum nature of the electromagnetic field in between the plates, more precisely, due to the effect of the plates on the zero-point energy of quantum electromagnetic field, the plates are attracted with the force

$$\frac{\text{Force}}{L^2} = -\hbar c \, \frac{\pi^2}{240\, a^4} \,,$$

where it is assumed that

$$L >> a >> \text{atomic distances.}$$

Casimir effect in KG theory

Let us study the nature of this effect using our scalar field theory, the KG theory, instead of the electrodynamics; the result is not going to be very different. Since the real photons are massless (and to make calculations simpler), we will assume that $m^2 = 0$. In macroscopic electromagnetic theory (valid at the scales $>>$ atomic distances) the presence of the conducting plates is taken into account by imposing the boundary conditions $\mathbf{E}_{\|} = 0$, $\mathbf{B} = 0$ at the surface of the conductor.

To mimic this situation we consider instead the KG field in the presence of two plates with the boundary condition

$$\varphi(\mathbf{x}) = 0 \qquad \text{at the surface.}$$

Let us concentrate attention on the field between the plates. In order to take into account the boundary conditions, in this case we must use, instead of the functions $f_{\mathbf{p}}(x)$ and $f^*_{\mathbf{p}}(x)$ above, the following complete set of solutions

$$f_{\mathbf{p}_\parallel,n}(\mathbf{x},\mathbf{t}) = \sqrt{\frac{\mathbf{2}}{\mathbf{a}}}\ \mathbf{e}^{\mathbf{i}\omega_{\mathbf{p},\mathbf{n}}\,\mathbf{t}-\mathbf{i}\mathbf{p}_\parallel\,\mathbf{x}_\parallel}\ \sin\left(\frac{\pi\,\mathbf{n}}{\mathbf{a}}\,\mathbf{x}_\perp\right),$$

and the corresponding complex-conjugated functions $f^*_{\mathbf{p}_\parallel,n}$. Here $\mathbf{x}_\parallel$ and $x_\perp$ are the components of $\mathbf{x}$ parallel and perpendicular to the plates, $\mathbf{x} = (\mathbf{x}_\parallel, x_\perp)$, and

$$\omega_{\mathbf{p}_\parallel,n} = \sqrt{\mathbf{p}_\parallel^2 + \frac{\pi^2\,n^2}{a^2}}$$

with positive integer and n,

$$n = 1, 2, 3, \ldots .$$

These functions solve the KG equation with $m^2 = 0$ as well as the boundary conditions

$$f_{\mathbf{p}_\parallel,n}\big|_{x_\perp=0} = 0\,, \qquad \text{and} \qquad f_{\mathbf{p}_\parallel,n}\big|_{x_\perp=a} = 0\,.$$

Exactly as we did in the infinite-volume case, we now introduce the operators

$$\hat{A}_{\mathbf{p}_\parallel,n} = \int_0^a dx_\perp \int d^2\mathbf{x}_\parallel \left[\hat{\pi}\, f_{\mathbf{p}_\parallel,n} - \hat{\varphi}\, \dot{f}_{\mathbf{p}_\parallel,n}\right],$$

and similarly for $\hat{A}^\dagger_{\mathbf{p}_\parallel,n}$, with $f^*_{\mathbf{p}_\parallel,n}$ replacing $f_{\mathbf{p}_\parallel,n}$. It is not difficult then to show that

$$[\hat{A}_{\mathbf{p}_\parallel,n}, \hat{A}^\dagger_{\mathbf{p}'_\parallel,n'}] = 2\omega_{\mathbf{p}_\parallel,n}\,\delta_{nn'}\,(2\pi)^2\,\delta(\mathbf{p}_\parallel - \mathbf{p}'_\parallel)\,,$$

and to compute the Hamiltonian of the part of the system residing between the plates,

$$\hat{H} = \sum_{n=1}^{\infty} \int \frac{d^2\mathbf{p}_\parallel}{(2\pi)^2}\,\frac{1}{4}\left(\hat{A}^\dagger_{\mathbf{p}_\parallel,n}\,\hat{A}_{\mathbf{p}_\parallel,n} + \hat{A}_{\mathbf{p}_\parallel,n}\,\hat{A}^\dagger_{\mathbf{p}_\parallel,n}\right) + a\,L^2\,(\varepsilon - \varepsilon_0)\,,$$

where the last term comes from vacuum energy counterterm

$$(\varepsilon - \varepsilon_0) \int_0^a dx_\perp \int d^2\mathbf{x}_\parallel = a\, L^2\, (\varepsilon - \varepsilon_0)\,.$$

We still have to demand that all the operators $\hat{A}_{\mathbf{p}_\parallel, n}$ nullify the ground state, since otherwise $\hat{H}$ is unbounded from below. Therefore the vacuum eigenvalue of this Hamiltonian is

$$E_{\rm vac} = \frac{L^2}{2} \sum_{n=1}^{\infty} \int \frac{d^2\mathbf{p}_\parallel}{(2\pi)^2} \sqrt{\mathbf{p}_\parallel^2 + \frac{\pi^2 n^2}{a^2}}\; \Phi\left(\frac{\mathbf{p}_\parallel^2 + \pi^2 n^2/a^2}{\Lambda^2}\right) - a\, L^2\, \varepsilon_0 + a\, L^2\, \varepsilon\,,$$

where I have introduced again the cutoff factor Φ - without it this expression is as divergent as it was in the absence of the boundaries.

Now, the term $-a\, L^2\, \varepsilon_0$ is the divergent part of the vacuum energy counterterm. It is

$$-aL^2\, \varepsilon_0 = -\frac{L^2}{2}\, a \int \frac{dp_\perp}{(2\pi)} \int \frac{d\mathbf{p}_\parallel}{(2\pi)^2} \sqrt{\mathbf{p}_\parallel^2 + p_\perp^2}\; \Phi\left(\frac{\mathbf{p}_\parallel^2 + p_\perp^2}{\Lambda^2}\right),$$

where I have separated the $dp_\perp$ and $d^2\mathbf{p}_\parallel$ parts of the integral over $d^3\mathbf{p}$. After the change of variables

$$p_\perp = \pi\, \tau / a$$

it takes the form

$$-\frac{L^2}{2} \int_0^\infty d\tau \int \frac{d^2\mathbf{p}_\parallel}{(2\pi)^2} \sqrt{\mathbf{p}_\parallel^2 + \frac{\pi^2\, \tau^2}{a^2}}\; \Phi\left(\frac{\mathbf{p}_\parallel^2 + \pi^2\, \tau^2/a^2}{\Lambda^2}\right).$$

We see that the interesting part of the ground-state energy (rather, of the energy per unit area of the plates) can be written as

$$\frac{E}{L^2} - a\, \varepsilon = \int \frac{d^2\mathbf{p}_\parallel}{(2\pi)^2} \left[\sum_{n=1}^{\infty} F_{\mathbf{p}_\parallel}(n) - \int_0^\infty F_{\mathbf{p}_\parallel}(\tau)\, d\tau\right],$$

where

$$F_{\mathbf{p}_\parallel}(\tau) = \frac{1}{2} \sqrt{\mathbf{p}_\parallel^2 + \pi^2\, \tau^2/a^2}\;\; \Phi\left(\frac{\mathbf{p}_\parallel^2 + \pi^2\, \tau^2/a^2}{\Lambda^2}\right).$$

The $\sum F(n) - \int F(\tau)$ expression above can be handled with the help of the Euler-McLaurin summation formula (see Appendix)

$$\sum_{n=1}^{\infty} F(n) - \int_0^\infty F(\tau)\, d\tau = -\frac{1}{2}\, F(0) + i \int_0^\infty \frac{F(it+0) - F(-it+0)}{e^{2\pi t} - 1}\, dt\,. \qquad (3.1)$$

which is valid provides $F(\tau)$ is analytic in the right half-plane of complex τ. The first term

$$-\frac{1}{2}\,F_{\mathbf{p}_\parallel}(0) = -\frac{1}{4}\int \frac{d^2\mathbf{p}_\parallel}{(2\pi)^2}\,\sqrt{\mathbf{p}_\parallel^2}\;\Phi\big(\mathbf{p}_\parallel^2/\Lambda^2\big)\,.$$

does not depend on the separation a; it is interpreted as the surface energy. It diverges, again signaling ambiguity. When there is a surface, a constant surface term can be added to the action. We ignore this term since, being independent of a it does not contribute to the force.

The interesting contribution is extracted from the second term, which involves

$$i\int_0^\infty \frac{F(it+0)-F(-it+0)}{e^{2\pi t}-1}\,dt$$

The the shift "+0" in the arguments in (EM) is important since the function $F_{\mathbf{p}_\parallel}(\tau)$ has square-root brunching points at the imaginary axis. Note that if it was regular, the integral would vanish in view of the obvious $\tau\to-\tau$ symmetry of $F_{\mathbf{p}_\parallel}(\tau)$. In fact, one can take advantage of this symmetry to write the integrand as

$$i\int_0^\infty \frac{F(it+0)-F(it-0)}{e^{2\pi t}-1}\,dt = -2\int_{|\mathbf{p}_\parallel|\,a/\pi}^\infty \frac{\Im m\,F(it+0)}{e^{2\pi t}-1}\,dt\,,$$

where the last form reflects the fact that the function $F_{\mathbf{p}_\parallel}(\tau)$ has the branch cut from $i|\mathbf{p}|a/\pi$ to $i\infty$ (and from $-i|\mathbf{p}|a/\pi$ to $-i\infty$). We have, at $|t|>a|p|/\pi$

$$\Im m\,F_{\mathbf{p}_\parallel}(it+0) = \frac{1}{2}\sqrt{\frac{\pi t^2}{a^2}-\mathbf{p}_\parallel^2}\;\;\Phi\left(\frac{\mathbf{p}_\parallel^2-\pi^2t^2/a^2}{\Lambda^2}\right) := \frac{1}{2}\,G_{\mathbf{p}_\parallel}(t)\,.$$

Note that the factor $(e^{2\pi t}-1)^{-1}$ makes this integral fast convergent at large t, even without the cutoff factor Φ. At large $\Lambda >> a^{-1}$ this factor has no effect, and can be dropped,

$$G_{\mathbf{p}_\parallel}(t)\to\sqrt{\frac{\pi^2t^2}{a^2}-\mathbf{p}_\parallel^2}\,.$$

The interesting contribution to the vacuum energy then is

$$-\int\frac{dt}{e^{2\pi t}-1}\int_{|\mathbf{p}_\parallel|\le\pi t/a}\frac{d^2\mathbf{p}_\parallel}{(2\pi)^2}\,\sqrt{\frac{\pi^2t^2}{a^2}-\mathbf{p}_\parallel^2}\,.$$

where I have interchanged the order in which the integrals are taken. The integral over $\mathbf{p}_\parallel$ is evaluated in closed form

$$\int_0^{\frac{\pi^2t^2}{a^2}}\frac{dp^2}{4\pi}\,\sqrt{\frac{\pi^2t^2}{a^2}-p^2} = \frac{1}{6\pi}\left(\frac{\pi t}{a}\right)^3,$$

The contribution to the energy is

$$-\frac{\pi^2}{6\,a^3}\int_0^\infty \frac{t^3\,dt}{e^{2\pi t}-1} = -\frac{\pi^2}{6\,a^3}\,\frac{\Gamma(4)\,\zeta(4)}{(2\pi)^4}$$

where I have used

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^x-1}\,dx = \Gamma(s)\,\zeta(s)\,.$$

($\zeta(4) = \pi^4/90$). Thus we find

$$\frac{E}{L^2} = a\,\varepsilon - \frac{1}{2}\,F(0) - \frac{\pi^2}{1440\,a^3}$$

The contribution $a\,\varepsilon$ is not too interesting - there is the same term in the vacuum energy density outside the space between the plates, and hence this term does not lead to any force. The interesting term $-\pi^2/1440\,a^3$ shows that the plates attract; the force (per unit plate area) is

$$\frac{\text{Force}}{L^2} = \frac{\pi^2\,\hbar c}{480\,a^4}\,.$$

Let me stress that this result is completely independent of the form of auxiliary cutoff factor Φ - this is exactly what we want from quantum field theory. Another useful observation is that in order to obtain this result we don't really need to send Λ to Planck's energies or something, it suffices to have $a^2 >> 1/\Lambda^2$.

In the real case of electromagnetic field the calculations can be done in very similar way. The resulting force is twice as large, due to the fact that photon field has two degrees of freedom for each $\mathbf{p}$, as the photon has two polarization states.

Few words about the a-independent term

$$-\frac{1}{2}\,F(0) = -\frac{1}{16\pi}\int_0^\infty d\xi\,\sqrt{\xi}\,\Phi\big(\xi/\Lambda^2\big)\,.$$

It does not contribute to the force, but still needs some interpretation. It is divergent, and it is easy to design special counter-term to absorb this divergence. But it is instructive to think about real case of electromagnetic field and real conducting pates made of atoms, conducting electrons, and all that. In this case the cutoff represents the limit of validity of the macroscopic electrodynamics, and the cutoff energy Λ in this expression should be taken around atomic energy scale. Having sufficient knowledge about the microscopic structure of the conductor we could in

principle calculate the cutoff factor and hence this contribution in terms of the atomic parameters. The nice feature of the above result is in that it predicts some attractive force between the plates which is completely independent of all these microscopic details. This points to another area where quantum field theory applies, when it is not that we can't know the short-distance physics, rather we don't want to know. In this area the quantum field theory describes universal, i.e. not too sensitive to microscopic details, properties of matter.

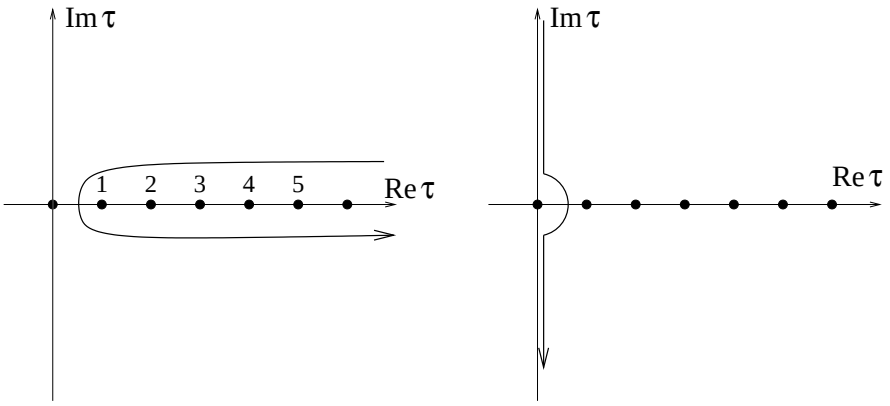
Appendix: Euler-Maclaurin formula

Let $F(\tau)$ be a function analytic in the right half-plane, and decaying at $\Re e\,\tau \to +\infty$ sufficiently fast, so that the integral $\int_0^\infty f(\tau)\,d\tau$ converges.

Start with the integral

$$I = \int_{C_0} \frac{F(\tau)\,d\tau}{e^{2\pi i\tau} - 1}\,,$$

where the integration is over the contour C which encloses all positive integer points $\tau = 1, 2, 3, ...$, as shown in the left figure.



The integrand has poles at the integer points $\tau = n$, with the residues $F(n)/2\pi i$. Therefore

$$I = \sum_{n=1}^{\infty} F(n)\,.$$

On the other hand, the integral can be split into two parts

$$I = \int_{C_0} = \int_{C_+} + \int_{C_-} = I_+ + I_-\,,$$

where C_+ is the part of the contour C_0 that lays above the real axis, and C_- is the part below the real axis. Note that the factor

$$\frac{1}{e^{2\pi i\tau}-1}$$

decays fast into the lower half-plane, hence the contour C_- can be rotated into the lower half-plane to go along the contour $\tilde{C}_-$,

$$I_- = \int_{\tilde{C}_-} \frac{F(\tau)\,d\tau}{e^{2\pi i\tau}-1} = -i\int_\epsilon^\infty \frac{F(-it+0)\,dt}{e^{2\pi t}-1} + \int_{C'_-} \frac{F(\tau)\,d\tau}{e^{2\pi i\tau}-1}\,,$$

where for the straight part of C_- I have changed

$$\tau = -it\,,$$

and C'_- is the small quarter-circle below the real axis, seen in the right part of the figure. It is easy to check that in the limit $\epsilon \to 0$ this part of the integral yields $-F(0)/4$.

On the other hand the factor $(e^{2\pi i\tau}-1)^{-1}$ does not decay into the upper half plane, but one can write

$$\frac{1}{e^{2\pi i\tau}-1} = -1 - \frac{1}{e^{-2\pi i\tau}-1}\,,$$

where the second term do decay into the upper half-plane, so that

$$I_+ = -\int_{C_+} F(\tau)\,d\tau + \int_{\tilde{C}_+} \frac{F(\tau)\,d\tau}{e^{-2\pi i\tau}-1} = \int_\epsilon^\infty F(\tau)\,d\tau + i\int_\epsilon^\infty \frac{F(it+0)\,dt}{e^{2\pi t}-1} + \int_{C'_+}\,,$$

where the second term in the last form represents the contribution of the straight part of C_+ (with $\tau = it+0$), and the last term is the contribution of the small quarter-circle above the real axis; again, it is easy to check that this last contribution is $-F(0)/4$ in the limit $\epsilon \to 0$. Adding I_+ to I_-, and taking the limit $\epsilon = 0$, yields (3.1).

4 Lecture 4

Quantum Klein-Gordon theory (Continued)

The space of states $\mathcal{H}$ of the quantum KG theory is the Fock space, defined as the linear envelope of the basic states

$$|\mathbf{p}_1, ..., \mathbf{p}_N\rangle = \hat{A}^\dagger_{\mathbf{p}_1} ... \hat{A}^\dagger_{\mathbf{p}_N} \mid 0\rangle \,,$$

where the vacuum $|0\rangle$ satisfies

$$\hat{A}_{\mathbf{p}} \mid 0\rangle = 0 \qquad \text{for all} \quad \mathbf{p} \,.$$

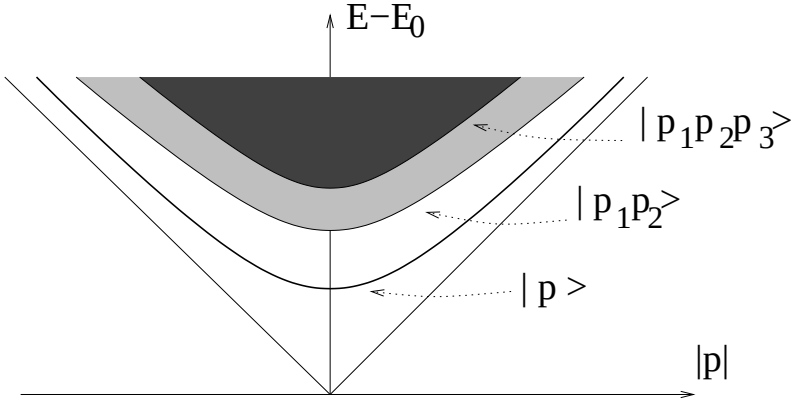
Here I have returned to the default case of the infinite Minkowski space-time, so that the 3-momenta $\mathbf{p_i}$ may take continuous unrestricted values.

The states $|\mathbf{p}_1, ..., \mathbf{p}_N\rangle$ are interpreted as N-particle states. Their scalar products follow from the commutation relations $[\hat{A}_{\mathbf{p}}, \hat{A}^\dagger_{\mathbf{p}'}] = 2\omega_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. In particular, for the one-particle states

$$\langle \mathbf{p} \mid \mathbf{p}' \rangle = 2\omega_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \,.$$

Since the operators $\hat{A}^\dagger_{\mathbf{p}}$ are Lorentz invariant (only the momentum $\mathbf{p}$ transforms under the Lorentz transformations), the particle have zero spin. Since the creation operators $\hat{A}^\dagger_{\mathbf{p}}$ commute, the N-particle states are symmetric with respect to permutations of the momenta $\mathbf{p}_i$ - the particles are bosons.

The spectrum of the energy-momentum operator $\left(\hat{H} - E_0, \hat{\mathbf{P}}\right)$ is the composition of spectra of the N-particle states. In the space of vectors $(E - E_0, \mathbf{P})$ the vacuum resides at the origin of coordinates, the one particle states form the mass-shell hyperboloid $(E - E_0)^2 - \mathbf{P}^2 = m^2$, the two-particle states fill the inside of the hyperboloid $(E - E_0)^2 - \mathbf{P}^2 = 4\,m^2$, etc.



Correspondingly, the spectrum of the operator $\hat{M}^2 = (\hat{H} - E_0)^2 - \hat{\mathbf{P}}^2$ has two discrete components, 0 and m^2, and continuous part from $4\,m^2$ to infinity.

Klein-Gordon propagator

By their definition, the creation and annihilation operators $\hat{A}^\dagger_{\mathbf{p}}, \hat{A}_{\mathbf{p}}$ are the Fourier components of the local field operators $\hat{\varphi}(\mathbf{x})$ and $\hat{\pi}(\mathbf{x})$,

$$\hat{\varphi}(\mathbf{x}) = \int d\mu(\mathbf{p}) \left[\hat{A}_{\mathbf{p}} + \hat{A}^\dagger_{-\mathbf{p}}\right] e^{i\mathbf{p}\mathbf{x}}\,,$$

$$\hat{\pi}(\mathbf{x}) = \int d\mu(\mathbf{p})\, i\omega_{\mathbf{p}} \left[-\hat{A}_{\mathbf{p}} + \hat{A}^\dagger_{-\mathbf{p}}\right] e^{i\mathbf{p}\mathbf{x}}\,,$$

where I have introduced special notation for the Lorentz-invariant measure on the mass shell,

$$d\mu(\mathbf{p}) = \frac{1}{2\omega_{\mathbf{p}}} \frac{d^3\mathbf{p}}{(2\pi)^3}\,.$$

Exercise: Check that the measure $d\mu(\mathbf{p})$ is Lorentz-invariant. An instructive way to do that is to check the identity

$$d\mu(\mathbf{p}) = \int_{p^0>0} dp^0\, \delta(p^\mu p_\mu - m^2)\,,$$

i.e. we have

$$\int d\mu(\mathbf{p})\, F(\mathbf{p}, \omega_{\mathbf{p}}) = \int_{p^0>0} \frac{d^4p}{(2\pi)^4}\; 2\pi\; \delta(p^\mu p_\mu - m^2)\, F(\mathbf{p}, p^0)\,.$$

The operators $\varphi(\mathbf{x})$ and $\pi(\mathbf{x})$ are the "Schroedinger", or the "equal time" field operators. As defined, they obey the canonical commutation relations

$$[\hat{\pi}(\mathbf{x}), \hat{\varphi}(\mathbf{x}')] = -i\,\delta^{(3)}(\mathbf{x} - \mathbf{x}')\,.$$

As is common in quantum mechanics, it is often useful to trade the time evolution of the states for the time evolution of the operators. This is known as the Heisenberg picture. In our case, introduce the Heisenberg field operator

$$\hat{\varphi}(x) = \hat{\varphi}(\mathbf{x}, t) = e^{i\hat{H}t}\, \hat{\varphi}(\mathbf{x})\, e^{-i\hat{H}t}\,, \qquad x = (\mathbf{x}, t).$$

In terms of the creation and annihilation operators it is

$$\hat{\varphi}(x) = \int d\mu(\mathbf{p}) \left[\hat{A}_{\mathbf{p}} \, f^*_{\mathbf{p}}(x) + \hat{A}^\dagger_{\mathbf{p}} \, f_{\mathbf{p}}(x) \right] ,$$

where as before $f_{\mathbf{p}}(x) = e^{i\omega_{\mathbf{p}} t - i\mathbf{p}\,\mathbf{x}}$. Let us compute the commutator $[\hat{\varphi}(x), \hat{\varphi}(x')]$. Then

$$\hat{\varphi}(\mathbf{x}) = \hat{\varphi}(\mathbf{x}, \mathbf{t} = \mathbf{0}) , \qquad \hat{\pi}(\mathbf{x}) = \partial_{\mathbf{t}} \hat{\varphi}(\mathbf{x}, \mathbf{t}) \mid_{\mathbf{t}=\mathbf{0}} .$$

Commutators

It is interesting and instructive to see how the canonical commutators of the "Schroedinger" operators extend to the Heisenberg field operators $\hat{\varphi}(x)$. For this calculation, it is useful to write

$$\hat{\varphi}(x) = \hat{\varphi}_-(x) + \hat{\varphi}_+(x) ,$$

where

$$\hat{\varphi}_-(x) = \int d\mu(\mathbf{p}) \, \hat{A}_{\mathbf{p}} \, f^*_{\mathbf{p}}(x) , \qquad \hat{\varphi}_+(x) = \int d\mu(\mathbf{p}) \, \hat{A}^\dagger_{\mathbf{p}} \, f_{\mathbf{p}}(x) ,$$

The operator $\hat{\varphi}_+(x)$ contains only the creation operators $\hat{A}^\dagger_{\mathbf{p}}$; it can be interpreted as an operator creating particle at the space-time point x; likewise, $\hat{\varphi}_-(x)$ absorbs the particle at the space-time point x. Since $\hat{\varphi}_-(x)$ obviously commutes with $\hat{\varphi}_-(x')$, and the same is true for the $\hat{\varphi}_+$ component, we have

$$[\hat{\varphi}(x), \hat{\varphi}(x')] = D_-(x - x') - D_+(x - x') ,$$

where

$$D_-(x - x') = [\hat{\varphi}_-(x), \hat{\varphi}_+(x')] ,$$
$$D_+(x - x') = [\hat{\varphi}_-(x'), \hat{\varphi}_+(x)] \equiv D_-(x' - x) .$$

Explicitly

$$D_-(\mathbf{x}, t) = \int d\mu(\mathbf{p}) \, e^{-i\omega_{\mathbf{p}} \, t + i\mathbf{p}\,\mathbf{x}} ,$$

$$D_+(\mathbf{x}, t) = \int d\mu(\mathbf{p}) \, e^{i\omega_{\mathbf{p}} \, t - i\mathbf{p}\,\mathbf{x}} ,$$

Some properties of these integrals are worth mentioning.

i) The integrals for $D_-(x)$ and $D_+(x)$ are Lorentz-invariant, i.e. if two 4-vectors x and x' are related by a Lorentz transformation, $x' = \Lambda x$, then $D(x) = D(x')$. Indeed, the integrals can be written in explicitly covariant form, say

$$D_-(x) = \int_{p^0>0} \frac{d^4p}{(2\pi)^4} \, 2\pi \, \delta(p^\mu p_\mu - m^2) \, e^{-i \, p_\mu \, x^\mu} .$$

Then the change of the integration variables $p = \Lambda^{-1}p'$ verifies that $D_-(x) = D_-(\Lambda x)$. Similar representation exists for $D_+(x)$.

ii) The integrals are not absolutely convergent. It is usually convenient to compute such integrals as some limiting values of absolutely convergent integrals. In our case it can be done by considering the integrals at complex values of the time variable t. For instance, the above integral defining $D_-(\mathbf{x}, t)$ converges absolutely if t has **negative imaginary part** (remember that $\omega_{\mathbf{p}}$ is positive), therefore this integral defines the function $D_-(\mathbf{x}, t)$ of complex variable t which is analytic in the lower half-plane of the complex t-plane. Then, for real t we need to take the limit $\lim_{\epsilon\to-0} D_-(\mathbf{x}, t+i\epsilon)$ (this is usually written as $D_-(\mathbf{x}, t-i0)$). Likewise, the integral for D_+ defines the function $D_+(\mathbf{x}, t)$ of complex t, analytic in the upper half-plane, and for real t we take $D_+(\mathbf{x}, t+i0)$.

Let us consider the case of real t such that the 4-vector $x^\mu = (t, \mathbf{x})$ is space-like,

$$x^\mu x_\mu = t^2 - \mathbf{x}^2 < 0$$

(in this consideration we assume that $\mathbf{x}$ is real). In the complex t-plane we are looking at the segment

$$-|\mathbf{x}| < t < |\mathbf{x}|$$

of the real axis. It is easy to show that at this segment the we have

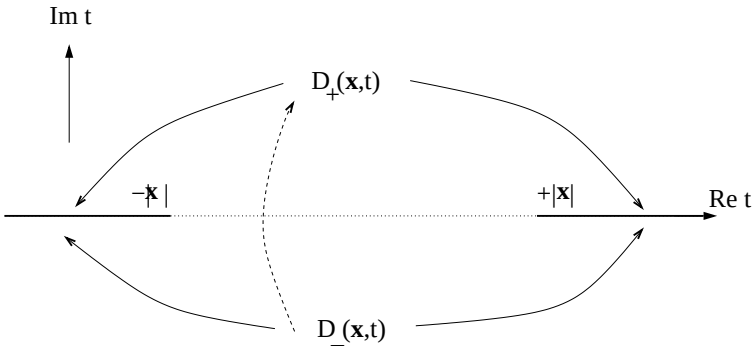
$$D_+(\mathbf{x}, t) = D_-(\mathbf{x}, t) \quad \text{for} \qquad -|\mathbf{x}| < t < |\mathbf{x}|$$

It suffices to verify this identity at $t = 0$, since one can always transform to the frame where $t = 0$, as long as x^μ is space-like. For $t = 0$ we have

$$D_-(\mathbf{x}, 0) = \lim_{\epsilon\to+0} \int e^{-\omega_{\mathbf{p}}\,\epsilon + i\mathbf{p}\,\mathbf{x}}\, d\mu(\mathbf{p}) \qquad \text{and} \qquad D_+(\mathbf{x}, 0) = \lim_{\epsilon\to+0} \int e^{-\omega_{\mathbf{p}}\,\epsilon - i\mathbf{p}\,\mathbf{x}}\, d\mu(\mathbf{p})\,,$$

and the identity is established by the change of the integration variables $\mathbf{p} \to -\mathbf{p}$. Note that this argument also shows that $D_\pm(\mathbf{x}, t)$ return real values when t takes values within the above segment of the real axis.

It follows that the function $D_-(\mathbf{x}, t)$, which was originally defined (by out integral) in the lower half-plane of t, in fact can be analytically continued, through the above segment of the real axis, to the upper half-plane, and there it coincides with $D_+(\mathbf{x}, t)$ (defined, again, through its integral representation written above).



In other words, in fact there is a single function of complex t (I will denote it as $D(\mathbf{x}, t)$), analytic in both the upper and lower half-planes, including the segment $-|\mathbf{x}| < t < |\mathbf{x}|$ of the real axis, and $D_+(\mathbf{x}, t)$ ($D_-(\mathbf{x}, t)$) is is given by its values in the upper (lower) half-plane. In particular, for real t

$$D_+(\mathbf{x}, t) = D(\mathbf{x}, t + i0)\,,$$

$$D_-(\mathbf{x}, t) = D(\mathbf{x}, t - i0)\,.$$

As we have seen, these two limits coincide when $-|\mathbf{x}| < t < |\mathbf{x}|$, but outside this segment they don't. That means the function $D(\mathbf{x}, t)$ has two branch cuts in the complex t-plane, from $-\infty$ to $-|\mathbf{x}|$, and from $|\mathbf{x}|$ to $+\infty$.

It is not difficult to identify the geometric meaning of the branching points $t = \pm|\mathbf{x}|$. They represent points on the future and past components of the light cone associated with the space-time point 0, that is the geometric place of all points which can be connected to 0 by light rays. At fixed $\mathbf{x}$ the real t line intersects the light cone at two points $\pm|\mathbf{x}|$. When t regarded as the complex variable, the intersection points become exactly the branching points in the complex t plane. At real t, when crossing the light cone one needs to choose the branch of the analytic function (similar to choosing a branch of $\sqrt{z}$ at negative z). The functions D_+ and D_- the two branches corresponding to two ways of going around the branching points in the complex t-plane.

Figure

Let us come back to the commutator $[\hat{\varphi}(x), \hat{\varphi}(0)] = D_-(x) - D_+(x)$. Here I have set $x' = 0$; in general case x has to be replaced by the difference $x - x'$. We see that the commutator is equal to the discontinuity

$$[\hat{\varphi}(\mathbf{x}, t), \hat{\varphi}(\mathbf{0}, 0)] = D(\mathbf{x}, t - i0) - D(\mathbf{x}, t + i0)\,.$$

When the separation $x = (t, \mathbf{x})$ is space-like the discontinuity vanishes, and we conclude that

$$[\hat{\varphi}(x), \hat{\varphi}(x')] = 0 \qquad \text{for all space}-\text{like separations} \qquad x - x'\,.$$

This is important conclusion. Recall that in quantum mechanics commutator of two operators allows one to determine how, in a given state, the measurement of one observable can affects results of the measurements of another observable. Our calculation shows that the measurement at some space-time point x can not affect measurements at the point x' as long as the separation $x - x'$ is space-like. It expresses the **causality** of the quantum field theory. This is general requirement of quantum field theory called the **local commutativity**. It states that for any local fields $\mathcal{O}_1(x)$, $\mathcal{O}_2(x')$ we must have

$$[\hat{\mathcal{O}}_1(x), \hat{\mathcal{O}}_2(x')] = 0 \qquad \text{for all space}-\text{like separations} \qquad x - x'\,.$$

The discontinuities across the brunch cuts $|t| > |\mathbf{x}|$ do not vanish, so when the separation is time-like the commutator takes non-zero values. We see that the commutator has its support inside the light cone.

Consider now the expectation value $\langle 0 \mid \hat{\varphi}(x)\,\hat{\varphi}(x') \mid 0\rangle$ of the product of two field operators. In the KG theory this expectation value can be expressed through the same analytic function $D(\mathbf{x}, t)$. Indeed, since $\hat{\varphi}_-(x) \mid 0\rangle = 0$ and $\langle 0 \mid \hat{\varphi}_+(x) = 0$, we have

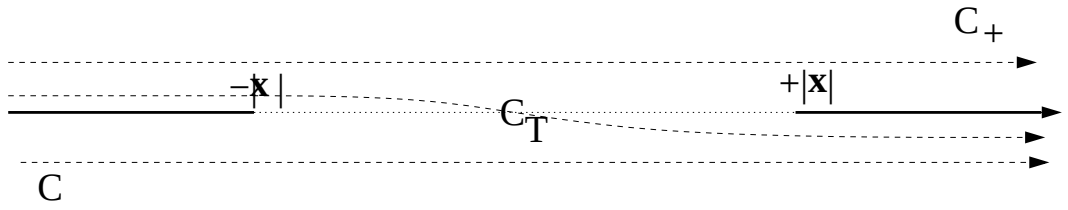
$$\langle 0 \mid \hat{\varphi}(x)\,\hat{\varphi}(x') \mid 0\rangle = \langle 0 \mid \hat{\varphi}_-(x)\,\hat{\varphi}_+(x') \mid 0\rangle = D_-(x - x') = D(\mathbf{x} - \mathbf{x}', t - t' - i0)\,.$$

If we interchange the positions of the operators, by similar calculation

$$\langle 0 \mid \hat{\varphi}(x')\,\hat{\varphi}(x) \mid 0\rangle = D_+(x - x') = D(\mathbf{x} - \mathbf{x}', t - t' + i0)$$

Thus, these expectation values are described by the function $D(\mathbf{x}, t)$ taken at the upper or lower edges of the light-cone branch cuts

$$\langle 0 \mid \hat{\varphi}(\mathbf{0},0)\,\hat{\varphi}(\mathbf{x},t) \mid 0\rangle$$



$$\langle 0 \mid \hat{\varphi}(\mathbf{x},t)\,\hat{\varphi}(\mathbf{0},0) \mid 0\rangle$$

Important role in our future studies will belong to the **time ordered** expectation values

$$D_{\rm F}(x-x') = D_{\rm F}(\mathbf{x}-\mathbf{x}', t-t') = \langle 0 \mid T\big(\hat{\varphi}(\mathbf{x},t)\,\hat{\varphi}(\mathbf{x}',t')\big) \mid 0\rangle\,,$$

where the symbol T signifies the time ordered product of the operators,

$$T\big(\hat{\varphi}(\mathbf{x},t)\,\hat{\varphi}(\mathbf{x}',t')\big) = \begin{cases} \hat{\varphi}(\mathbf{x},t)\,\hat{\varphi}(\mathbf{x}',t') & \text{for} \quad t > t' \\ \hat{\varphi}(\mathbf{x}',t')\,\hat{\varphi}(\mathbf{x},t) & \text{for} \quad t < t' \end{cases}$$

This expectation value is also called the **Feynman propagator**. One can visualize it in the above drawing by first considering complex values of t along the contour $C_{\rm F}$, and then taking the limit when this contour is brought to the real axis; $D_{\rm F}$ corresponds to the values of D along $C_{\rm F}$.

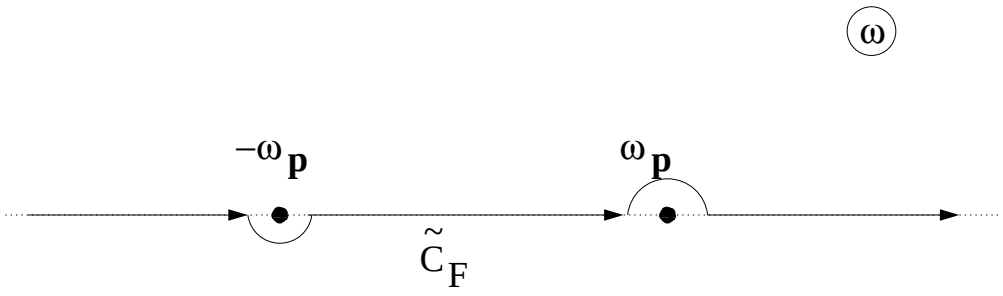
The Feynman propagator can be written as

$$D_{\rm F}(\mathbf{x},t) = \begin{cases} D_-(\mathbf{x},t) = \int d\mu(\mathbf{p})\, e^{-i\omega_{\mathbf{p}}\, t + i\mathbf{p}\,\mathbf{x}} & \text{for} \quad t > 0 \\ D_+(\mathbf{x},t) = \int d\mu(\mathbf{p})\, e^{\,i\,\omega_{\mathbf{p}}\, t - i\mathbf{p}\,\mathbf{x}} & \text{for} \quad t < 0 \end{cases},$$

but the most convenient representation is given by the 4-dimensional integral

$$D_{\rm F}(\mathbf{x},t) = \int_{\tilde{C}_{\rm F}} \frac{d\omega}{2\pi} \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \frac{i\, e^{-i\,\omega\, t + i\,\mathbf{p}\,\mathbf{x}}}{\omega^2 - \mathbf{p}^2 - m^2}\,,$$

The integrand has two poles, at $\omega = \pm\,\omega_{\mathbf{p}}$, and the integration contour $\tilde{C}_{\rm F}$ for ω goes around these poles like this



If $t > 0$ the integrand decays exponentially into the lower half-plane of the variable ω, and one can perform the integration over this variable by closing the contour around $\omega = \omega_{\mathbf{p}}$. The integral reduces to the residue at this pole, which yields $D_-(\mathbf{x}, t)$. Similarly, when $t < 0$ the integration contour can be closed around the pole $\omega = -\omega_{\mathbf{p}}$, and the residue calculation yields $D_+(\mathbf{x}, t)$.

This integration prescription is usually expressed as

$$D_{\rm F}(x) = \int \frac{d\omega}{2\pi} \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{i\, e^{-i\,\omega\, t + i\,\mathbf{p}\,\mathbf{x}}}{\omega^2 - \mathbf{p}^2 - m^2 + i0} = \int \frac{d^4p}{(2\pi)^2} \frac{i\, e^{-i\,p_\mu\, x^\mu}}{p_\mu p^\mu - m^2 + i0}\,.$$

This form of the Feynman propagator is explicitly covariant. It also shows that $D_{\rm F}(x)$ is the Green's function of the KG equation,

$$\left(\partial_\mu \partial^\mu + m^2\right) D_{\rm F}(x) = -i\,\delta^{(4)}(x)\,.$$

Note that the functions $D_\pm(x)$ satisfy homogeneous KG equation

$$\left(\partial_\mu \partial^\mu + m^2\right) D_\pm(x) = 0\,,$$

since in the KG theory the Heisenberg operator $\hat\varphi(x)$ itself satisfies homogeneous KG equation.

Exercise. The Feynman propagator is defined as

$$D_{\rm F}(\mathbf{x}, t) = \theta(t)\,\langle 0 \mid \hat\varphi(\mathbf{x}, t)\,\hat\varphi(\mathbf{0}, 0) \mid 0\rangle + \theta(-t)\,\langle 0 \mid \hat\varphi(\mathbf{0}, 0)\,\hat\varphi(\mathbf{x}, t) \mid 0\rangle\,.$$

Applying the differential operator $\partial_\mu \partial^\mu + m^2$ and using the equal-time commutation relations

$$[\hat\pi(\mathbf{x}, t), \hat\varphi(\mathbf{x}', t)] = -i\,\delta^{(3)}(\mathbf{x} - \mathbf{x}')\,,$$

confirm that

$$\left(\partial_\mu \partial^\mu + m^2\right) D_{\rm F}(x) = -i\,\delta^{(4)}(x)\,.$$

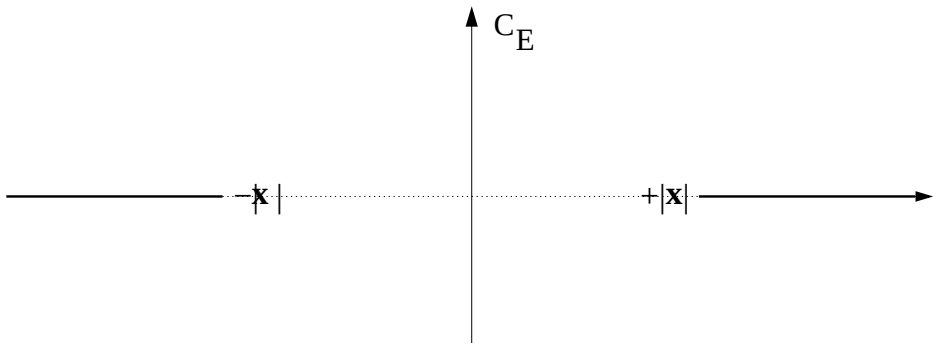
Finally, let us consider the "euclidean", or "imaginary time" correlation function. It is defined in terms of the function $D(\mathbf{x}, t)$ taken at pure imaginary values of the time variable. Namely, let

$$t = -i\, x_4\,,$$

with real x_4. Then

$$D_{\rm E}(\mathbf{x}, x_4) = D(\mathbf{x}, -i\, x_4)\,,$$

i.e. $D_{\rm E}$ is the function D with t taken along the contour $C_{\rm E}$ in the complex t-plane.



Note that this contour can be obtained from $C_{\rm F}$ by 90^o rotation.

As is well known, by taking pure imaginary values of t one converts the Minkowski space-time into 4-dimensional Euclidean space, since

$$-d\tau^2 \to ds^2 = dx_4^2 + d\mathbf{x}^2\,,$$

and the Lorentz symmetry becomes the orthogonal group $O(4)$ of rotations of the the Euclidean 4-vectors

$$x_{\rm E} = (\mathbf{x}, x_4)$$

The function $D_{\rm E}(x_{\rm E})$ is called the Euclidean correlation function of the KG theory. We will discuss its significance later. For now, let us observe that since $D_{\rm E}$ is related to $D_{\rm F}$ simply by rotating the time contour, it satisfies the Euclidean-space version of the inhomogeneous KG equation

$$\big(m^2 - \frac{\partial^2}{\partial x_4^2} - \nabla^2\big)\, D_{\rm E}(\mathbf{x}, x_4) = \delta(x_4)\, \delta^{(3)}(\mathbf{x})\,,$$

or in symmetric form

$$\big(m^2 - \Delta_{\rm E}\big)\, D_{\rm E}(x_{\rm E}) = \delta^{(4)}(x_{\rm E})\,,$$

where $\Delta_{\rm E}$ stands for the 4-dimensional Laplacian. This equation is easily solved by Fourier transformation

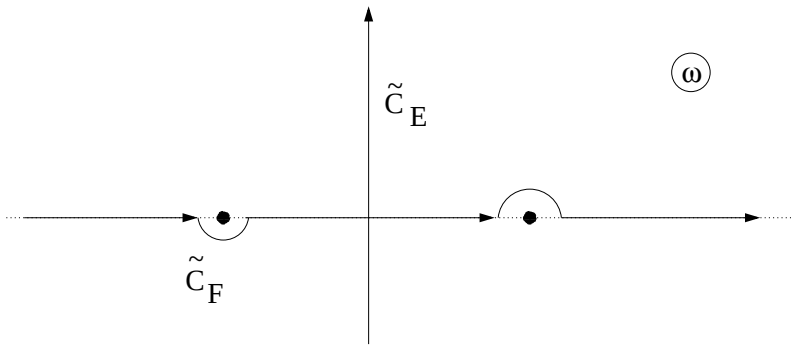
$$D_{\rm E}(x_{\rm E}) = \int \frac{d^4p_{\rm E}}{(2\pi)^4}\,\frac{e^{i\,p_{\rm E}\,x_{\rm E}}}{p_{\rm E}^2+m^2}\,,$$

where the integration is over the 4-dimensional Euclidean momentum space, $p_{\rm E} = (\mathbf{p}, p_4)$, and $p_{\rm E}^2 = \mathbf{p}^2 + p_4^2$. Note that now the integrand does not have singularities in the integration domain. Also, unlike the Minkowski space version of this equation, the solution is unique if we demand that $D_{\rm E}(x_{\rm E})$ decays at the Euclidean infinity $x_{\rm E} \to \infty$. By introducing auxiliary integration, one can write this function as

$$D_{\rm E}(x_{\rm E}) = \int_0^\infty d\tau \int \frac{d^4p_{\rm E}}{(2\pi)^4}\,e^{-\tau\,(p_{\rm E}^2+m^2)}\,e^{i\,p_{\rm E}\,x_{\rm E}}\,, \tag{4.1}$$

This form is known as the Schwinger's proper-time representation.

Clearly, the above Euclidean momentum space integral for $D_{\rm E}(x_{\rm E})$ is related to the covariant 4-momentum integral for the Feynman propagator. The correlation function $D_{\rm E}(\mathbf{x}, x_4)$ is the analytic continuation of $D_{\rm F}(\mathbf{x}, t)$ obtained by rotating the time contour $C_{\rm F}$ to $C_{\rm E}$. Correspondingly, the integration contour $\tilde{C}_{\rm F}$ in the complex ω-plane is rotated by 90^o to the contour $\tilde{C}_{\rm E}$.



This example suggests that some analytic characteristics of the KG theory, and perhaps more general QFT, may take simpler and more symmetric form when continued to pure imaginary time. This idea can be refined with the help of the path integral formulation of quantum mechanics, which we consider next.

5 Lecture 5

Let me briefly review the method of path integral in ordinary quantum mechanics. Here I will combine it with the idea of analytic continuation in the time variable. We will start with the "time evolution" in pure imaginary values of the time parameter. (Besides technical advantage of improving convergence of the integral, the imaginary-time path integral makes evident remarkable relation to classical statistical mechanics.) The real-time transition amplitudes can be recovered by analyticity. For derivation directly in real time see Sect.9.1 of PS.

Path integral in Quantum Mechanics

Given a quantum mechanical system, with the space of states $\mathcal{H}$ and the time-independent Hamiltonian $\hat{H}$, most problems can be reduced to the calculations of appropriate matrix elements of its time-evolution operator,

$$\langle f \mid e^{-\frac{i}{\hbar}\hat{H}\,t} \mid i \rangle \,.$$

In terms of the stationary states $\mid n\rangle$ this can be written as

$$\sum_n \langle f \mid n\rangle \, \langle n \mid i\rangle \, e^{-\frac{i}{\hbar}E_n\,t} \,.$$

In a typical problem, the energy spectrum E_n is bounded from below, $E_n \geq E_0$, and E_n grow to infinity with n. The above expression is a sum of oscillating terms, and we would like to have an efficient mathematical tool to sort them out. It is useful to consider this sum at complex values of t. If the number of states does not grow exponentially with E (and in ordinary quantum system it never does), this sum converges absolutely if t has negative imaginary part; thus it defines an analytic function of t in the lower half-plane. The real-time matrix element is the limiting value of this analytic function. One can start with pure imaginary t,

$$t = -i\,\tau \,,$$

and try to evaluate the matrix elements of the corresponding bounded Hermitian operator

$$\langle f \mid e^{-\frac{1}{\hbar}\hat{H}\,\tau} \mid i \rangle \,,$$

anticipating that the real-time answer will be then obtained by analytic continuation.

Consider a system with one degree of freedom q, and assume that the Hamiltonian operator is "local" in the basis which diagonalizes $\hat{q}$; this means that the matrix elements

$$\langle q \mid \hat{H} \mid q' \rangle$$

have a support restricted to $q = q'$. Most typical Hamiltonian belong to this class, for instance for $H = \frac{1}{2}\, p^2 + V(q)$ the above matrix element is ($\hbar = 1$ from now on)

$$-\frac{1}{2}\, \delta''(q - q') + V(q)\, \delta(q - q')\,.$$

In such cases it is possible to develop a path-integral representation for the matrix element

$$\langle q_f \mid e^{-\hat{H}\,\tau} \mid q_i \rangle\,.$$

The composition property

$$\langle q_f \mid e^{-\hat{H}\,(\tau_1+\tau_2)} \mid q_i \rangle = \int \langle q_f \mid e^{-\hat{H}\,\tau_1} \mid q \rangle\, \langle q \mid e^{-\hat{H}\,\tau_2} \mid q_i \rangle\, dq$$

can be applied repeatedly to reduce the above matrix element to the composition of the matrix elements with arbitrarily small "time" intervals $\Delta\tau$,

$$\langle q_f \mid e^{-\hat{H}\,\tau} \mid q_i \rangle = \int \prod_{k=1}^{n-1} dq_k \prod_{k=1}^{n} \langle q_k \mid e^{-\hat{H}\,\Delta\tau} \mid q_{k-1} \rangle\,,$$

where $q_0 = q_i$, $q_n = q_f$, and $\Delta\tau = \tau/n$. In the limit of small $\Delta\tau$, as the consequence of the "locality" of $\hat{H}$ in the q-space, only the matrix elements $\langle q' \mid \exp(-\hat{H}\Delta\tau) \mid q \rangle$ with q' close to q will bring significant contributions to the integral.

To see this, consider again the Hamiltonian of the form

$$\hat{H} = \frac{1}{2}\, \hat{p}^2 + V(\hat{q})\,.$$

It is straightforward to show that in this case the matrix element

$$G(q, q'|\tau) = \langle q \mid e^{-\hat{H}\tau} \mid q' \rangle$$

satisfy the differential equation of the diffusion type

$$-\frac{\partial}{\partial\tau}\, G(q, q'|\tau) = \left(-\frac{1}{2}\, \frac{\partial^2}{\partial q^2} + V(q) \right) G(q, q'|\tau)\,,$$

with the initial condition

$$G(q,q'|0) = \delta(q-q')\,.$$

For $V(q) = 0$ the problem is easily solved,

$$G(q,q'|\tau) = \frac{1}{\sqrt{2\pi\,\tau}}\, e^{-\frac{(q-q')^2}{2\,\tau}}\,.$$

This function indeed decays fast at $|q-q'| >> \sqrt{\tau}$. In the case of nonzero potential V the problem still can be examined in the limit of small τ. We write

$$G(q,q'|\tau) = e^{-\sigma(q,q'|\tau)}$$

and look for σ of the form of the following small-τ expansion

$$\sigma(q,q'|\tau) = \frac{\sigma_0(q,q')}{\tau} + a\,\log\tau + b + \tau\,\sigma_1(q,q') + \dots\,.$$

Substituting this ansatz into the above diffusion equation one finds (I suggest doing it as an **Exercise**)

$$\sigma_0(q,q') = \frac{(q-q')^2}{2}\,, \qquad a = 1/2\,,$$

$$\sigma_1(q,q') = \frac{1}{q-q'}\int_{q'}^{q} V(y)\,dy\,, \qquad \text{etc}\,.$$

The leading terms do not depend on V, and are the same as for the free case, and σ_1 is the potential V averaged over the interval $[q',q]$. The constant $b = \frac{1}{2}\log 2\pi$ is determined by the initial condition at $\tau = 0$.

Because of the factor $\exp(-\sigma_0(q,q')/\tau)$ the function $G(q,q'|\tau)$ at small τ is sharply peaked around $|q-q'| = 0$ with the width $\sqrt{\tau}$. This is typical for the *Brownian motion*, where $\bar{R} \sim \sqrt{t}$ (our equation *is* the diffusion equation). If the potential V is differentiable (which we assume), we have

$$\sigma_1(q,q') = \frac{1}{2}\,V(q) + \frac{1}{2}\,V(q') + O(\tau)$$

(I have used the fact that $(q-q')^2 \sim \tau$). Therefore, for finite τ but large n (i.e. small $\Delta\tau = \tau/n$) we can write

$$\langle q_f \mid e^{-\hat{H}\,\tau} \mid q_i\rangle = \lim_{n\to\infty}\left(\frac{1}{2\pi\,\Delta\tau}\right)^{n/2}\int\left(\prod_{k=1}^{n-1} dq_k\right) e^{-\mathcal{A}\{q_k\}}\,,$$

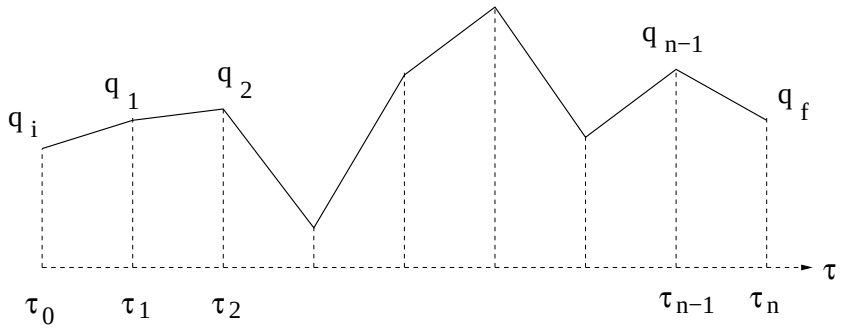
where

$$\mathcal{A}\big\{q_k\big\} = \sum_{k=1}^{n} \left[\frac{(q_k - q_{k-1})^2}{2\,\Delta\tau} + \Delta\tau\, \frac{V(q_k) + V(q_{k-1})}{2} \right],$$

and $\Delta\tau = \tau/n$, and it is understood that

$$q_0 = q_i\,, \quad \text{and} \quad q_n = q_f\,.$$

Before the limit $n \to \infty$ is taken, the integration in this formula can be understood as going over the piecewise linear "trajectories" $q(\tau)$ running from q_i to q_f.



When $n \to \infty$, $\Delta\tau \to 0$, the function $\mathcal{A}\big\{q_k\big\}$ in the exponential has formal limit

$$\mathcal{A}[q(\tau)] = \int_0^\tau d\tau \left[\frac{1}{2} \big(dq(\tau)/d\tau\big)^2 + V\big(q(\tau)\big) \right].$$

It coincides with the "imaginary time action", i.e. the action associated with our system, with t replaced by $-i\,\tau$,

$$e^{\frac{i}{\hbar}\,S[q(t)]} \;:\quad -\quad (t \to -i\,\tau) \quad\to\quad e^{-\mathcal{A}[q(\tau)]}\,.$$

It is conventional to write the $n \to \infty$ limit of the above n-fold integral as

$$\langle q_f \mid e^{-\hat{H}\,(\tau_f - \tau_i)} \mid q_i \rangle = \int \big[Dq(\tau)\big]_{\substack{q(\tau_i)=q_i \\ q(\tau_f)=q_f}} e^{-\mathcal{A}[q(\tau)]}\,,$$

and to call it the (imaginary time) path integral, since the r.h.s involves the continuous path $q(\tau)$ from q_i to q_f.

Few remarks.

1. The path integral method makes the most explicit use of the *superposition principle* of quantum theory. The latter states that if some process can go several distinct ways, the full transition amplitude is the sum of the amplitudes associated with each possible way. Here (when returning back to real time) we sum the amplitudes associated with all distinct paths from q_i to q_f.

2. What kind of paths enter the path integral in the limit $n \to \infty$? Although it is correct to think of the limiting paths as continuous functions $q(\tau)$, absolute majority of the paths entering the integral are not smooth curves. This can be observed directly from the fact that for $\tau_1 - \tau_2 \to 0$ we have $|q(\tau_1) - q(\tau_2)| \sim \sqrt{|\tau_1 - \tau_2|}$, not $\sim |\tau_1 - \tau_2|$ as should be true for a continuously differentiable function. From this point of view the two terms in the action play significantly different role. The "kinetic" term $1/2 \int (dq/d\tau)^2 \, d\tau$ selects the class of the paths entering the integral, i.e. those paths for which $(q(\tau + \Delta\tau) - q(\tau))^2/\Delta\tau$ remains finite as $\Delta\tau \to 0$; in this sense the factor $\exp\big(-1/2 \int (dq/d\tau)^2 \, d\tau\big)$ should be considered as a part of the functional measure. The factor $\exp\big(-\int V(q(\tau))\, d\tau\big)$ weights the paths according to their average potential energy.

3. Note that the symbol

$$[Dq(\tau)] = \lim_{n\to\infty} \left(\frac{1}{2\pi\,\Delta\tau}\right)^{n/2} \prod_{k=1}^{n-1} dq_k$$

contains divergent (in the limit) factor

$$\left(\frac{1}{2\pi\,\Delta\tau}\right)^{n/2} = e^{-\frac{\log(2\pi\Delta\tau)}{\Delta\tau}\,L}\,,$$

where $L = (\tau_f - \tau_i) = n\,\Delta\tau$ is the "volume" of the imaginary time interval. This factor is reminiscent of the irritating infinity we had for the vacuum energy in the KG theory, and absorbing it into $[Dq(\tau)]$ is analogous to subtracting E_0 from $\hat{H}$.

4. We have derived the above path integral representation of the evolution operator in the imaginary time τ. But one can repeat the above calculation step by step, considering the paths in the real time t instead of τ (this is the form in which the path integral is usually introduced in quantum mechanics, see for instance §9.1 of the textbook). In this way one arrives at the expression

$$\langle\, q_f \;\mid\; e^{-\frac{i}{\hbar}\,H\,T} \;\mid\; q_i \,\rangle = \int [Dq(t)]\, e^{\frac{i}{\hbar}\,S[q(t)]}\,,$$

with the phase $e^{\frac{i}{\hbar}S}$ replacing $e^{-\mathcal{A}}$., This of course is equivalent approach, although, unlike the imaginary time scheme, the integrals involved are not absolutely convergent. Usually, the most efficient way to handle such integrals is through analytic continuation (in variables or parameters) which makes the integral absolutely convergent, with subsequent taking the limiting values. The imaginary-time approach simply makes this procedure explicit.

5. Generalization of the above construction to the systems with many degrees of freedom q_a is straightforward, if the Hamiltonian is of the form

$$H = \sum_a \frac{1}{2}\, p_a^2 + V(q_a)\,;$$

one obtains

$$\int [Dq_a]\, e^{-\mathcal{A}[q_a(\tau)]}$$

for the evolution operator. In more complicated cases the path integral in the phase space can be useful; this is briefly discussed in Sect.9.1 of PS.

Relation to Classical Statistical Mechanics

At this point, let me deviate to discuss remarkable relation which exists between the *Path Integral in Quantum Mechanics* (in its imaginary time version) and the *Classical Statistical Mechanics*. Let us recall some generalities of the classical statistical mechanics.

Suppose we have a classical dynamical system with the phase-space coordinates $\{P_i, Q_i\}$ (I will use capitals here because the relation between these Q_i and q_α appearing in the path integral is not going to be straightforward), and the Hamiltonian $\mathcal{H}(P_i, Q_i)$. In a thermal equilibrium state at a temperature T the probability distribution of microscopic states is given by the Gibbs formula

$$\mathcal{P}(P_i, Q_i) \prod_i dP_i dQ_i = Z^{-1}\; e^{-\beta\, \mathcal{H}(P_i,Q_i)}\; \frac{1}{N!} \prod_i dP_i dQ_i\;,$$

where $\beta = 1/kT$, and Z is the partition function

$$Z = \int \frac{\prod_i\, dP_i dQ_i}{N!}\, e^{-\beta\, \mathcal{H}(P_i,Q_i)}\;.$$

Calculating the partition function is one of the main problems in statistical mechanics because the thermodynamic quantities are expressed through Z, say the free energy is $\beta F = -\log Z$. Again, a typical form of the Hamiltonian is

$$\mathcal{H}(P_i, Q_i) = \sum_i \frac{1}{2} P_i^2 + U(\{Q_i\}) \ .$$

In this case the integration over P_i is easy to perform,

$$Z = \frac{1}{N!} \left(\frac{2\pi}{\beta} \right)^{\frac{N}{2}} Z_{\text{conf}} \ ,$$

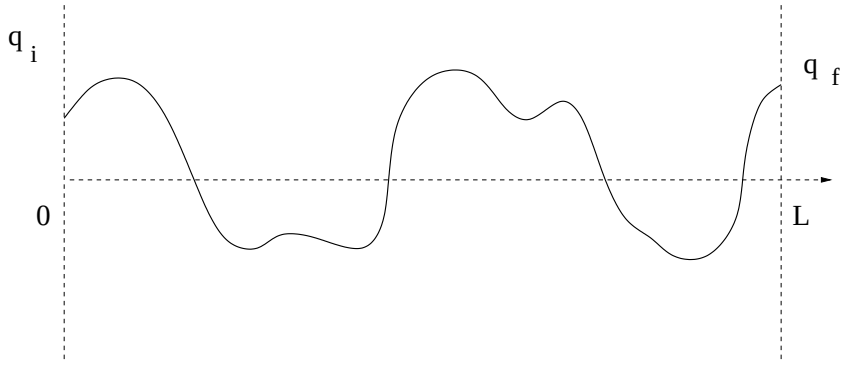
where N is the number of the degrees of freedom, $i = 1, 2, \cdots, N$, and Z_{conf} is the configuration-space integral

$$Z_{\text{conf}} = \int \prod_i dQ_i \ e^{-\beta U(\{Q_i\})}$$

which contains most of interesting physics.

As a special example of such system, consider an elastic string whose configuration is described by a function $q(\tau)$, $\tau \in [0, L]$; the values $q(\tau)$ for all τ within this interval play the role of the configuration-space coordinates,

$$\{Q_i\} \to q(\tau) \ .$$



We also assume that the string lays in a potential well $V(q)$, so that the potential energy of the string is

$$U[q(\tau)] = \int_0^L d\tau \left[V\big(q(\tau)\big) + \frac{1}{2} \left(\frac{dq(\tau)}{d\tau} \right)^2 \right],$$

where the last term accounts for the elastic energy; to simplify the equations I have assumed that the elastic string tension equals 1.

To evaluate the configuration part of the partition function, we must integrate over the functions $q(\tau)$ satisfying given boundary conditions at the ends of the interval $[0, L]$, say

$$q(0) = q_i\,, \quad q(L) = q_f\,,$$

with the statistical weights $\exp(-\beta\, U[q(\tau)])$. Thus

$$Z_{\rm conf} = \int [Dq(\tau)]\, e^{-\beta\, U[q(\tau)]}\,.$$

With the above form of $U[q(\tau)] = \mathcal{A}[q(\tau)]$, this expression is identical to the path integral for the imaginary time transition amplitude,

$$Z_{\rm conf} \sim \langle q_f \mid e^{-\frac{1}{\hbar}\hat{H}\,L} \mid q_i \rangle\,,$$

provided we also identify

$$\beta = 1/\hbar\,.$$

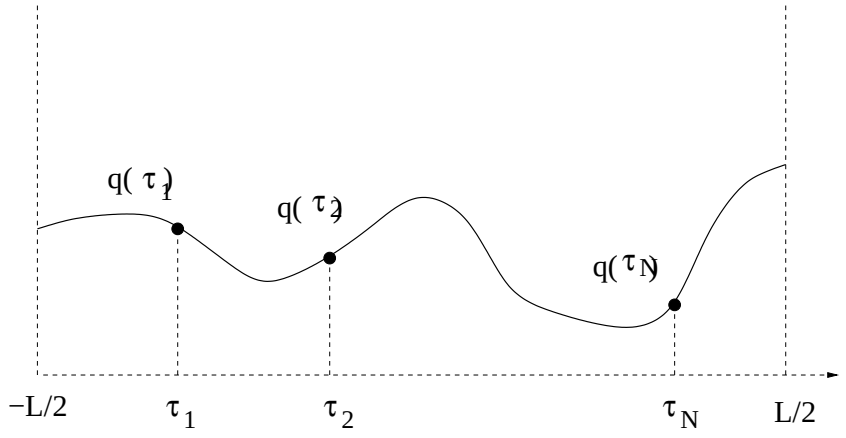
We see that quantum mechanics with a single degree of freedom q, and the Hamiltonian operator

$$\hat{H} = \frac{1}{2}\,\hat{p}^2 + V(\hat{q})\,,$$

is related in this way to classical statistical theory of a system with continuously many degrees of freedom $q(\tau)$.

Correlation functions

In statistical mechanics one may also be interested in *correlation functions*. In our example one may ask how the string positions $q(\tau)$ at different points, say at τ_1, τ_2, ..., τ_N, are correlated in the thermal equilibrium state. In the drawing below I made a trivial shift of the variable τ, so that from now on $\tau \in [-L/2, L/2]$.



The correlations are described by the averaged products $q(\tau_1)q(\tau_2)...q(\tau_N)$, which in turn are given by the integrals

$$\langle\, q(\tau_1)\, q(\tau_2)\, \ldots\, q(\tau_N)\,\rangle = Z^{-1} \int q(\tau_1)q(\tau_2)\ldots q(\tau_N)\, e^{-\mathcal{A}[q(\tau)]}\, [Dq(\tau)]\,,$$

because $Z^{-1}\, e^{-\mathcal{A}[q(\tau)]}\, [Dq(\tau)]$ is the statistical weight of the microstate $q(\tau)$. It is instructive to find out how these correlation functions are interpreted in associated quantum mechanical problem.

Without loss of generality we can assume that

$$-L/2 \le \tau_1 \le \tau_2 \le \ldots \le \tau_N \le L/2\,.$$

The integration over all microstates $q(\tau)$ can be performed in two steps: First, we fix the values of $q(\tau)$ at the points $\tau_1, \tau_2, ..., \tau_N$ to be

$$q(\tau_1) = q_1\,, \quad q(\tau_2) = q_2\,, \quad \ldots \quad q(\tau_N) = q_N\,,$$

and integrate over all $q(\tau)$ subject to these constraints. After that, the integration over $q_1, q_1, ..., q_N$ is performed to complete the integral. Comparing this with the path integral representation for the evolution operator, say

$$\langle\, q_1 \mid e^{-\hat{H}\,(\tau_2-\tau_1)} \mid q_2\,\rangle = \int [Dq(\tau)]_{\substack{q(\tau_1)=q_1 \\ q(\tau_2)=q_2}}\, e^{-\mathcal{A}[q(\tau)]}$$

we find

$$\langle\, q(\tau_1)q(\tau_2)...q(\tau_N)\,\rangle = Z^{-1} \int \left[\prod_{k=1}^{N} dq_k\right] \times q_1 q_2 ... q_N \times$$

$$\langle q_f \mid e^{-\hat{H}(L/2-\tau_N)} \mid q_N\rangle\langle q_N \mid e^{-\hat{H}(\tau_N-\tau_{N-1})} \mid q_{N-1}\rangle ... \langle q_1 \mid e^{-\hat{H}(\tau_1+L/2)} \mid q_i\rangle$$

Recall that in the coordinate representation the operator $\hat{q}$ acts by multiplication

$$\hat{q}\Psi(q) = q\Psi(q)\,,$$

so that in this expression we can replace

$$\int dq \mid q\rangle\, q\, \langle q \mid = \hat{q}\,.$$

Then the above expression can be written as

$$\langle q_f \mid e^{-\hat{H}(L/2-\tau_N)}\, \hat{q}\, e^{-\hat{H}(\tau_N-\tau_{N-1})}\, \hat{q}\, ...\, \hat{q}\, e^{-\hat{H}(\tau_1+L/2)} \mid q_i\rangle$$

which is best expressed in terms of the (imaginary time version of) Heisenberg operators

$$\hat{q}_E(\tau) = e^{\hat{H}\tau}\, \hat{q}\, e^{-\hat{H}\tau}$$

($\tau = it$), allowing one to write the above formula as

$$\langle\, q_f \mid\, e^{-\hat{H}\frac{L}{2}}\, \hat{q}_E(\tau_N)\, \hat{q}_E(\tau_{N-1})\, ...\, \hat{q}_E(\tau_1)\, e^{-\hat{H}\frac{L}{2}} \mid\, q_i\,\rangle$$

To simplify things further, let's take the limit $L \to \infty$, thus turning to the correlation functions of the infinite string. In this limit the exact boundary conditions, e.g. the values q_i and q_f, become unimportant. Indeed, the states $\mid q_i\rangle$ and $\mid q_f\rangle$ can be expanded in the stationary states $\mid n\rangle$ of the Hamiltonian $\hat{H}$, say

$$\mid q_i\rangle = \sum_n \mid n\rangle\, \langle n \mid q_i\rangle = \sum_n \mid n\rangle\, \Psi_n^*(q_i)\ ,$$

where it is assumed that

$$\langle n \mid m\rangle = \delta_{n,m}\,.$$

In the limit $L \to \infty$ the contribution of the ground state $\mid 0\rangle$ dominates, since all other states are suppressed by the factor $\sim e^{-(E_n-E_0)\frac{L}{2}}$. Thus we have

$$\langle\, q(\tau_1)q(\tau_2)...q(\tau_N)\,\rangle = \frac{\langle q_f \mid 0\rangle\langle 0 \mid q_i\rangle}{Z}\, e^{-E_0 L}\ \langle 0 \mid \hat{q}_E(\tau_N)\, \hat{q}_E(\tau_{N-1})\, ...\, \hat{q}_E(\tau_1) \mid 0\rangle$$

Furthermore, note that

$$Z = \langle q_f \mid e^{-\hat{H}\,L} \mid q_i\rangle$$

hence the first factor in the above equation equals 1 (in the limit $L = \infty$), and we have

$$\langle\, q(\tau_1)...q(\tau_N)\,\rangle = \langle 0 \mid \hat{q}_E(\tau_N)\,\hat{q}_E(\tau_{N-1})...\hat{q}_E(\tau_1) \mid 0\rangle\,.$$

Note that the operators here are arranged in the order of τ increasing from the left to the right. Let me stress that the Euclidean Heisenberg operators can not be arranged any other way, as otherwise the product of the operators (the sum over the intermediate states) does not converge. More generally, if we consider complex values of τ_i (which we will), the above product of the operators makes direct sense in the domain

$$\Re e\,\tau_1 \leq \Re e\,\tau_2 \leq \ ...\ \leq \Re e\,\tau_N$$

where the sums over the intermediate states converge (hence the sums yield analytic function).

To relate the above Euclidean correlation function to the real-time ones one analytically continues to pure imaginary values of τ,

$$\tau_k = it_k + 0\,,$$

where "+0" shift indicates that if there is a branch cut along the imaginary axis the values at the right edge are to be taken.

6 Lecture 6

Previously, we have discussed the path integral representation in ordinary quantum mechanics. The real-time transition amplitude is represented as a suitably defined integral

$$\langle\, q_f \;\mid\; e^{-\frac{i}{\hbar}\,H\,T} \;\mid\; q_i \,\rangle = \int [Dq(t)]_{\substack{q(0)=q_i \\ q(T)=q_f}}\; e^{\frac{i}{\hbar}\,S[q(t)]}\,,$$

over all "paths" $q(t)$ from q_i to q_f. Now I want to discuss how the idea of the path integral can be applied to a relativistic particle.

Path integral for a relativistic particle

We have seen that the KG theory is the theory of spinless Bose particles, and we expect that the Feynman propagator $D_{\rm F}(x_i - x_f)$ (and related Euclidean correlation function $D_{\rm E}$) admit an interpretation in terms of amplitude associated with the particle propagation from x_i to x_f. Can we write such amplitude as an integral over the "paths"?

In classical mechanics the action of a relativistic particle travelling from x_i to x_f is

$$S = -m \int_{x_i}^{x_f} \sqrt{dx_\mu\, dx^\mu} = -m \int_{(t_i,\mathbf{x}_i)}^{(t_f,\mathbf{x}_f)} \sqrt{dt^2 - d\mathbf{x}^2}\,,$$

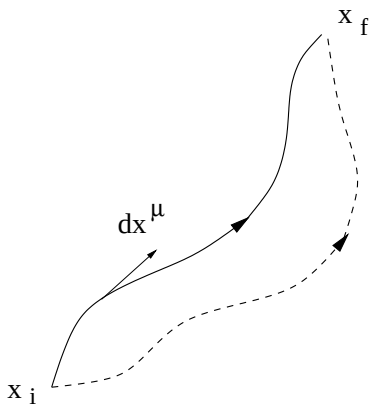
where the integration is performed along a given path (generally described by a parametric curve $x^\mu(s)$) from x_i to x_f. In classical theory it is assumed that $(dx^0)^2 \geq d\mathbf{x}^2$ along the whole path (the particle velocity never exceeds the speed of light) the time component $t = x^0$ of x^μ can serve as the parameter of the path, and we can write

$$S[\mathbf{x}(t)] = -m \int_{t_i}^{t_f} dt\, \sqrt{1 - \dot{\mathbf{x}}^2}\,.$$

In quantum theory we may attempt to write for the transition amplitude

$$\int_{\text{paths}(i\to f)} e^{\frac{i}{\hbar}\,S}\,[D\mathbf{x}(t)]\,,$$

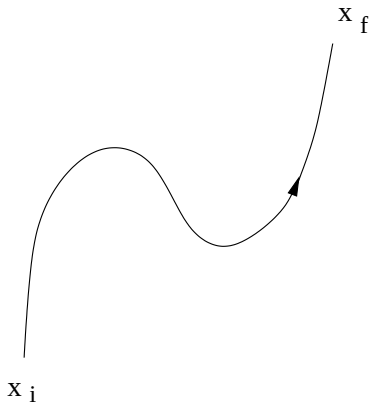
where the integration is over "all paths" from x_i to x_f.



The problems appear when we want to give a precise meaning to this expression. On the formal side, we observe that the Hamiltonian associated with the problem,

$$H = \sqrt{m^2 + \mathbf{p}^2}$$

(where $\mathbf{p}$ is the momentum conjugated to $\mathbf{x}$), is not "local" in the sense we discussed the last time; the matrix elements $\langle \mathbf{x} \mid H \mid \mathbf{x}' \rangle$ do not vanish when $\mathbf{x} \neq \mathbf{x}'$ (check it against your solution of the Problem 1'). More fundamentally, the problem is about the class of the paths which we need to include into the path integral. The expression $dx_\mu \, dx^\mu$ in the square root in S can be either positive or negative, depending on whether dx^μ is time-like or space-like. One can try to exclude from the integration the paths having space-like dx^μ anywhere, but it does not seem to be right idea if we want to describe the propagator D_{F} this way. Indeed, unlike the commutator, the function $D_{\mathrm{F}}(\mathbf{x}, t)$ does not vanish outside the light cone (at $|t| < |\mathbf{x}|$). It is exponentially small if you go far outside the light cone, but it is not zero. This is similar to a nonrelativistic particle in the classically unaccessible region of its configuration space - the wave function is exponentially small but does not vanish ("quantum tunneling"). Thus we have to admit the paths having space-like dx^μ somewhere along them. Ones such paths are admitted, the Lorentz invariance requires that paths in which dx^0 is allowed to take negative values (i.e. the paths with the folds going "backward in time") are admitted as well:



For such paths $\mathbf{x}(t)$ is not a function, and the integral over $[D\mathbf{x}(t)]$ does not apply. Besides, whereas for the case of space-like dx^μ there is a natural choice of the branch of the square root $\sqrt{dx_\mu dx^\mu}$ in the action (e^{iS} must receive exponentially small contributions from the space-like parts of the path), we have no clear intuition which sign to choose in the case of the time-like "backward in time" paths.

All these ambiguities are resolved by adopting the Euclidean (= imaginary time) formulation of the path integral. Replacing as before

$$t = -i\, x_4\,,$$

we have $iS \to -\mathcal{A}$, where the Euclidean action is

$$\mathcal{A} = m \int \sqrt{(dx_{\rm E})^2}\,,$$

with explicitly positive expression

$$dx_{\rm E}^2 = dx_4^2 + d\mathbf{x}^2$$

under the square root. In the subsequent discussion I will often omit the subscript E for the Euclidean 4-vectors. With this, one can define the Euclidean amplitude as the integral

$$\int_{{\rm paths}(x_i \to x_f)} e^{-\mathcal{A}[{\rm path}]}$$

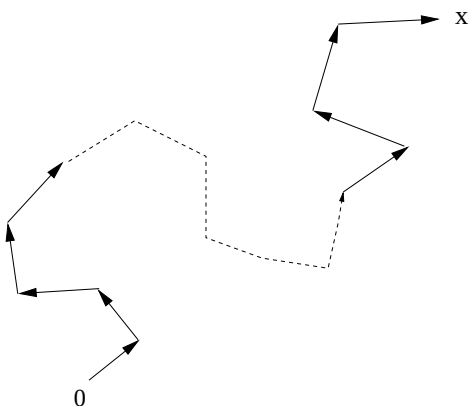
over all paths connecting two Euclidean points x_i and x_f; one expects that the real-time propagator will be accessible through analytic continuation, as we previously discussed.

Note that

$$\mathcal{A}[\text{path}] = m_0\, L[\text{path}]\,,$$

where $L[\text{path}]$ is just the Euclidean length of the path. I have replaced m by the "bare mass" m_0; we will see that this parameter does not necessarily coincide with the physical mass but requires "renormalization".

By analogy with our definition of the path integral in ordinary quantum mechanics, let us define this path integral as a limit of finite-dimensional integrals. Namely, for given (large) number n, let us consider the piecewise linear paths from $x_i = 0$ to $x_f = x$, each linear piece having fixed length Δ.



The overall length of such path is

$$L_n = n\,\Delta\,.$$

Let ν_k be the unit 4-vector in the direction of the k-th piece; $|\nu_k| = 1$. In this discrete approximation the path integral can be written as

$$\sum_{n=0}^{\infty} e^{-m_0\,\Delta\,n} \int \left(\prod_{k=1}^{n} d\mu(\nu_k)\right) \delta^{(4)}\big(\Delta \sum_{k=1}^{n} \nu_k - x\big)\,.$$

Here $d\mu(\nu_k)$ stands for the usual measure on S^3, and the delta-function ensures that the vectors $\Delta\,\nu_k$ add up to x, i.e. the path indeed is from 0 to $x = x_f - x_i$. Finally, the sum over n must be performed since we want to include all paths, with arbitrary length L.

The delta-function can be represented as

$$\delta^{(4)}\big(\Delta \sum \nu_k - x\big) = \int \frac{d^4P}{(2\pi)^4}\, e^{i\,P\,(x - \Delta \sum \nu_k)}\,.$$

For fixed P the integration over $\prod\, d\mu(\nu_k)$ splits into n identical integrals $I(P\Delta) = \int\, d\mu(\nu)\, e^{-i\,\Delta\,\nu\,P}$,

$$\prod_{k=1}^{n} \int\, d\mu(\nu_k)\, e^{-i\,\Delta\,\nu_k\,P} = \big[I(P\,\Delta)\big]^n\,.$$

Certainly, the integral I can be evaluated explicitly in terms of the Bessel's function. However, there is no need in its explicit form. We are going to send $\Delta \to 0$ (in which limit a significant contribution comes from large n). Therefore only the behavior of I at small values of its argument is important. We find

$$I(P\,\Delta) = A - A\xi\,P^2\,\Delta^2 + O(\Delta^4)\,,$$

where exact value of the constants $A = \int_{S^3}\, d\mu(\nu) = 4\pi^2$ and $\xi = 1/8$ are not important. We have

$$I^n \sim A^n\, e^{-\Delta^2\,n\,\xi\,P^2}\,.$$

Now the integration over P can be performed, and we obtain for given n

$$\big(4\pi\,n\Delta^2\xi\big)^{-2}\; e^{-\frac{x^2}{4\xi\,n\,\Delta^2}}\,.$$

Note that Δ enters here in the combination $n\,\Delta^2$, not $n\,\Delta$ as one could naively expect. Attempting to take the naive limit

$$\Delta \to 0\,, \quad n \to \infty \qquad \text{with} \quad L = n\,\Delta = \text{finite}$$

would lead to the propagation amplitude which vanishes at any finite $|x|$. This result is not surprising. Our calculation was nearly identical to the well known probability calculation for a Brownian particle. We know that a Brownian particle with typical microscopic velocity $\bar{v}$ will be found, after the elapsed time T, at the distance

$$\bar{x} = \bar{v}\,\sqrt{T\,\Delta t}$$

away from its initial position. Here Δt is the typical time between collisions. In this analogy

$$\Delta \sim \bar{v}\,\Delta t\,, \qquad n \sim T/\Delta t\,.$$

Thus we expect to have

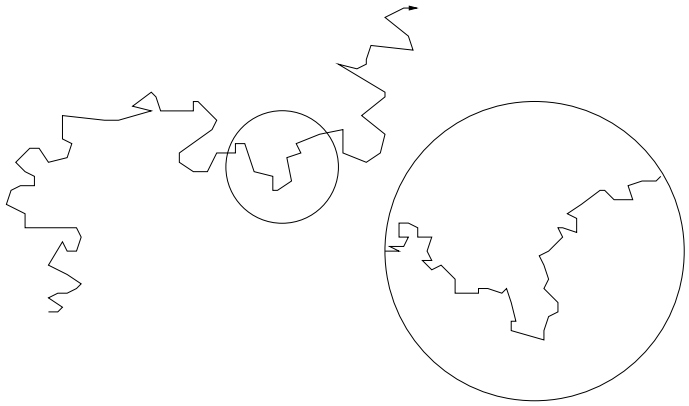
$$\bar{x}^2 = n\,\Delta^2\,,$$

in agreement with our calculation.

We conclude that the right continuous limit is achieved by taking

$$\Delta \to 0\,, \quad n \to \infty \qquad \text{with} \quad s = \xi\,n\,\Delta^2 = \text{finite}\,,$$

and it is the parameter s that characterizes the geometry of the typical path; the microscopic length $L = n\,\Delta$ becomes infinitely large in this limit. The absolute majority of the paths contributing to the functional integral are extremely crumpled at all scales, as the drawing illustrates:



The parameter s has the dimension of the [length]^2; it is said that the typical path has the fractal dimension 2.

Finally, there are two unpleasant factors to be dealt with before we can take the limit $\Delta \to 0$. These are

$$A^n\, e^{-m_0\, n\, \Delta} = e^{-n\,(m_0\,\Delta - \log A)}\,.$$

Consider m_0 as a parameter which can carry some dependence on Δ, and is free to be adjusted (in the limit $\Delta \to 0$) so that

$$\Delta\, m_0(\Delta) - \log A \;\to\; \xi\, m^2\, \Delta^2 \qquad \text{as} \qquad \Delta \to 0\,,$$

where it is m^2, not m_0, which remains constant in this limit. Then the above two factors combine into

$$e^{-m^2\,\xi\, n\, \Delta^2} \to e^{-m^2\, s}\,.$$

For large n the sum is replaced by the integral over s, and one obtains

$$\int_0^\infty ds\, e^{-m^2\, s} \int \frac{d^4P}{(2\pi)^4}\, e^{-s\,P^2}\, e^{iPx}\,,$$

which is exactly the Schwinger's proper-time representation of the Euclidean correlation function $D_{\rm E}(x)$ of the KG theory. Thus this correlation function admits interpretation as the integral over the paths in Euclidean space-time. As we know,

the Feynman propagator $D_{\rm F}$ coincides with the analytic continuation of this correlation function back to the real time, $x_4 = it$.

Remark: In our calculation above we have assumed that the linear pieces of the discretized paths have equal lengths Δ. This is not essential, and is done for simplicity only. One can change details of the finite-dimensional approximations, but in the limit the same correlation function $D_{\rm E}$ comes out (though the relation between the "bare" and "renormalized" mass parameters m_0 and m does depend on the details of the approximation). This is an elementary example of *universality*.

In principle, it is possible to develop a theory of interacting relativistic particles based on the idea of integrating over all the particles paths, including the paths of many particles, with interaction events - "vertices". But much more universal and powerful approach is based on quantization of the field variables. In the associated path integral, one integrates not over the paths of the particles, but over the paths of the field variables, i.e. over all the "trajectories" $\Phi(\mathbf{x}, \mathbf{t})$ of the fields. Let us elaborate this idea in the KG theory.

Path integral in Klein-Gordon Theory

Let us now apply the idea of the (Euclidean) path integral to the KG field theory. In this case the role of the generalized coordinates q is taken by the field degrees of freedom $\varphi(\mathbf{x})$, thus we have to replace

$$q \;\to\; \varphi(\mathbf{x}) \;, \qquad q(\tau) \;\to\; \varphi(\mathbf{x}, \tau) = \varphi(x_{\rm E}) \;,$$

where τ is the "imaginary time" which I will denote x_4 below, and $x_{\rm E} = (\mathbf{x}, x_4)$. The euclidean action is obtained by substituting $t = -i\, x_4$ in the KG action, i.e.

$$\mathcal{A} = \int dx_4\, d\mathbf{x}\, \frac{1}{2} \left[(\partial_4 \varphi)^2 + (\nabla \varphi)^2 + m^2\, \varphi^2 \right] =$$

$$\int d^4 x_E \left[\frac{1}{2}\, (\partial_\mu \varphi)^2 + \frac{m^2}{2}\, \varphi^2 \right] ,$$

where $\mu = 1, 2, 3, 4$. We are interested in the path integral

$$Z = \int [D\varphi(x)]\; e^{-\mathcal{A}[\varphi(x)]} \,.$$

More generally, we may be interested in the correlation functions defined as the ratios

$$\langle\varphi(x_1)\varphi(x_2)\,\ldots\,\varphi(x_N)\rangle = \frac{1}{Z}\int [D\varphi]\,\varphi(x_1)\varphi(x_2)\,\ldots\,\varphi(x_N)\,e^{-\mathcal{A}[\varphi]}\,. \qquad (6.1)$$

Here and below I drop the subscript E and use the notation x_i for the *Euclidean* vectors, unless stated otherwise. As we discussed before, when continued to complex values of $(x_i)_4$ and specialized to $(x_i)_4 = it_i$, these correlation functions yield the time-ordered expectation values

$$\langle 0 \mid T(\hat\varphi(\mathbf{x}_1,t_1)\,\ldots\,\hat\varphi(\mathbf{x}_N,t_N)) \mid 0\rangle\,,$$

where $\hat\varphi(\mathbf{x},t)$ denotes the real-time Heisenberg field operator.

Few words about the definition of the path integral in the case of fields. For the path integral in quantum mechanics we had a definition in terms of a limit $\Delta\tau\to 0$ of a finite-dimensional integral with discretized "time" τ. There are many ways to build finite-dimensional approximations in quantum field theory. For example, the 4-dimensional space can be replaced by, say, the hypercubic lattice, in which x^μ take discrete values

$$x^\mu = n^\mu\,\Delta\,, \qquad n^\mu\in\mathbf{Z}\,.$$

The derivatives and the integration in the action can be replaced by finite differences and summations, i.e.

$$\partial_\mu\varphi(x)\to\frac{1}{\Delta}\,(\varphi(x+e^\mu\,\Delta)-\varphi(x))\,; \qquad \int d^4x\to\Delta^4\sum_{e^\mu}$$

(e^μ is the unit vector in the direction μ); with this one can set

$$[\mathcal{D}\varphi] = \lim_{\Delta\to 0}\prod_x d\varphi(x)\,.$$

In the Klein-Gordon theory such definition works more or less straightforwardly. In more interesting cases defining the integral over $[\mathcal{D}\varphi]$ can become highly non-trivial problem, which is closely related to the problem of *critical behavior*; many subsequent lectures will be devoted to this subject. Let me also note here that in the context of field theory the integrals like those above are usually referred to as *functional integrals* (the term "path integral" is also used).

The action $\mathcal{A}$ of the KG is quadratic in φ. This makes it possible to evaluate the above path integrals explicitly. I am not going to explain how these integrals are

actually evaluated; this calculation is presented in § 9.2 in the textbook. Instead I will derive the above correlation functions using formal properties of the functional integral. The resulting formula is known as the

Wick's Theorem:

- The correlation function of any odd number of fields vanishes,

$$\langle \varphi(x_1) \ldots \varphi(x_{2N+1}) \rangle = 0 \,.$$

This follows from the fact that the action $\mathcal{A}$ is an even functional of φ, while in this case the product of an odd factors φ in the integrand is odd.

- The two-point correlation function is

$$\langle \varphi(x_1)\varphi(x_2) \rangle = D_{\rm E}(x_1 - x_2) \,,$$

where $D_{\rm E}(x)$ is the familiar function

$$D_{\rm E}(x) = \int \frac{d^4p}{(2\pi)^4} \, \frac{1}{p^2 + m^2} \, e^{ipx} \,.$$

From now on I will usually drop the subscript E, so that $D(x)$ will stand for this function.

- The $2N$-point correlation function

$$\langle \varphi(x_1) \ldots \varphi(x_{2N}) \rangle$$

is a sum of $(2N-1)!! \equiv 1 \cdot 3 \cdot \ldots \cdot (2N-1)$ terms, each term corresponding to one of $(2N-1)!!$ distinct ways of "pairing" (or "contracting") between $2N$ field insertions $\varphi(x_1) \ldots \varphi(x_{2N})$. A pairing of the insertions $\varphi(x_i)$ with $\varphi(x_j)$ is usually denoted by an overbrace

$$\ldots \overbrace{\varphi(x_i) \ldots \varphi}(x_j) \ldots \,.$$

Every contraction gives rise to the factor $D(x - x')$,

$$\overbrace{\varphi(x)\,\varphi}(x') \quad = D(x - x')$$

The last rule of pairing can be expressed in a compact form, in terms of the following

Wick's Recursion Relation:

$$\langle\varphi(x_1)\varphi(x_2)\cdots\varphi(x_N)\rangle = \sum_{k=2}^{N} D(x_1-x_k)\,\langle\varphi(x_2)\cdots\cancel{\varphi(x_k)}\cdots\varphi(x_N)\rangle\,, \qquad (6.2)$$

where the "cancel" slash indicates that this insertion is dropped. This relation expresses N-point correlation function in terms of the $N-2$-point ones. When supplemented with the condition $\langle\varphi(x)\rangle = 0$, it contains all the statements of the Wick's theorem ($\langle 1\rangle = 1$ by definition). Let us derive this relation.

Consider the functional integral representing the $N-1$-point correlation function,

$$Z^{-1}\int [D\varphi]\,\varphi(x_2)\cdots\varphi(x_N)\,e^{-\mathcal{A}[\varphi]}\,.$$

Note that the product of the insertions starts with $\varphi(x_2)$, and involves $N-1$ terms. Make the following change of the integration variables

$$\varphi(x)\to\varphi'(x) = \varphi(x)+\epsilon(x)\,,$$

where $\epsilon(x)$ is an arbitrary infinitesimal function. The action changes as follows

$$\mathcal{A}[\varphi+\epsilon] = \mathcal{A}[\varphi] + \int d^4x_1\,\epsilon(x_1)\left[m^2-\Delta_{x_1}\right]\varphi(x_1) + O(\epsilon^2)\,,$$

where I have assumed that the function $\epsilon(x)$ decays sufficiently fast making the integration by parts possible. Here

$$\Delta_x := \sum_{\mu=1}^{4}\left(\frac{\partial}{\partial x^\mu}\right)^2$$

is 4D Laplace operator. Since the change of integration variables does not change the value of the integral, we have the identity

$$0 = \sum_{k=2}^{N}\epsilon(x_k)\int [D\varphi]\,\varphi(x_2)\cdots\cancel{\varphi(x_k)}\cdots\varphi(x_N)\,e^{-\mathcal{A}} -$$

$$-\int d^4x_1\,\epsilon(x_1)\left(m^2-\Delta_{x_1}\right)\int [D\varphi]\,\varphi(x_1)\varphi(x_2)\cdots\varphi(x_N)\,e^{-\mathcal{A}}\,.$$

As the function $\epsilon(x)$ is arbitrary, we must have

$$\big(m^2 - \Delta_{x_1}\big)\langle\varphi(x_1)\varphi(x_2)\cdots\varphi(x_N)\rangle = \sum_{k=2}^{N} \delta^{(4)}(x_1 - x_k)\,\langle\varphi(x_2)\cdots\cancel{\varphi(x_k)}\cdots\varphi(x_N)\rangle\,.$$

Assuming that the $N-2$-point correlation functions in the right hand side are known, we can think of this identity as of the equation for the N-point correlation function in the left hand side. The solution has the form of our recursion relation (6.2), where the function $D(x-x')$ solves the equation

$$\big(m^2 - \Delta_x\big)D(x-x') = \delta^{(4)}(x-x')\,.$$

The Wick's theorem can be derived in many other ways. Direct evaluation of the Gaussian functional integral is performed in the Section 9.2 of PS. Derivation of equivalent rules in the operator formalism is explained in Sect.4.3 of PS.

The rules of the Wick's theorem are conveniently represented in terms of diagrams. With every factor $D(x-x')$ one associates a line connecting the points x and x',

$D(x-x') =$ x ——— x'

The diagram consists of the lines representing the contractions. For example

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = \overbrace{\varphi(x_1)\varphi(x_2)}\;\overbrace{\varphi(x_3)\varphi(x_4)} \quad +$$

$$+ \quad \text{two other ways of contractions} \quad =$$

x_1 ——— x_2 $\qquad$ x_1 $\quad$ x_2 $\qquad$ x_1 $\quad$ x_2

$\qquad\qquad +$ $\qquad\qquad +$

x_3 ——— x_4 $\qquad$ x_3 $\quad$ x_4 $\qquad$ x_3 $\quad$ x_4

7 Lecture 7

Interaction

The KG theory describes free particles. Its action is quadratic in the field variables. We want to add interactions. This is done by adding new terms to the action. It should be done in a way that preserves the locality. We can modify the Lagrangian density by adding local terms involving higher powers of the field(s). Let us consider an example known as the

φ^4 theory

It is described by the Euclidean action

$$\mathcal{A}[\varphi] = \int d^4x \left[\frac{1}{2} \left(\partial_\mu \varphi\right)^2 + \frac{m^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 \right].$$

Here λ is a new parameter, the "coupling constant".

As in the KG theory, one can be interested in the correlation functions

$$\langle \varphi(x_1) \cdots \varphi(x_N) \rangle = Z^{-1} \int [D\varphi] \, \varphi(x_1) \cdots \varphi(x_N) \, e^{-\mathcal{A}[\varphi]} .$$

Now the action is not quadratic in the integration variables φ, and the integral cannot be evaluated in an explicit form.

Exercise: Performing the change of variables $\varphi(x) \to \varphi(x) + \varepsilon(x)$ in the functional integral defining the N-point correlation functions in φ^4 theory, derive the equation

$$\left(m^2 - \partial_\mu^2\right) \langle \, \varphi(x) \, \varphi(x_1) ... \varphi(x_N) \rangle + \frac{\lambda}{3!} \langle \, \varphi^3(x) \, \varphi(x_1) ... \varphi(x_N) \rangle =$$

$$= \sum_{i=1}^{N} \delta^{(4)}(x - x_i) \, \langle \varphi(x_1) ... \varphi(x_i) ... \varphi(x_N) \rangle$$

Equations of this kind are generally known as the *Schwinger- Dyson equations.*

Unlike the KG theory, the Schwinger-Dyson equation in the interacting theory is not closed equation for the correlation function $\langle \varphi(x) ... \rangle$, since it involves $\langle \varphi^3(x) ... \rangle$

which has no useful expression in terms of the function $\langle\varphi(x)...\rangle$ itself. Even the definition of the composite field φ^3 may be a non-trivial problem (we are going to consider it later on).

Practical way of computations is the *perturbation theory*, which generates a power series expansion in the coupling parameter λ.

Perturbation Theory

Perturbation theory allows one to evaluate the correlation functions as the power series in the coupling parameter λ. One splits the action into the free-field part (quadratic in the field variables φ), and the interaction part, which includes the terms of higher order:

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_I \,.$$

The exponential $\exp(-\mathcal{A})$ is then expanded as

$$e^{-\mathcal{A}} = e^{-\mathcal{A}_0} \left(1 - \mathcal{A}_I + \frac{1}{2}\,\mathcal{A}_I^2 + ...\right).$$

Since $\mathcal{A}_0$ is quadratic, each term can be evaluated using the Wick's theorem.

To see how it works, consider the φ^4 theory. In this case the natural way of splitting is

$$\mathcal{A}_0 = \mathcal{A}_{\rm KG}\,, \qquad \mathcal{A}_I = \frac{\lambda}{4!}\int \varphi^4(x)\, d^4x$$

(later we will see that in actual calculations it is more convenient to include some parts of the quadratic action - the so called counterterms - into $\mathcal{A}_I$, but at the moment we ignore this detail). With this splitting, the power series expansion of $\exp(-\mathcal{A}_I)$ generates expansion in the powers of λ.

Suppose we want to calculate such expansion of the two-point correlation function $\langle\varphi(x_1)\varphi(x_2)\rangle$. By its definition

$$\langle\varphi(x_1)\varphi(x_2)\rangle = Z^{-1}\int [D\varphi]\,\varphi(x_1)\varphi(x_2)\, e^{-\mathcal{A}_0}\, e^{-\mathcal{A}_I}\,.$$

Expanding the last exponential as prescribed, we have

$$Z^{-1}\int [D\varphi]\,\varphi(x_1)\varphi(x_2)\left(1 - \frac{\lambda}{4!}\int \varphi^4(x)\, d^4x + O(\lambda^2)\right) e^{-\mathcal{A}_0}\,.$$

The zero-order term is just the functional integral for the two-point function of the KG theory. We can rewrite the previous expression as

$$\langle\varphi(x_1)\varphi(x_2)\rangle = \frac{Z_0}{Z}\left[\langle\varphi(x_1)\varphi(x_2)\rangle_0 - \frac{\lambda}{4!}\int\langle\varphi(x_1)\varphi(x_2)\varphi^4(x)\rangle_0\, d^4x + ...\right], \quad (7.1)$$

where $Z_0 = \int D[\varphi]\, e^{-\mathcal{A}_0}$ and $\langle\cdots\rangle_0$ stand for the partition function and correlation functions in the free theory. This expression represents the general situation: the perturbation theory allow one to write the correlation functions of interaction theory as a power series in the coupling parameter, the coefficients being certain integrals of the correlation functions of the free theory. The role of the factor $\frac{Z_0}{Z}$ will be clarified later.

The correlation functions of the free theory are computed using the Wick's rules. In the above example, the first-order in λ term involves the d^4x integral of the correlation function

$$\langle\varphi(x_1)\varphi(x_2)\,\varphi^4(x)\rangle_0 = \langle\varphi(x_1)\varphi(x_2)\varphi(x)\varphi(x)\varphi(x)\varphi(x)\varphi(x)\rangle_0\,.$$

This is a six-point correlation function of the free theory, with four points coincident at x. According to the Wick's rule, it can be written as the sum of

$$5!! = 15$$

terms corresponding to different ways of pairwise contractions of the six fields involved. Many different contractions lead to identical contributions. There are two classes of such contractions with essentially different contributions:

(i) First, one can contract $\varphi(x_1)\varphi(x_2) \to \overbrace{\varphi(x_1)\varphi}(x_2)$, and then contract four factors $\varphi(x)$ among themselves. There are three possibilities for contraction of the first factor, and once that is chosen, there is only one way to contract the remaining two insertions:

$$\begin{array}{cc} 3 \text{ ways} & 1 \text{ way} \\ \overbrace{\varphi(x)\varphi}(x)\,\overbrace{\varphi(x)\varphi}(x) & \end{array}$$

We find that such contractions brings in the contribution

$$3 \times \overbrace{\varphi(x_1)}\varphi(x_2) \quad \overbrace{\varphi(x)}\varphi(x) \quad \overbrace{\varphi(x)}\varphi(x) = 3\; D(x_1 - x_2)\, D(x - x)\, D(x - x)\,.$$

(ii) Otherwise, we can contract $\varphi(x_1)$ with one of the four identical factors $\varphi(x)$ (4 ways), and $\varphi(x_2)$ with any one of the remaining three factors $\varphi(x)$ (3 ways); this leaves just two insertions $\varphi(x)$, with only one way of contractions. Therefore there are $4 \times 3 = 12$ identical contributions

$$12 \times \overbrace{\varphi(x_1)\,\overbrace{\varphi(x_2)\;\;\varphi(x)}\,\varphi(x)}\;\overbrace{\varphi(x)\,\varphi(x)} = 12\; D(x_1 - x)\, D(x_2 - x)\, D(x - x)\,.$$

The resulting contribution $\sim \lambda$ in the brackets in (7.1) then reads

$$-\frac{\lambda}{4!}\left[3\, D(x_1 - x_2) \int d^4x\, D^2(x - x) + 12 \int d^4x\, D(x_1 - x)\, D(x_2 - x)\, D(x - x)\right] \quad (7.2)$$

Depicting contractions as solid lines

$$\overbrace{\varphi(x)\,\varphi(x')} \;\; = \;\; x \;\bullet\!\!\!\!-\!\!\!\!-\!\!\!\!\bullet\; x' \;\; \to \;\; D(x - x')$$

the two terms are represented by two diagrams

$$3 \times \Big[\; x_1 \;\bullet\!\!\!\!-\!\!\!\!-\!\!\!\!\bullet\; x_2 \;\; \bigcirc\!\!\!\!\bullet \;\Big] \;+\; 12 \times \Big[\; x_1 \;\bullet\!\!\!\!-\!\!\!\!\bigcirc\!\!\!\!-\!\!\!\!\bullet\; x_2 \;\Big] \qquad (7.3)$$

where the loop represents the contraction

$$\overbrace{\varphi(x)\varphi(x)} \;\; = \;\; \bigcirc\!\!\!\!\bullet \;\; \to \; D(x - x)$$

Recalling the overall factor

$$-\frac{\lambda}{4!} = -\frac{\lambda}{24}$$

in (7.1), we find that the two diagrams in (7.3) come with the factors $-\lambda/S_1$ and $-\lambda/S_2$, respectively, where the so called "symmetry factors" here are

$$\frac{1}{S_1} = \frac{1}{8}\,, \qquad \frac{1}{S_2} = \frac{1}{2}\,.$$

Similar structure is found in the higher orders of the λ-expansion. The n-th order term has the form

$$\frac{1}{n!}\left(-\frac{\lambda}{4!}\right)\,\langle\varphi(x_1)\varphi(x_2)\,\left[\int\varphi^4(y)\,d^4y\right]^n\rangle_0 \tag{7.4}$$

which involves the n-fold 4-space integration

$$\left[\int\varphi^4(y)\,d^4y\right]^n=\int\varphi(y_1)\varphi(y_2)...\varphi(y_n)\,d^4y_1...d^4y_n\,.$$

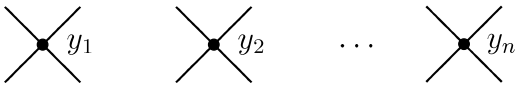
The correlation function in the integrand is that of the product of $2+4n$ factors φ, two at x_1 and x_2, and $4n$ at the integration points $y_1,...,y_n$, four factors at each point.

Contractions of these field factors are represented by diagrams. The diagrams are built from $n+1$ contraction lines connecting two types of "vertices":

There are two "external vertices" associated with the points x_1 and x_2, with exactly one of the contraction lines attached to each,

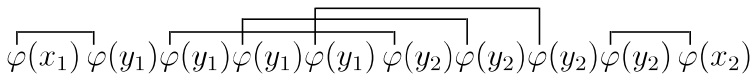
External points : 

and n "interaction vertices", associated with n integration points $y_1,y_2,...,y_n$, with four contraction lines connected to each of them:

Interaction points : 

Thus, in the order λ^2 the diagrams involve two external and two interaction vertices, for example

Many distinct contractions are represented by identical diagrams. For instance, the above diagram may represent the contraction



as well as any one of $4 \cdot (4!)$ equivalent contractions related to this one by interchanging of the contracted factors φ inside $\varphi^4(y_1)$ and $\varphi^4(y_2)$. Every such contraction corresponds to the same product of the propagators

$$D(x_1 - y_1)\, D^3(y_1 - y_2)\, D(y_2 - x_2)\,,$$

and since the contributions involve the integrations over the interaction points

$$\int d^4y_1 d^4y_2 \; (...)$$

the diagrams related by the interchange $y_1 \leftrightarrow y_2$ yield identical contributions. Therefore the above contraction comes with the multiplicity factor

$$2 \cdot 4 \cdot 4!\,.$$

This factor almost entirely cancels the overall factor $(1/n!)(-\lambda/4!)^n$ in (7.4), which for $n = 2$ is

$$\frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2\,,$$

leaving behind the overall factor $(-\lambda)^2/3!$. Finally, total contribution of the contractions corresponding to the diagram in this example is

$$\frac{(-\lambda)^2}{S}\int d^4y_1 d^4y_2\, D(x_1 - y_1)\, D^3(y_1 - y_2)\, D(y_2 - x_2)\,,$$

with the "symmetry factor" $1/S = 1/3!$.

This example illustrates the general rules of the diagrammatic representations of the contributions in the perturbation theory, known as the

Feynman Rules (for φ^4)

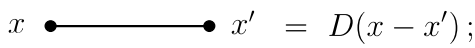
To find λ^n contribution to

$$\langle\, \varphi(x_1) \cdots \varphi(x_N)\, \rangle$$

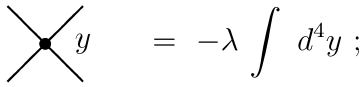
1. Draw all diagrams with N external vertices associated with the points $x_1, \cdots, x_N$, and n interaction vertices. Exclude disconnected diagrams with "vacuum fragments" (see below).

2. For each diagram, associate

   i.



   ii.



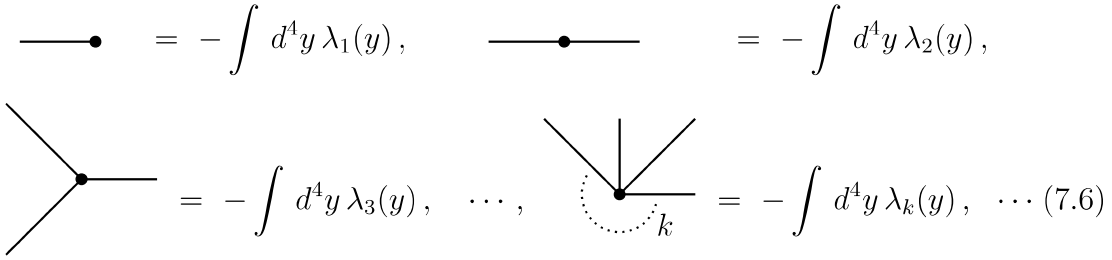
   iii. Multiply by the "symmetry factor" $1/S$.

3. Sum the values of all the diagrams.

Remark: These rules are generalized in a straightforward way to interactions involving any powers of the field φ,

$$\mathcal{A}_I = \int d^4y \left[\lambda_1(y)\,\varphi(y) + \frac{\lambda_2(y)}{2!}\,\varphi^2(y) + \frac{\lambda_3(y)}{3!}\,\varphi^3(y) + \frac{\lambda_4(y)}{4!}\,\varphi^4(y) + \cdots\right]. \quad (7.5)$$

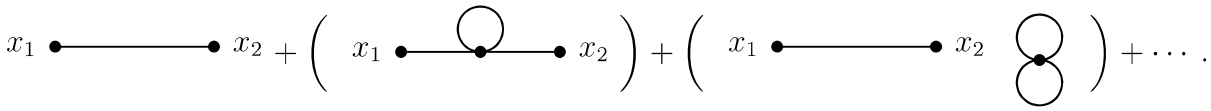
Here I allowed the coupling parameters $\lambda_k = \lambda_k(y)$ to be arbitrary functions of y (this generally breaks space-time translation and rotation symmetries of the theory). The interaction term may also include the terms linear and quadratic in φ. (Later, these generalizations will allow us to use diagrammatic representations of the generating functional and effective action.) The Feynman rules generalize to the case (7.5) by including k-leg vertices



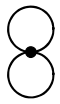
Returning back to the homogeneous φ^4 theory, let us take another look at the diagrams contributing to the two-point function. We write

$$\langle\,\varphi(x_1)\varphi(x_2)\,\rangle = \left[\frac{1}{Z_0}\int [D\varphi]\,\varphi(x_1)\varphi(x_2)\,e^{-\mathcal{A}_0-\mathcal{A}_I}\right] \times \left[\frac{Z_0}{Z}\right], \quad (7.7)$$

where Z_0 is the partition function of the free theory defined by $\mathcal{A}_0 = \mathcal{A}_{KG}$. The diagrams we wrote down in the previous lecture actually correspond to the first factor in (7.2). For this first factor we have



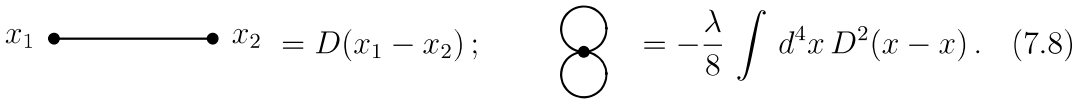
Observe that there are diagrams containing disconnected pieces, like the diagram (c) above. This diagram has a fragment



which is not connected to any of the "external" points (x_1, x_2 in this case). Such fragments are called "vacuum diagrams" (or "vacuum fragments"). The value

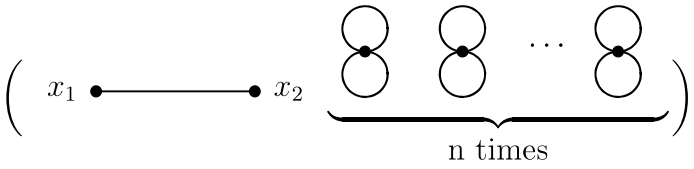
$$D(x_1 - x_2) \times \left(-\frac{\lambda}{8} \int d^4x \, D(x-x) D(x-x) \right)$$

associated with the diagram (c) above is the product of the values of its parts,

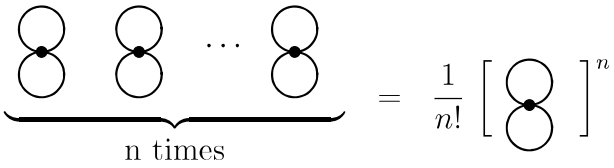


(7.8)

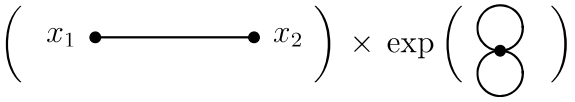
As follows directly from the Feynman rules above, this holds in general: the value of any disconnected diagram is the product of the values of its connected fragments. In fact, in computing the correlation function (7.7) the vacuum fragments should be ignored because their contributions are canceled exactly by the second factor in the r.h.s. of (7.7), which contains the partition function Z of the full theory. To see this, consider again the diagram (c) above. In higher orders of the perturbation theory we obtain infinitely many diagrams in which the vacuum fragment in (c) is repeated many times,



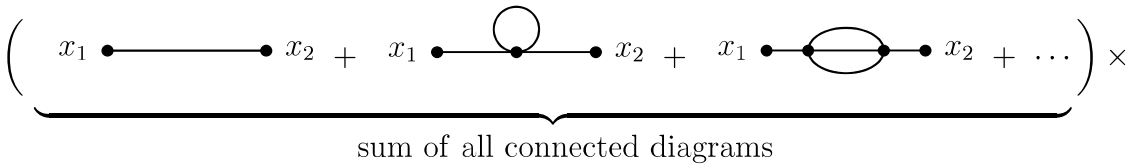
The value corresponding to n copies of the vacuum fragment is just

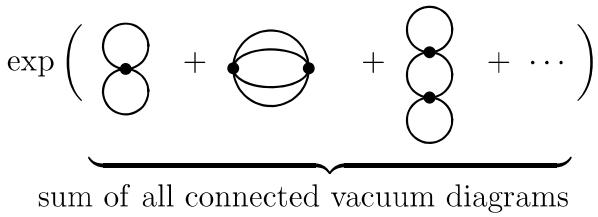


the factor $1/n!$ being the extra symmetry factor associated with the permutations of the fragments. The sum over n therefore yields



This simple analysis can be repeated with any other vacuum fragment. As the result we obtain for the first factor in (7.7)





Consider now the second factor in (7.7),

$$\frac{Z}{Z_0} = \frac{1}{Z_0} \int [D\varphi]\, e^{-\mathcal{A}_0}\, e^{-\mathcal{A}_I}\,.$$

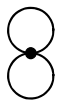
Expanding in $\mathcal{A}_I$ we find that this quantity is given by the sum of all vacuum diagrams, which again exponentiates in terms of the connected vacuum diagrams. That is

$$Z/Z_0 = \exp\left(\text{sum of all connected vacuum diagrams}\right) \tag{7.9}$$

so that the second factor in (7.7) cancels the contributions of all vacuum fragments. Thus, if we are interested in the correlation function, the factor Z^{-1} in the definition of the correlation function can indeed be ignored if we simultaneously drop all the diagrams with disconnected vacuum fragments.

Note that the above analysis of the "vacuum fragments" generalizes straightforwardly to the case of interaction (7.5) with arbitrary x-dependent coupling constants. In particular it is still true that the partition function of this more general theory exponentiates in terms of the connected vacuum diagrams, as in (7.9). Of course in this case the diagrams involve all the vertices in (7.6).

Let me comment on the interpretation of the above formula (7.9) for Z. The first of the connected vacuum diagrams is given by the vacuum fragment in (7.8), i.e.

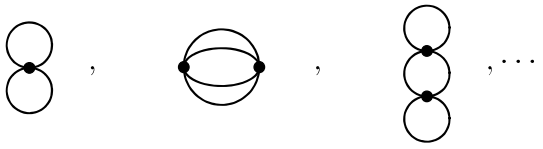


$$= -\frac{\lambda}{8} D^2(0) \int d^4x$$

Clearly, it contains an infinite factor

$$\int d^4x \;=\; V^{(4)}\,,$$

where $V^{(4)}$ is the volume of the 4-dimensional space. This is the common feature of the connected vacuum diagrams



As a consequence of translational invariance each of them contains this volume factor. Therefore

$$Z = Z_0 \, \exp\Big(-\varepsilon_I \, V^{(4)} \Big) \tag{7.5}$$

where ε_I is just the sum of all connected vacuum diagrams with the volume factored out.

The functional integral

$$Z = \int [D\varphi]\, e^{-\mathcal{A}}$$

can be interpreted as a (configuration) partition function of a classical statistical mechanics. The classical system in this interpretation is a field $\varphi(x)$ in 4-dimensional (Euclidean) space, with the potential energy $U[\varphi]$,

$$\beta\, U[\varphi] := \frac{1}{\hbar}\, \mathcal{A}[\varphi]\,,$$

where β is interpreted as the inverse temperature. (β can be set equal to 1, again by suitable choice of the energy units.) This is a spatially homogeneous system with infinitely many degrees of freedom associated with all points in the 4-dimensional space. Naturally, its free energy is an intensive quantity proportional to $V^{(4)}$. We see that the quantity

$$\varepsilon_0 + \varepsilon_I\ , \qquad \text{where} \qquad \varepsilon_0 = -\log Z_0/V^{(4)}$$

is interpreted as the specific (per unit volume) free energy of this system, the first term being the specific free energy of the free theory and ε_I incorporating all corrections due to the interaction.

Another way to look at the formula (7.9) is to recall the relation between the path integrals and the matrix elements of the (imaginary) time evolution operator. The path integral defining Z can be thought of as the $T \to \infty$ limit of the path integral

$$\int [D\varphi]_{\substack{\varphi(\vec{x},-T/2)=\varphi_i(\vec{x}) \\ \varphi(\vec{x},+T/2)=\varphi_f(\vec{x})}}\ e^{-\mathcal{A}}$$

which represents the matrix element

$$\langle \varphi_f(\vec{x}) \mid\ e^{-T\,\hat{H}}\ \mid \varphi_i(\vec{x})\rangle$$

Here φ_i and φ_f are arbitrary "initial" and "final" field configurations. The evolution operator inside can be written as

$$e^{-T\,\hat{H}} = \sum_n \mid n\rangle\langle n \mid\ \ e^{-T\,E_n}\,,$$

where $\mid n\rangle$ is a complete set of the stationary states; its $T \to \infty$ limit then is

$$e^{-E_0\,T}\ \mid 0\rangle\langle 0 \mid,$$

where

$$E_0 = \varepsilon\, V^{(3)}\,,$$

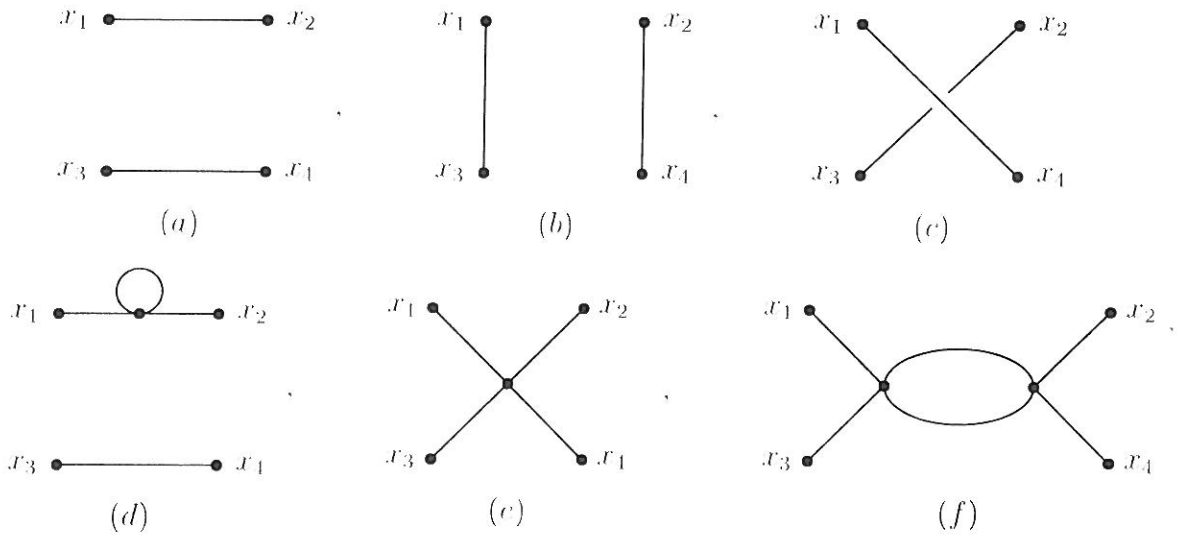
($V^{(3)}$ is the volume of the 3-dimensional "equal time" section of our 4-dimensional space). Because $V^{(4)} = T \times V^{(3)}$, comparison with the above formula (7.9) shows that vacuum diagrams represent the expansion of the ground state energy in the powers of λ (hence the term "vacuum diagrams"). We will see later that all the vacuum diagrams are highly divergent at short distances. This is similar to the short-distance (high momenta) divergence of the vacuum energy of the Klein-Gordon theory which we observed before. Proper understanding of the physical content of these divergent terms is important in certain problems like the Casimir effect or interaction to gravity.

8 Lecture 8

Continuing our analysis of the diagrammatic representation of the correlation functions let us consider the diagram expansion of the four- point function

$$\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) \rangle$$

in the φ^4 theory. First, by applying the same arguments as before it is easy to show that in this case (and indeed for any correlation function) the "vacuum fragments" are again get canceled by the expansion of Z in the denominator. However, in this case the disconnected diagrams remain even after we drop the vacuum fragments. Consider the first few diagrams for the above four-point function.



The first three diagrams here represent the four point function in the free theory: they consist of the propagators which being connected to the external points are not connected to each other. These contributions are expressed in a simple way through the free-field two-point functions D. The diagram (d) and similar diagrams are also disconnected: they represent corrections to those two-point functions. The diagrams (e) and (f) represent the class of truly connected diagrams. By the structure of the above diagram series it is natural to write

$$\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) \rangle =$$

$$\langle \varphi(x_1)\varphi(x_2) \rangle \langle \varphi(x_3)\varphi(x_4) \rangle + \langle \varphi(x_1)\varphi(x_3) \rangle \langle \varphi(x_2)\varphi(x_4) \rangle + \langle \varphi(x_1)\varphi(x_4) \rangle \langle \varphi(x_2)\varphi(x_3) \rangle +$$
$$\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) \rangle_c$$

where the first three terms combine the contributions of the disconnected diagrams; they are expressed in terms of the two-point functions. The connected diagrams are incorporated in the last term which is called the *connected* correlation function.

Similar pattern is observed for the higher-order correlation functions. There are disconnected diagrams which sum up into the lower-order correlation functions, and besides there are truly connected ones, their sums define the connected correlation functions.

In fact the disconnected diagrams can be efficiently sorted out by using the so called Generating Functional.

Generating Functional (PS Sect.9.2, 9.3)

Perturbation theory represents any correlation function as the sum of the Feynman diagrams. There are connected and disconnected diagrams. The vacuum bubbles can be ignored as they are canceled out by the expansion of the normalization factor Z^{-1}. The remaining disconnected diagrams are efficiently sorted out by using the method of Generating Functional. Although the following analysis is done for the case of φ^4 theory, it is useful to keep in mind that all the arguments below can be repeated step by step for the generic interaction.

Let us define the Generating Functional as

$$\zeta[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, \langle \, \varphi(x_1) \cdots \varphi(x_n) \, \rangle \; J(x_1) \cdots J(x_n) \, . \tag{8.1}$$

It depends on the auxiliary function J, and incorporates all n-point correlation functions. By the definition

$$\langle \, \varphi(x_1) \cdots \varphi(x_n) \, \rangle = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \, \zeta[J] \, \Big|_{J=0} \, . \tag{8.2}$$

The generating functional can be written as

$$\zeta[J] = \langle \exp \left(\int J(x) \varphi(x) \, d^4x \right) \rangle$$

and therefore it is expressed as the ratio

$$\zeta[J] = Z[J]/Z[0] \, ;$$

where

$$Z[J] = \int [\mathcal{D}\varphi]\, e^{\int J(x)\varphi(x)d^4x}\, e^{-\mathcal{A}[\varphi]} \tag{8.3}$$

is a partition function associated with the action

$$\mathcal{A}_J = \mathcal{A} - \int J(x)\varphi(x)\, d^4x\,.$$

For this reason the function $J(x)$ is usually referred to as an "external source".

The source-dependent action above is a particular case of generic theory with x-dependent coupling $\lambda_1(x) = -J(x)$,

$$\mathcal{A}_J = \int d^4x \left[-J(x)\varphi(x) + \frac{\lambda}{4!}\varphi^4(x)\right].$$

Therefore its partition function $Z[J]$ (more precisely $Z[J]/Z_0$) is given by a sum of all vacuum diagrams which now contain two kinds of vertices.

$$= \int d^4x\, J(x)\,, \qquad$$

$$= -\lambda \int d^4x\,.$$

By taking the variational derivatives with respect to $J(x)$ we just convert the one-leg vertices above into the external legs - this is the meaning of the formula (8.2) above. Now, the sum of all the vacuum diagrams exponentiate in terms of the sum of connected vacuum diagrams. Defining

$$\zeta[J] = \exp W[J] \tag{8.4}$$

we conclude that the expansion of this new functional $W[J]$ in terms of J will generate the *connected* correlation functions, namely

$$W[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n\, W^{(n)}(x_1, \cdots, x_n)\; J(x_1)\cdots J(x_n)\,.$$

where

$$W^{(n)}(x_1, \cdots, x_n) = \langle\, \varphi(x_1)\cdots\varphi(x_n)\,\rangle_{\rm conn}$$

The functional $W[J]$ is called the generating functional for the *connected* correlation functions. In the KG theory one can evaluate the Gaussian functional integral (8.3) for $Z[J]$ explicitly and find

$$W_{KG}[J] = \frac{1}{2} \int d^4x\, d^4x'\, J(x) D(x - x') J(x') ,$$

which shows again that in the free theory all connected correlation functions but the two-point one vanish.

Exercise: Using $\zeta[J] = \exp W[J]$, check by direct calculation that

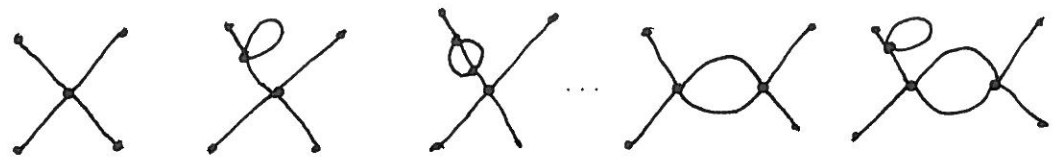
$$W^{c(2)}(x_1, x_2) = \langle\, \varphi(x_1)\varphi(x_2)\,\rangle ,$$

$$W^{c(4)}(x_1, \cdots, x_4) = \langle\, \varphi(x_1)\cdots\varphi(x_4)\,\rangle -$$

$$\langle\, \varphi(x_1)\varphi(x_2)\,\rangle\langle\, \varphi(x_3)\varphi(x_4)\,\rangle \; - \text{two other disconnected terms} .$$

In many cases it is useful to decompose the connected correlation functions further into more elementary "blocks". Consider the diagrams for the connected four-point function

$$\langle\, \varphi(x_1)\cdots\varphi(x_4)\,\rangle_c =$$



(a) (b) (c) (d) (e)

It is clear that some of these diagrams (like (b), (c), and (e)) represent the corrections to the "external legs". This shows that the connected correlation functions can be expressed through so-called "amputated" correlation functions,

$$W^{c(n)}(x_1, \cdots, x_n) = \int \left[\prod_{i=1}^{n} d^4y_i\, W(x_i - y_i) \right] W^{c(n)}_{amp}(y_1, \cdots, y_n)$$

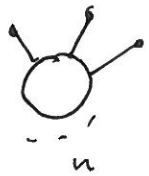
where I used the shorthand

$$W(x - y) \equiv W^{c(2)}(x, y) = \langle\, \varphi(x)\varphi(y)\,\rangle$$

Let us give a graphical representation of this relation. Denoting the connected n-point correlation function by an empty blob with n legs.

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_{\text{conn}} =$$

we have



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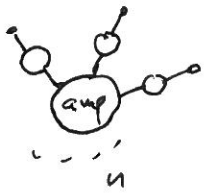

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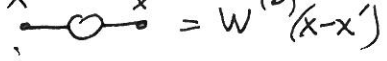


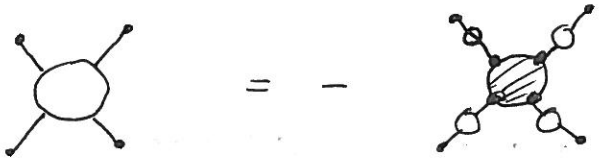
According to this definition the amputated correlation functions do not contain neither external legs nor the corrections to them. Note that by this definition $W^{(2)}_{amp}(x,y) = S(x-y)$ is inverse (in the operator sense) to the two-point function, i.e.

$$\int d^1y\, S(x-y)W(y-x') = \delta^{(1)}(x-x')\,.$$

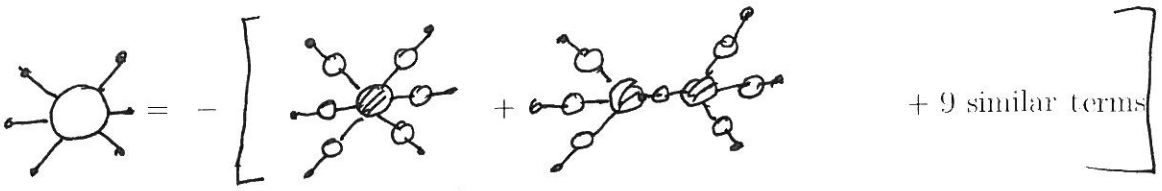
In the free theory this is the inverse of the function D i.e. just the (Euclidean) Klein-Gordon operator

$$S_0(x-y) = (m^2 - \Delta_x^2)\delta^{(1)}(x-y)\,.$$

Finally, let us define the so called *proper vertices* (also referred to as the "one-particle irreducible correlation functions"). For $n > 2$ the n-point proper vertex $-\Gamma^{(n)}(y_1, \cdots, y_n)$ is the sum of all diagrams contributing to $W^{(n)}(y_1, \cdots, y_n)$ which cannot be made disconnected by cutting just one line (please pay attention to the minus sign in this definition). In particular, in the φ^4 theory $\Gamma^{(4)} = -W^{(4)}_{amp}$ (why?), i.e. diagrammatically



but for the 6-point functions we have in graphic representation



where each shadowed blob with n fat points on it stands for the proper vertex $\Gamma^{(n)}$.

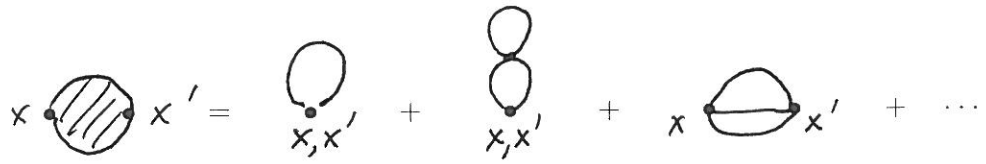
For the reasons that will become clear later the two-point proper vertex deserves special definition, namely

$$\Gamma^{(2)}(x, x') = S(x - x') .$$

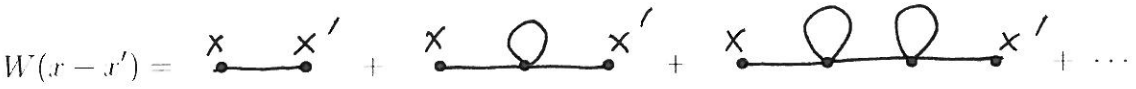
Its relation to the one-particle irreducible diagrams is as follows. Write

$$S(x - x') = S_0(x - x') + \Sigma(x - x') .$$

The function $\Sigma(x - x')$ is often referred to as the mass operator. It is the function $-\Sigma$ (again note the sign) which is equal to the sum of all one-particle irreducible diagrams for the two-point correlation function. Indeed, denoting $-\Sigma$ by the two-point shadowed blob we have



The full set of diagrams contributing to W,



can be obtained by by repeating the one-particle irreducible diagrams in a chain-like manner,

Explicitly,

$$W = D - D * \Sigma * D + D * \Sigma * D * \Sigma * D - \cdots$$

Here $A * B$ denotes a convolution

$$(A * B)(x - x') = \int d^1y\, A(x - y)B(y - x')\,.$$

Using

$$S_0 * D = I$$

where $I = \delta^1(x - y)$ is the unit operator, it is strightforward to check that $S_0 + \Sigma$ is indeed an inverse of W.

Similarly to the above generating functionals ζ and W we can define a generating functional of the proper vertices. This functional, denoted $\Gamma[\phi]$, is called the

Effective Action (Sect 11.3-11.5)

It turns out that $\Gamma[\phi]$ is related to the functional $W[J]$ by the functional version of the *Legendre transform*. To give some motivation and also to remind what the Legendre transform is, let us consider a magnet which in the absence of an external field is described by a Hamiltonian H. If we add an external field B, its internal energy is

$$U(B) = \langle\, H \,\rangle - M\, B$$

where M is the magnetization,

$$M = -\frac{\partial F}{\partial B}$$

and $F = U - TS$ is the free energy. One can solve the above equation for B: $B = B(M)$, and define the thermodynamic potential

$$\Phi(M) = \Big[F(B) + MB\Big]_{B=B(M)} = \langle\, H \,\rangle_{B(M)} - TS\,,$$

so that

$$B(M) = \frac{\partial \Phi(M)}{\partial M}\,.$$

In particular the equation

$$\frac{\partial \Phi}{\partial M} = 0$$

determines spontaneous magnetization in the absence of external field. The transformation relating $\Phi(M)$ to $F(B)$ is the Legendre transform.

Note that the above functional $Z[J]$ is interpreted as the partition function of a field statistical system with the Hamiltonian

$$\mathcal{A}[\varphi] - \int J(x)\varphi(x)\, d^4x\,,$$

i.e. the source $J(x)$ plays the role of the external field in the above example. The generating functional $W[J] = \log \zeta[J]$ is then interpreted as minus free energy,

$$W[J] = -F[J]\,.$$

In particular

$$\phi(x,[J]) = \langle\, \varphi(x)\,\rangle_J = \frac{\delta W[J]}{\delta J(x)}\,. \tag{8.5}$$

where $\langle \cdots \rangle_J$ denotes the expectation value in the presence of the source J. Let us assume that the above equation can be solved for J, defining the inverse functional

$$J = J(x,[\phi]) \;:\quad J(x,[\phi(x,[J_0(x)]]) = J_0(x)\,,\quad \phi(x,[J(x,[\phi_0]]) = \phi_0(x)\,,$$

for any functions $J_0(x)$ or $\phi_0(x)$. One can define then the analog of the thermodynamic potential

$$\Gamma[\phi] = \left[-W[J] + \int J(x)\phi(x)\, d^4x \right]_{J(x)\,=\,J(x,[\phi])}$$

It is not difficult to show that for $\Gamma[\phi]$ thus defined the relation holds

$$J(x,[\phi]) = \frac{\delta\Gamma[\phi]}{\delta\phi(x)}\,. \tag{8.6}$$

Indeed, taking the variation of

$$\Gamma[\phi] + W[J] - \int J(x)\phi(x)\, d^4x = 0$$

(where it is understood that ϕ and J are related through (7.5)) we have

$$\int d^4x \left[\frac{\delta\Gamma}{\delta\phi(x)}\,\delta\phi(x) + \frac{\delta W}{\delta J(x)}\,\delta J(x) - \phi(x)\,\delta J(x) - J(x)\,\delta\phi(x) \right] = 0$$

The second and the third terms here cancel each other in view of (7.5) and we obtain (7.6).

It can be shown that the functional $\Gamma[\phi]$ coincides with the generating functional of the proper vertices. Namely, its functional Taylor expansion in ϕ.

$$\Gamma[\phi] = \sum_{n=2}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, \Gamma^{(n)}(x_1, \cdots, x_n)\phi(x_1) \cdots \phi(x_n)$$

has the proper vertices $\Gamma^{(n)}$ as its coefficient functions. Let us see how it works for lower-order correlation functions, using φ^4 theory as an example. We have

$$W[J] = \frac{1}{2} \int d^4x_1 \, d^4x_2 \, W(x_1 - x_2) \, J(x_1) J(x_2) +$$

$$\frac{1}{4!} \int \left[\prod_{i=1}^{4} d^4x_i \right] W^{(4)}(x_1, \cdots, x_4) \, J(x_1) J(x_2) J(x_3) J(x_4) \; + \; O(J^6)$$

and

$$\Gamma[\phi] = \frac{1}{2} \int d^4x_1 \, d^4x_2 \, \Gamma^{(2)}(x_1 - x_2) \, \phi(x_1) \phi(x_2) +$$

$$\frac{1}{4!} \int \left[\prod_{i=1}^{4} d^4x_i \right] \Gamma^{(4)}(x_1, \cdots, x_4) \, \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \; + \; O(\phi^6) \, .$$

Using the equation $\phi(x, [J]) = \delta W / \delta J(x)$ we can express ϕ in terms of J as

$$\phi(x, [J]) = \int d^4x_1 \, W(x - x_1) \, J(x_1) +$$

$$\frac{1}{3!} \int \left[\prod_{i=1}^{3} d^4x_i \right] W^{(4)}(x, x_1, x_2, x_3) \, J(x_1) J(x_2) J(x_3) \; + \; O(J^5) \, .$$

Similarly, using $J(x, [\phi]) = \delta \Gamma / \delta \phi(x)$, J is expressed through ϕ as

$$J(x, [\phi]) = \int d^4x_1 \, \Gamma^{(2)}(x - x_1) \, \phi(x_1) +$$

$$\frac{1}{3!} \int \left[\prod_{i=1}^{3} d^4x_i \right] \Gamma^{(4)}(x, x_1, x_2, x_3) \, \phi(x_1) \phi(x_2) \phi(x_3) \; + \; O(\phi^5) \, .$$

These two relations must be inverse one to another. Substituting the first into the second we have, schematically

$$J(x) = \int_{y,y'} \Gamma^{(2)}(x - y) W(y - y') J(y') +$$

$$\frac{1}{3!}\int_{y,y_1,y_2,y_3}\Gamma^{(2)}(x-y)W^{(4)}(y,y_1,y_2,y_3)J(y_1)J(y_2)J(y_3)\ +$$

$$\frac{1}{3!}\int_{\substack{y_1',y_2',y_3'\\ y_1,y_2,y_3}}\Gamma^{(4)}(x,y_1',y_2',y_3')W(y_1'-y_1)W(y_2'-y_2)W(y_3'-y_3)J(y_1)J(y_2)J(y_3)+O(J^5)\,.$$

As this must be an identity, we find again

$$\Gamma^{(2)}(x-x')=S(x-x')\,,$$

and

$$W^{(4)}(x_1,....x_4)=-\int_{y_1,\cdots,y_4}\Gamma^{(4)}(y_1,\cdots,y_4)W(y_1-x_1)W(y_2-x_2)W(y_3-x_3)W(y_4-x_4)\,,$$

which coincide with our definitions of the proper vertices $\Gamma^{(2)}$ and $\Gamma^{(4)}$.

Exercise: Carry out the Legendre transform to the order 6 in J and ϕ. Find $W^{(6)}$ in terms of $\Gamma^{(6)}$, $\Gamma^{(4)}$, $\Gamma^{(2)}$.

9 Lecture 9

There is a useful way to reorganize the diagrams contributing to $W[J]$ or $\Gamma[\phi]$ known as the

Semiclassical (or Loop) Expansion

Generally, classical limit appears as the leasing term if one tries to estimate the functional integral using the saddle point approximation. Then, the integral is dominated by the field configurations close to the stationary point determined by solving the equation $\delta\mathcal{A}[\varphi] = 0$, i.e. the equations of motion of classical field theory defined by the action $\mathcal{A}[\varphi]$. With this in mind, let us restore the quantum parameter h (previously set to 1) in the definition of the generating functional

$$Z[J] = \int [\mathcal{D}\varphi]\, e^{-\frac{1}{h}\left(\mathcal{A}[\varphi] - \int_x J(x)\varphi(x)\right)} \tag{9.1}$$

The semiclassical approximation becomes exact in the limit $h \to 0$, while corrections to the leading semiclassical contribution can be developed in the form of a power series in h. Note that as in statistical mechanics interpretation $h \sim kT$, and we can also talk here about the low-temperature expansion. For this reason it is convenient to retain h as the formal parameter[1].

For the purpose of the present analysis I will consider a generic action of the form

$$\mathcal{A} = \int \left[\frac{1}{2}(\partial\varphi)^2 + \frac{m^2}{2}\varphi^2 + V(\varphi)\right], \tag{9.2}$$

with $V(\varphi)$ of general form

$$V(\varphi) = \frac{\lambda_3}{3!}\varphi^3 + \frac{\lambda_4}{4!}\varphi^4 + \cdots$$

with constants λ_k, although I will often give illustrations in terms of φ^4 theory.

[1] It of course is always possible to choose units in which the actual Plank's constant is 1. Then formal parameter named h in (9.1) is dimensionless, and it looks like an additional parameter of the theory. In fact, such parameter can always be traded for coupling constants determining the strength of interaction. For example, the φ^4 action can be written as

$$\mathcal{A} = \frac{1}{\lambda}\int \left(\frac{1}{2}(\partial\chi)^2 + \frac{m^2}{2}\chi^2 + \frac{1}{4!}\chi^4\right) d^4x$$

in terms of the renormalized field $\chi = \sqrt{\lambda}\,\varphi$.

The leading (tree) approximation

.

For small h we calculate the above integral (9.1) for $Z[J]$ using a saddle-point approximation. That is, one first finds the minimum of the action functional

$$\mathcal{A}_J[\varphi] = \mathcal{A}[\varphi] - \int_x J(x)\varphi(x)\,.$$

i.e. the solution of the classical equation of motion

$$\frac{\delta \mathcal{A}[\varphi]}{\delta \varphi(x)} = J(x)\,.$$

The solution depends on J; I will denote it as

$$\varphi_c(x) = \varphi_c(x,[J])\,.$$

Then in the leading approximation

$$\log Z[J] = \frac{1}{h}\left(-\mathcal{A}[\varphi_c] + \int_x J(x)\varphi_c(x)\right) + O(h^0)$$

When h is explicit it is convenient to normalize the generating functional $W[J]$ so that

$$\frac{Z[J]}{Z[0]} = e^{\frac{1}{h}W[J]}\,.$$

We write

$$W[J] = W_0[J] + h\,W_1[J] + \cdots\,.$$

In this notations we have

$$W_0[J] = -\mathcal{A}[\varphi_c] + \int J(x)\varphi_c(x)\,, \tag{9.3}$$

where it is understood that $\varphi_c = \varphi_c[J]$.

The relation of this leading semiclassical approximation to the ordinary perturbation theory (expansion in λ_i) is recovered by expanding the solution φ_c in the powers of J. In view of the chosen form of the action (9.2) the classical equation has the form

$$(m^2 - \Delta_x)\varphi_c(x) + V'(\varphi_c(x)) = J(x)\,.$$

This can be transformed to the integral equation

$$\varphi_c(x) = \int_{x'} D(x-x')J(x') - \int_{x'} D(x-x')V'(\varphi_c(x'))$$

which can be solved iteratively, with the last term with V' taken as the perturbation. For example, if

$$V = \frac{\lambda}{4!}\varphi^4, \qquad V' = \frac{1}{3!}\varphi^3,$$

we find

$$\varphi_c(x,[J]) = \int_{x'} D(x-x')J(x') -$$

$$\frac{\lambda}{3!}\int_{x,x_1,x_2,x_3} D(x-x')D(x'-x_1)D(x'-x_2)D(x'-x_3)J(x_1)J(x_2)J(x_3) + \cdots .$$

Obviously, this generates the expansion in J. Also, this series is represented diagrammatically as

$\varphi_c(x,[J]) =$  $+ \cdots$

Here I mark the "external point" x where $\varphi_c(x)$ sits by a cross to distinguish it from the one-leg vertex

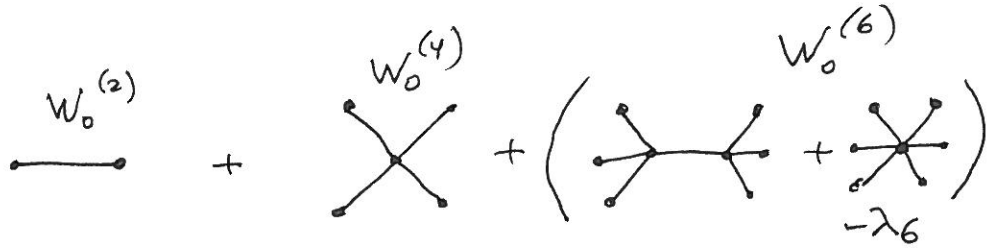
 $= \int d^4x J(x)$.

It is easy to see that the diagrams appearing here have no loops (i.e. one cannot travel along the diagram and come back the starting point never going the same line twice). Such diagrams are called the *tree diagrams*.

The easiest way to obtain the functional $W_0[J]$ (9.3) is to observe that

$$\frac{\delta W_0}{\delta J} = \left[-\frac{\delta \mathcal{A}}{\delta \varphi_c} + J\right]\frac{\delta \varphi_c}{\delta J} + \varphi_c = \varphi_c ,$$

where the fact that φ_c solves the classical equation is used. Integrating the above expansion of $\varphi_c(x,[J])$ term by term we find

$W_0[J] =$ 

i.e. the leading semiclassical approximation for W is just the sum of all connected tree diagrams. For this reason the leading approximation is also called the *tree approximation.*

Let us find the effective action in this approximation. By the definition of the Legendre transform we have to set

$$\phi = \frac{\delta W_0}{\delta J} = \varphi_c[J] .$$

Then the effective action in this approximation is

$$\Gamma_0[\phi] = -W_0[J] + J\phi = \mathcal{A}[\varphi_c] - J\varphi_c + J\varphi_c = \mathcal{A}[\phi] ,$$

i.e. in the leading semiclassical approximation the effective action $\Gamma_0[\phi]$ coincides with the action $\mathcal{A}[\phi]$. In particular

$$\Gamma_0^{(2)}(x,x') = S_0(x-x') = (m^2 - \Delta_x)\delta(x-x') ;$$

and assuming for example that $V(\varphi) = \frac{\lambda_4}{4!}\varphi^4 + \frac{\lambda_6}{6!}\varphi^6 + \cdots$,

$\Gamma_0^{(4)} = -$  ; $\Gamma_0^{(6)} = -$  ; $\cdots$

where I have shown where the amputated legs used to be by the dotted lines. Clearly, these are the simplest one-particle irreducible amputated diagrams for $\Gamma^{(n)}$.

The first order (one-loop) correction

To evaluate the first correction to the effective action one must take into account the gaussian fluctuations around the classical solution φ_c. In the functional integral (9.1) defining $Z[J]$ set

$$\varphi(x) = \varphi_c(x) + \chi(x)$$

and expand the action $\mathcal{A}_J$ to the terms quadratic in χ.

$$\mathcal{A}_J[\varphi] = \mathcal{A}[\varphi_c] - J\varphi_c + \left.\frac{\delta\mathcal{A}}{\delta\varphi}\right|_{\varphi=\varphi_c} \chi - J\chi + \int_{x,x'} \frac{1}{2}\left.\frac{\delta^2\mathcal{A}}{\delta\varphi(x)\delta\varphi(x')}\right|_{\varphi=\varphi_c} \chi(x)\chi(x') + O(\chi^3)\,.$$

As usual, the terms linear in χ cancell in view of the classical equation φ_c satisfies, and we obtain in this approximation

$$Z[J] = e^{\frac{i}{\hbar}W_0[J]} \int [\mathcal{D}\chi]\, e^{-\frac{i}{\hbar}\mathcal{A}_1[\chi]}\,,$$

where

$$\mathcal{A}_1[\chi] = \int d^4x \left[\frac{1}{2}(\partial\chi)^2 + \frac{m^2}{2}\chi^2 + \frac{1}{2}V''(\varphi_c)\chi^2\right] \tag{9.4}$$

Since the higher-order terms in χ are neglected the integral over χ is gaussian and it can be evaluated explicitly in terms of the determinant of the operator

$$K(x,x') = \frac{1}{2}\left.\frac{\delta^2\mathcal{A}}{\delta\varphi(x)\delta\varphi(x')}\right|_{\varphi=\varphi_c} = (m^2 - \Delta_x)\delta(x-x') + V''(\varphi_c)\delta(x-x')\,.$$

Exercise: Evaluate n-dimensional gaussian integral

$$\prod_{i=1}^{n}\left[\int_{-\infty}^{\infty} d\xi_i\right] \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{n} s_{ij}\,\xi_i\xi_j\right\}$$

where s_{ij} are elements of a symmetric positive matrix, and show that it equals to

$$(2\pi)^{n/2}\,[\det(s_{ij})]^{-1/2}\,.$$

Actually, we are interested in the ratio $Z[J]/Z[0]$, therefore we need to evaluate the ratio of the determinants

$$\left[\frac{\det\left(S_0(x-x') + \delta(x-x')V''(x')\right)}{\det\left(S_0(x-x')\right)}\right]^{-\frac{1}{2}} =$$

$$\exp\left[-\frac{1}{2}\log\det\left(\delta(x-x') + D(x-x')V''(\varphi_c(x'))\right)\right]$$

This can be transformed with the help of the identity, valid for any positive Hermitian operator M:

$$\log\big(\det M\big) = \mathrm{tr}\big(\log M\big)\,,$$

which is established by bringing M to the diagonal form.

$$\log\big(\prod_i m_i\big) = \sum_i \log m_i\,.$$

Using this property, we obtain

$$W_1[J] = -\frac{1}{2}\,\mathrm{tr}\,\log\big(I + A(\varphi_c)\big)\,. \tag{9.5}$$

where A is an integral operator with the kernel

$$A(x,x') = D(x-x')\,V''\big(\varphi_c(x)\big)\,.$$

and I is the identity operator, $I(x,x') = \delta(x-x')$. The logarithm expands as

$$\log\big(I+A\big) = A - \frac{A^2}{2} + \frac{A^3}{3} - \cdots - \frac{(-A)^n}{n} + \cdots\,.$$

When taking the trace we obtain

$$\mathrm{tr}A = \int_x D(x-x)\,V''(x)\,.$$

$$\mathrm{tr}A^2 = \int_{x,x'} D(x-x')\,V''(x')\,D(x'-x)\,V''(x)\,.$$

etc

Here $V''(x)$ is the shorthand for $V''(\varphi_c(x))$. Obviously, this expansion can be represented in terms of diagrams

$$W_1[J] = -\frac{1}{2}\,\mathrm{tr}\log\big(I+A\big) =$$

with the line standing for D, as usual, but with the special two-leg vertex

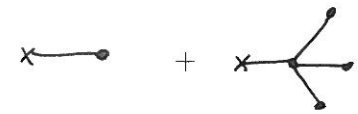
$$= - \int d^1x\, V'''(\varphi_c(x)) .$$

A diagram with n such vertices comes with the factor

$$\frac{1}{2n} = \frac{1}{2}\frac{1}{n} ,$$

$1/2$ already being a factor in (9.5), and $1/n$ coming from the expansion coefficient of the log function; this factor can be interpreted as the symmetry factor of the n-vertex diagram. Of course the above diagram expansion can be obtained directly from the Feynman rules for the action (9.4).

To obtain $W_1[J]$, substitute

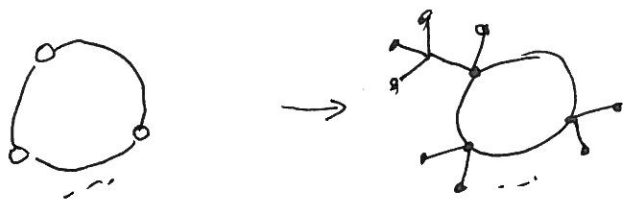
$\varphi_c = \varphi_c[J] =$  $+ \cdots ,$

which, as we have seen, is a sum of the tree diagrams. For φ^4 theory

$$V'''(\varphi_c) = \frac{1}{2}\varphi_c^2$$

and so

Therefore W_1 is the sum of the diagrams of the form



which have exactly one loop. This is why the $\sim h$ correction to W is called the "one-loop correction".

It is not difficult to find the effective action in this approximation. We have

$$\Gamma[\phi] = -W_0[J] - \hbar W_1[J] + \int J\phi + O(\hbar^2)\,,$$

where we have to substitute $J = J[\phi]$. Note that definition of the Legendre transform, the r.h.s. of this equation is stationary with respect to variations of J with ϕ fixed. Therefore the correction $\sim \hbar$ to the relation between J and ϕ will produce a change of the order of $\hbar^2$ to the r.h.s. of the above equation. Hence at order $\hbar$ we still can replace J by the solution of the equation

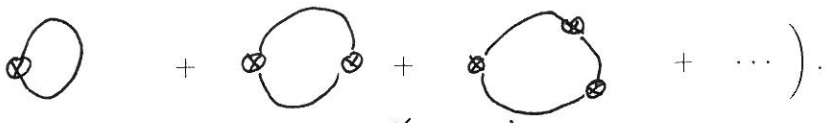
$$\phi(x) = \varphi_c(x, [J])\,.$$

Then, again $-W_0 + \int J\phi = \mathcal{A}[\phi]$, and

$$\Gamma[\phi] = \mathcal{A}[\phi] + \hbar\Gamma_1[\phi] + O(\hbar^2)\,,$$

where

$$\Gamma_1[\phi] = \frac{1}{2}\,\mathrm{tr}\log\big(I + A(\phi)\big) =$$

$-\Big($  $+\cdots\Big)$.

with $A(x,x') = D(x-x')\,V''(\phi(x'))$, and

 $= -V'''(\phi)$.

Note that all these diagrams are one-particle irreducible.

Similar analysis can be carried out in higher orders in $\hbar$. One finds that at the order $\hbar^n$ only the diagrams with exactly n loops contribute to $W[J]$.

Effective Action vs Generating Functional

We can now clarify the nature of the relation between $W[J]$ and $\Gamma[\phi]$. Write again

$$\Gamma[\phi] = \sum_{n=2}^{\infty} \int \Gamma^{(n)}(x_1, \ldots, x_n)\,\phi(x_1)\ldots\phi(x_n)\,.$$

and represent $\Gamma^{(n)}$ as an n-point blob

$\Gamma^{(n)} =$ 

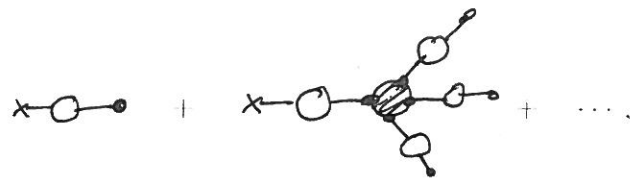
The generating functional $W[J]$ is related to $\Gamma[\phi]$ by a functional Legendre transform.

$$W[J] = \left[-\Gamma[\phi] + \int_x J(x)\phi(x) \right]_{\phi=\phi[J]}$$

One first solves the equation

$$J(x) = \frac{\delta\Gamma[\phi]}{\delta\phi(x)}$$

with respect to $\phi(x)$. Exactly as was done when solving the classical equations of motion, the solution is represented as a sum of tree diagrams

$\phi(x,[J]) =$  $+ \cdots ,$

this time with $W(x-x')$ (the inverse of $\Gamma^{(2)}$, i.e. the full two-point function) standing instead of the propagators D, and the functions $\Gamma^{(n)}$ playing the role of the vertices. In drawing the above diagrams I have assumed that the interaction was even, $V(\varphi) = V(-\varphi)$, so that odd order vertices are absent; of course, this limitation is not significant. Again, integrating the equation

$$\frac{\delta W[J]}{\delta J(x)} = \phi(x,[J])$$

term by term in J, one finds that $W[J]$ is a sum of all tree diagrams, again with W for propagators and $\Gamma^{(n)}$ for vertices.

$W[J] =$  $+$ $+$ all tree diagrams

This is exactly what we expect to be the relation between the sum of all connected diagrams and the sum of all one-particle irreducible diagrams: one takes one-particle

irreducible diagrams and connect them in a tree-like manner to produce all connected diagrams.

10 Lecture 10

Now we turn to actual evaluation of the diagram contributions. For this purpose in most cases it is much more convenient to use the Feynman rules in the momentum space, evaluating the Fourier transformed correlation functions

$$\int W^{\tau(n)}(x_1,\cdots,x_n)\, e^{-ip_1x_1}\cdots e^{-ip_nx_n}\, d^4x_1\cdots d^4x_n =$$

$$(2\pi)^4\,\delta^{(4)}(p_1+\cdots+p_n)\,\tilde{W}^{\tau(n)}(p_1,\cdots,p_n)\ . \qquad (10.1)$$

Here again $W^{\tau(n)}$ denotes *connected* n-point correlation function. The delta function in the r.h.s. appears because of the translation invariance: $W^{\tau(n)}(x_1,\cdots,x_n)$ depends only on the differences $x_i - x_j$.

I will skip detailed derivation of the diagram rules in the momentum space (look up Sect.4.4 of PS for the details). Very briefly, in any diagram one uses the fourier expansions

$$D(x-x') = \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2+m^2}\, e^{ip(x-x')}$$

of all propagators. Then the integral over all x associated with vertices, as well as over all "external" points x_i, can be evaluated explicitly yielding the momentum delta-functions. As the result one obtains the

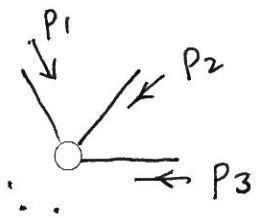
Momentum-space Feynman rules:

(i) "Internal" propagators carry (directed) momentum to be integrated over,

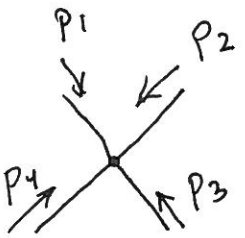
$$\underline{\qquad p\ \rightarrow \qquad} \quad = \quad \underline{\qquad \leftarrow\ -p \qquad} \quad = \quad \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2+m^2}\ .$$

(ii) "External" legs carry fixed momenta

$$\tilde{W}^{\tau(n)}(p_1,\cdots,p_n) \quad =$$



(iii) The vertices conserve the momentum



$$= \quad -\lambda\,(2\pi)^4\,\delta^{(4)}(p_1+\cdots+p_4)$$

(iv) Integration $\int \frac{d^4p_i}{(2\pi)^4}$ is performed over all the momenta associated with the "internal" propagators.

Some properties of $\tilde{W}^{c(n)}$

In the momentum representation many operations simplify. For example the "amputation" takes very simple form

$$\tilde{W}^{c(n)}(p_1,\cdots,p_n) = \left[\prod_{i=1}^{n}\tilde{W}^{c}(p_i)\right]\tilde{W}^{c(n)}_{\rm amp}(p_1,\cdots,p_n)\ ,$$

where $\tilde{W}^{c}(p)$ is the fourier transform of $W^{c}(x-x')$. Similar simplification occurs to the relation to proper vertices.

Exercise: Express $\tilde{W}^{c(6)}(p_1,\cdots,p_6)$ in terms of $\tilde{\Gamma}^{(6)}(p_1,\cdots,p_6)$, $\tilde{\Gamma}^{(4)}$ and $\tilde{W}^{c}$.

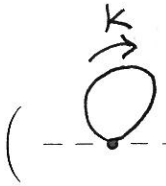
The momentum space $\tilde{\Gamma}^{(2)}(p)$ is just the inverse of $\tilde{W}^{c}(p)$ in usual sense,

$$\tilde{\Gamma}^{(2)}(p) = \frac{1}{\tilde{W}^{c}(p)} = p^2 + m^2 + \tilde{\Sigma}(p)\ ,$$

where $-\tilde{\Sigma}(p)$ is the sum of all one-particle irreducible two-leg diagrams in the momentum space.

Leading contribution to $\tilde{\Sigma}$. Mass renormalization

Let us calculate some perturbative contributions to $\tilde{\Sigma}(p)$. In the leading order we have



$$\tilde{\Sigma}(p) \;=\; -\Big(\quad\Big) \;=\; \frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+m^2} = \frac{\lambda}{8\pi^2}\int_0^\infty \frac{|k|^3 d|k|}{|k|^2+m^2}\ .$$

This integral is badly divergent as $k \to \infty$. What should we do about it? We can get rid of the meaningless expression by introducing a "cutoff", say, replacing

$$\frac{1}{k^2 + m^2} \quad \to \quad \frac{1}{k^2 + m^2}\, \Phi(k^2/\Lambda^2)\,.$$

with the "cutoff factor" $\Phi(\xi)$ decaying at $\xi \to \infty$ fast enough to make the integral convergent. Then

$$\frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2 + m^2}\, \Phi(k^2/\Lambda^2) = \frac{\lambda}{2}\, F(\Lambda^2/m^2)\,.$$

Just as in the case of KG vacuum energy, this move alone brings no real progress - it replaces something meaningless by something undetermined.

In this simple diagram one can note that the contribution has no momentum dependence, and enters the quantity $\tilde{\Gamma}^{(2)}(p)$ in the combination

$$\tilde{\Gamma}^{(2)}(p) = p^2 + m^2 + \frac{\lambda}{2}\, m^2\, F(\Lambda^2/m^2) + O(\lambda^2)\,.$$

We see that no matter what Φ and Λ are, the parameter m in our action is not actual mass of the particle; the interaction leads to a shift that happens to depend on the cutoff (We will see later that actual mass $m_{\rm ph}$ of the particle in interacting theory is determined through the equation $\tilde{\Gamma}^{(2)}(p)\big|_{-p^2 = m_{\rm ph}^2} = 0$). Therefore it is reasonable to change the notations, denoting m_0^2 the coefficient in front of φ^2 in the action. Then, up to terms $\sim \lambda^2$,

$$\tilde{\Gamma}^{(2)}(p) = p^2 + \underbrace{m_0^2 + \frac{\lambda}{2}\, m_0^2\, F(\Lambda^2/m_0^2)}_{m^2}$$

Recall that we have already observed similar phenomenon in our path integral representation of the free propagator D. We started with a discrete approximation of the path integral (an analog of the cutoff), and we were forced to take some parameter m_0 instead of m in the cutoff action

$$\mathcal{A} = m_0\, \text{Length}\,;$$

we then found that m_0 had to be given some dependence on the cutoff parameter Δ (which plays the same role as Λ^{-1} here) to keep actual mass m finite in the limit $\Delta \to 0$.

Similar procedure can be applied here. Assume that the parameter m_0^2 in the action (m_0 is usually called the "bare mass") depends on Λ is such a way that the actual mass

$$m^2 = m_0^2 + \frac{\lambda}{2} m_0^2 F(\lambda^2/m_0^2) + O(\lambda^2)$$

remains finite as $\Lambda \to \infty$. Then, to this order,

$$\tilde{\Gamma}^{(2)}(p) = p^2 + m^2 + O(\lambda^2)$$

has finite limit. The result is expressed in terms of the actual mass; all ambiguities associated with the cutoff are "absorbed" by the "bare" mass parameter.

This idea can be refined as follows. We start with the action

$$\mathcal{A} = \int \left[\frac{1}{2} \left(\partial\varphi\right)^2 + \frac{m_0^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 \right].$$

Let us write

$$m_0^2 = m^2 + \delta m^2 .$$

where m^2 is assumed to coincide with actual mass squared; at this point the term δm^2 is just our way to acknowledge that the mass parameter in the action need not be equal to m^2. We then split the action into the free and interaction parts as follows

$$\mathcal{A} = \int \left[\underbrace{\left(\partial\varphi\right)^2 + \frac{m^2}{2} \varphi^2}_{\mathcal{A}_0} + \underbrace{\frac{\delta m^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4}_{\mathcal{A}_I} \right].$$

Now the propagator is

$$\xrightarrow{\quad p \quad} \quad = \quad \frac{1}{p^2 + m^2} ,$$

not $\frac{1}{p^2+m_0^2}$, as in the original perturbation theory, but we have an additional vertex

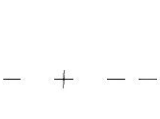
$$\xrightarrow{p_1} \times \xleftarrow{p_2} \quad = \quad -\delta m^2 \left(2\pi\right)^4 \delta^{(4)}(p_1 + p_2) ,$$

which we will rather write in a shorter form

$$\xrightarrow{p} \times \xrightarrow{p} \quad = \quad -\delta m^2 .$$

reflecting the fact that in this case the momentum delta function is easily integrated out.

The term $\frac{\delta m^2}{2}\varphi^2$ in the action is the simplest *counterterm*, and the associated two-leg vertex is called the counterterm vertex. In this modified perturbation theory we have (up to terms $\sim \lambda^2$ and higher)

$\tilde{\Sigma}(p)$ = $-\Big($  +  $\Big)$ =

$$= \frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+m^2}\,\Phi(k^2/\Lambda^2) + \delta m^2 = \frac{\lambda}{2}\,F(\Lambda^2/m^2) + \delta m^2\,.$$

Since we insist that m is the actual mass, we have to set

$$\delta m^2 = -\frac{\lambda}{2}\,m^2\,F(\Lambda^2/m^2) + O(\lambda^2)\,.$$

Note that this looks different from the relation

$$m_0^2 = m^2 - \frac{\lambda}{2}\,m_0^2\,F(\Lambda^2/m_0^2) + O(\lambda^2)$$

we have obtained within the original perturbation theory; however these two relations are consistent since the difference is $\sim \lambda^2$. The relation between the original and modified perturbation theories is illustrated by the solution of the following

As we have seen, in the modified perturbation theory, to the order $\sim \lambda$ the counterterm diagram exactly cancels the cutoff dependent bubble diagram,

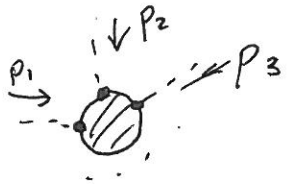
by choice of the parameter δm^2. The result has no dependence on Λ and Φ.Note also that this cancellation occurs inside more complicated diagrams, like

The idea behind this modified perturbation theory can be extended to obtain the so called *renormalized perturbation theory*, in which all divergences of original perturbation theory are eliminated. But first let's find out exactly what diagrams diverge.

Divergences in φ^4

Consider n-leg proper vertex

$\tilde{\Gamma}^{(n)}(p_1, \cdots, p_n)$ =



How to characterize diagrams which contribute to this vertex? Assume that the diagram contains

V vertices 

P propagators

I momentum integrals $\int \frac{d^4k}{(2\pi)^4}$.

These numbers are not all independent. First, V vertices must absorb exactly $4V$ lines, $2P$ coming from the propagators, and n from the external (amputated) legs, i.e. $4V = 2P + n$, or

$$2P = 4V - n\,.$$

Next, there is a 4-momentum flowing in each propagator, but there is also a momentum delta-function associated with each vertex. Each of these delta-functions reduces the number of 4-momentum integrations by 1. This is true for all delta-constraints but one, because the momentum-space connected correlation functions must have one overall delta function left, namely

$$(2\pi)^4\,\delta^{(4)}(p_1 + \cdots + p_n)\,.$$

Therefore

$$I = P - V + 1 = \big(P = 2V - n/2\big) = V - n/2 + 1\,.$$

Thus we have two equations

$$\begin{cases} 2P = 4V - n \\ 4I = 4V - 2n + 4 \end{cases}$$

Now, the expression for the diagram has the following general form

$$\int \frac{d^4k'_1 \cdots d^4k'_I}{(q_1^2 + m^2)\cdots(q_P^2 + m^2)}\,,$$

where $k'_1, \cdots, k'_I$ are the integration momenta left over after eliminating the delta-functions, and $q_1, \cdots, q_P$ are some linear combinations of k'_j and the external momenta p_i. For our present analysis it is irrelevant if we have m^2 or m_0^2 in the propagators.

We are concerned with possible divergence as $k'_j \to \infty$ (which generally imply $q_i \to \infty$). Roughly speaking, the above integral diverges if the overall power of the momenta in the integration measure in the numerator exceeds or equals the overall power in the denominator. Therefore let us define the *superficial degree of divergence* D as

$$D = 4I - 2P\,.$$

The integral definitely diverges if $D \geq 0$ (but $D < 0$ does not guarantee convergence, see below). If we introduce a cutoff momentum Λ, the diagram of the superficial degree of divergence D is $\sim \Lambda^D$.

Substituting the above expressions for P and I, we find

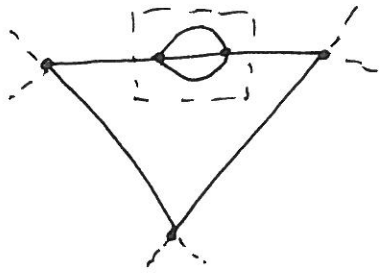
$$D = 4V - 2n + 4 - 4V + n = 4 - n\,.$$

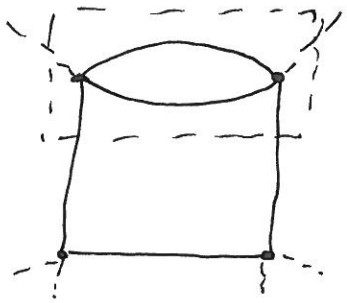
We see that the number of the vertices V cancels, and the superficial degree of divergence depends on n only. In addition, we see that only finite number of the proper vertices ($n = 0$, $n = 2$ and $n = 4$) are superficially divergent.

Incidentally, if a perturbation theory which has finitely many superficially divergent vertices (as the φ^4 perturbation theory does), it is called *renormalizable*, for the reason which will become clear later. Otherwise, when infinitely many proper vertices have are superficially divergent, we are dealing with *non-renormalizable* perturbation theory.

Exercise: Analyze the superficial degree of divergence for all proper vertices in φ^6 theory.

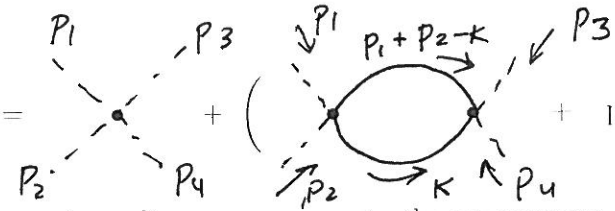
In the φ^4 perturbation theory, the proper vertices with $n > 4$ are superficially convergent. This does not mean all the diagrams for these vertices are given by convergent integrals (that is why we use the name "superficial" for that, essentially dimensional, analysis). The reason is that the diagrams for $\Gamma^{(n)}$ with $n \geq 6$ still can have 2- or 4-leg subdiagrams. For example, the diagram series for $\Gamma^{(6)}$ contains the diagrams like these





which contain divergent blocks with two or four legs. We will come back to decide how to handle such subdiagrams later.

The vertices with $D \geq 0$ (the so called *primitive divergences*) are $\Gamma^{(0)}$ and $\Gamma^{(4)}$. We have already observed the simplest divergent diagram for $\Gamma^{(2)}$. The diagrams for $\Gamma^{(4)}$ have the superficial degree of divergence $D = 0$: this means these diagrams are *logarithmically divergent*. To clarify the meaning of this term, let us consider the simplest connected diagrams for this vertex.

$-\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) =$ 

$+ \cdots$

The explicitly shown loop diagram represents the momentum integral

$$\frac{(-\lambda)^2}{2} \int \frac{d^4k}{(k^2 + m^2)((p_1 + p_2 - k)^2 + m^2)} .$$

When $|k| >> |p_1 + p_2|$ and $|k| >> m$ this reduces to

$$\frac{\lambda^2}{2\,(2\pi)^4} \int \frac{d^4k}{k^2\,k^2} .$$

Note that $d^4k = d\Omega_4\,|k|^3\,d|k|$, where $d\Omega_4$ is the surface element of a unit 3-sphere S^3, $\int_{S^3} d\Omega_4 = 2\pi^2$. We have

$$\frac{\lambda^2\,2\pi^2}{2\,(2\pi)^4} \int \frac{|k|^3\,d|k|}{|k|^4} = \frac{\lambda^2}{16\pi^2} \int \frac{d|k|}{|k|} .$$

The integral diverges at large $|k|$ (one shouldn't worry about its divergence at $|k| \to 0$: the above simplified form is not valid at small $|k|$). If one introduces a cutoff, say replaces

$$\frac{1}{k^2+m^2} \;\to\; \frac{1}{k^2+m^2}\,\Phi(k^2/\Lambda^2)$$

with some $\Phi(x)$ such that $\Phi(x) = 1$ for $x << 1$, but decays fast at $x >> 1$, the integral evaluates as

$$\frac{\lambda^2}{16\pi^2}\left(\log\frac{\Lambda^2}{m^2} + f(p_{12})\right),$$

where $f(p_{12})$ (I use $p_{12} = p_1 + p_2$) has finite limit as $\Lambda \to \infty$. Including the "permutations" terms, we then have

$$\tilde{\Gamma}^{(4)} = \lambda - \frac{\lambda^2}{16\pi^2}\left(3\,\log\frac{\Lambda^2}{m^2} + f(p_{12}) + f(p_{13}) + f(p_{14})\right).$$

One can observe that the divergent term carries no dependence of the external momenta, and in this respect comes on the same footing as the leading contribution λ. Therefore at least at the order λ^2 the Λ-dependent term can be absorbed into some redefinition of the coupling constant λ. This can be formulated more precisely as follows. Just as in the case of the mass renormalization discussed above, we should start with the action

$$\mathcal{A} = \int\left[\frac{1}{2}\left(\partial\varphi\right)^2 + \frac{m_0^2}{2}\,\varphi^2 + \frac{\lambda_0}{4!}\,\varphi^4\right].$$

with some "bare" coupling constant λ_0 (as well as the "bare" mass parameter m_0^2). Then it is possible to give λ_0 certain dependence on the cutoff momentum Λ,

$$\lambda_0 = \lambda_0(\Lambda)\,,$$

such that the limit $\Lambda \to \infty$ of $\tilde{\Gamma}^{(4)}$ in this order of the perturbation theory is finite. Namely, let us set

$$\lambda_0(\Lambda) = \lambda + \frac{3\,\lambda^2}{16\pi^2}\left(\log\frac{\Lambda^2}{m^2} + C\right) + O(\lambda^3)\,,$$

where new parameter λ is called the *renormalized* coupling constant, and C is arbitrary number. We then have

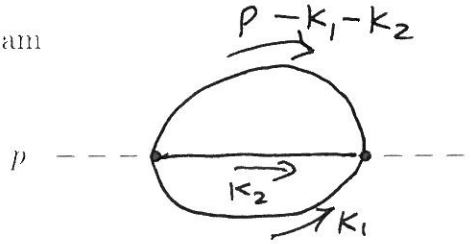
$$\tilde{\Gamma}^{(4)} = \lambda_0 - \frac{\lambda_0^2}{16\pi^2}\left(3\,\log\frac{\Lambda^2}{m^2} + f'(p)\right) + O(\lambda_0^3) =$$

$$\lambda + \frac{3\,\lambda^2}{16\pi^2}\left(\log\frac{\Lambda^2}{m^2} + C\right) - \frac{\lambda^2}{16\pi^2}\left(3\,\log\frac{\Lambda^2}{m^2} + \tilde{f}(p)\right) + O(\lambda^3)\,.$$

(Here $\tilde{f}(p) = f(p_{12}) + f(p_{13}) + f(p_{14})$). The Λ-dependent term cancels out. Assuming that λ is fixed in the limit $\Lambda \to \infty$, the above proper vertex has finite limit when the cutoff is removed.

Later we shall formulate this renormalization prescription with full precision, and in particular explain the meaning of this constant C. But for now we still have one more divergence to deal with.

Consider the diagram



contributing to $-\tilde{\Sigma}(p)$ in the order λ^2. I will denote this contribution as $-\tilde{\Sigma}_2(p)$. According to the general counting above, this diagram has superficial degree of divergence $D = 2$, i.e. with the cutoff introduced it behaves as Λ^2 as $\Lambda \to \infty$. We can see this writing the contribution explicitly,

$$-\tilde{\Sigma}_2(p) = \frac{(-\lambda)^2}{3!}\int \frac{d^4k_1}{(2\pi)^4}\,\frac{d^4k_2}{(2\pi)^4}\,\frac{1}{(k_1^2+m^2)(k_2^2+m^2)((p-k_1-k_2)^2+m^2)}$$

(note the symmetry factor 1/3!). For $k_1, k_2 >> p, m$ the integrand simplifies as

$$\sim \int \frac{d^4k_1\,d^4k_2}{k_1^2\,k_2^2\,(k_1+k_2)^2} \sim \Lambda^2\,.$$

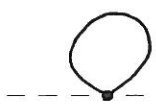
Let us write

$$\tilde{\Sigma}_2(p) = \tilde{\Sigma}_2(0) + \left(\tilde{\Sigma}_2(p) - \tilde{\Sigma}_2(0)\right).$$

Here

$$\tilde{\Sigma}_2(0) = -\frac{\lambda^2}{3!}\int \frac{d^4k_1\,d^4k_2}{(2\pi)^8}\,\frac{1}{(k_1^2+m^2)(k_2^2+m^2)((k_1+k_2)^2+m^2)}\,.$$

This divergent term does not depend on p; in this respect it is similar to the contribution of the diagram



.

Clearly, it also can be absorbed into the mass renormalization, by suitable modification of the counterterm

$$\frac{\delta m^2(\Lambda)}{2}\,\varphi^2\,.$$

The difference

$$\tilde{\Sigma}_2(p) - \tilde{\Sigma}_2(0) =$$

$$-\frac{\lambda^2}{3!}\int \frac{d^4k_1\,d^4k_2}{(2\pi)^8}\,\frac{1}{(k_1^2+m^2)(k_2^2+m^2)}\,\frac{2(pk_1)+2(pk_2)-p^2}{((k_1+k_2)^2+m^2)((k_1+k_2-p)^2+m^2)}$$

still diverges (logarithmically). If the cutoff is introduced, this contribution is written as

$$\sim p^2\,\log\frac{\Lambda^2}{m^2} + \text{finite}\,.$$

This divergent term cannot be absorbed by the mass renormalization, since it depends on the momentum p. Note however that this dependence has simple form $\sim p^2$.

Recall that the proper vertex $\Gamma^{(2)}$ has the form

$$\tilde{\Gamma}^{(2)}(p) = p^2 + m^2 + \text{loop corrections}\,.$$

where the p^2 term originates from the term

$$\frac{1}{2}\left(\partial\varphi\right)^2$$

term in the original action. This suggests that the above divergent term $\sim p^2$ can be understood in terms of the "field renormalization". Originally, the field φ appeared in the theory as the integration variable in the functional integral. Let us rename this integration variable as φ_0, and call it the "bare field". The new divergent term shows that the correlation functions of this bare field retains the dependence on the cutoff parameter Λ, even after the mass and coupling renormalizations are carried out. The way this dependence appears (through the p^2 term) suggests that one can define renormalized field,

$$\varphi(x) = Z^{-1/2}(\Lambda)\,\varphi_0(x)\,,$$

with certain cutoff-dependent renormalization constant $Z(\Lambda)$, such that the correlation functions

$$\langle\varphi(x_1)\cdots\varphi(x_n)\rangle$$

admit finite $\Lambda\to\infty$ limit.

11 Lecture 11

We are ready to formulate the *renormalization program*.

Renormalization program (φ^4 theory)

1. We start with the action

$$\mathcal{A} = \int d^4x \left[\frac{1}{2} \left(\partial\varphi_0\right)^2 + \frac{m_0^2}{2} \varphi_0^2 + \frac{\lambda_0}{4!} \varphi_0^4 \right],$$

which involves the bare field φ_0, the bare mass m_0, and the bare coupling constant λ_0.

2. We introduce some cutoff, with the cutoff momentum Λ, to make all diagram contributions finite but cutoff-dependent (This can be done in many ways, as we shall discuss later).

3. The bare parameters m_0^2, λ_0, and the field renormalization constant Z should be given certain dependence on the cutoff momentum Λ,

$$m_0^2(\Lambda), \quad \lambda_0(\Lambda), \quad Z(\Lambda),$$

such that the correlation functions of the renormalized field $\varphi(x) = Z^{-1/2} \varphi_0(x)$ will have finite $\Lambda \to \infty$ limit.

In practice, one uses the perturbation theory to evaluate the correlation functions. The above program can be further reformulated as the

Renormalized perturbation theory

Let us rewrite the action as

$$\mathcal{A} = \int d^4x \left[\underbrace{\frac{1}{2} \left(\partial\varphi\right)^2 + \frac{m^2}{2} \varphi^2}_{\text{free part}} \underbrace{- \frac{\lambda}{4!} \varphi^4}_{\text{interaction part}} + \underbrace{\frac{\delta Z}{2} \left(\partial\varphi\right)^2 + \frac{\delta m^2}{2} \varphi^2 + \frac{\delta\lambda}{4!} \varphi^4}_{\text{counterterms}} \right],$$

where m is the actual mass and λ is suitably defined renormalized coupling constant (to be discussed later), and φ is the renormalized field. The last three terms are the *counterterms*. The identity with the original action implies

$$1 + \delta Z = Z.$$

$$m^2 + \delta m^2 = Z\, m_0^2 ,$$
$$\lambda + \delta\lambda = Z^2\, \lambda_0 .$$

In perspective, the renormalized perturbation theory is an expansion in the renormalized coupling constant λ. We should assume that the counterterm coefficients δZ, δm^2, and $\delta\lambda$ themselves depend perturbatively (i.e. as the power series) on λ,

$$\delta Z = z_2\,\lambda^2 + z_3\,\lambda^3 + \cdots , \qquad \delta m^2 = b_1\,\lambda + b_2\,\lambda^2 + \cdots , \qquad \delta\lambda = a_2\,\lambda^2 + a_3\,\lambda^3 + \cdots ,$$

with Λ-dependent coefficients. We must show that these coefficients can be adjusted in such a way that the correlation functions of the renormalized field φ are Λ-independent order by order in λ.

Let me stress that this renormalization program is based on our observation that the φ^4 perturbation theory contains only finitely many basic types of divergent diagrams (the "primitive divergences"). Namely, the primitively divergent diagrams are in $\Gamma^{(2)}$ and $\Gamma^{(4)}$. The diagrams for higher vertices $\Gamma^{(n)}$ can be divergent only through subdiagrams of the $\Gamma^{(2)}$ and $\Gamma^{(4)}$ types. The idea of the renormalization program is that the above three primitive divergences can be absorbed in three renormalizations

$$m_0^2(\Lambda) = Z^{-1}(\Lambda)\,(m^2 + \delta m^2(\Lambda)) ; \quad \lambda_0(\Lambda) = Z^{-2}(\Lambda)\,(\lambda + \delta\lambda(\Lambda)) ; \quad \varphi_0 = Z^{\frac{1}{2}}(\Lambda)\,\varphi ,$$

order by order in the renormalized coupling constant λ.

Before we turn to execution of this program in φ^4 theory let us briefly discuss divergences and renormalizations in a scalar field theory whose interaction term is more general polynomial in φ.

Divergences in Scalar Theories

It is instructive to consider such field theory in the space of d dimensions, with generic d. We will consider general scalar theories with the actions of the form

$$\mathcal{A} = \int d^d x \left[\frac{1}{2}\,(\partial\varphi_0)^2 + \frac{m_0^2}{2}\,\varphi_0^2 + \sum_{k=3}^{N} \frac{\lambda_{0\,k}}{k!}\,\varphi^k \right] . \tag{11.1}$$

The Feynman rules remain the same as in 4-dimensional theory, except for the momentum integrations go over d-dimensional momentum space,

$$\frac{d^4k}{(2\pi)^4} \to \frac{d^dk}{(2\pi)^d} ,$$

and of course the diagrams contain k-leg vertices associated with the couplings $\lambda_{0\,k}$.

As before, we can consider a generic diagram contributing to $\tilde{\Gamma}^{(n)}$, containing P propagators and I d-momentum integrations. Because now the momenta are d-dimensional, the superficial degree of divergence of this diagram is

$$D = d \cdot I - 2 \cdot P \,.$$

Assume that the diagram contains V_k k-leg vertices. Analysis similar that we have made for φ^4 theory reveals two identities

$$2P - n = \sum_k k \cdot V_k \,,$$

$$I = P - \sum_k V_k + 1 \,.$$

Combining these equations one finds

$$D = \sum_k \left(\frac{d-2}{2} k - d \right) V_k - \frac{d-2}{2} n + d \,.$$

This formula admits simple interpretation in terms of dimensional counting. Note that in our units $c = \hbar = 1$ there is only one independent unit, which we take to be mass unit. I will denote by $[X]$ the mass dimension of a quantity X, for example

$$[mass] = 1 \,, \quad [length] = -1 \,.$$

The action is dimensionless and therefore it follows from its form that

$$[\varphi_0] = \frac{d-2}{2} \,, \quad [m_0^2] = 2 \,, \quad [\lambda_{0\,k}] = d - \frac{d-2}{2} k \,. \tag{11.2}$$

I want to note in passing that this simple dimensional analysis is applied to the *bare* quantities. We will see later that because of renormalization constant $Z(\Lambda)$ the renormalized field φ can have different dimension. This is why the above dimensions in (11.2) are often referred to as canonical (or "engineering") dimensions.

It is easy to check that

$$[\tilde{\Gamma}^{(n)}] = d - \frac{d-2}{2} n \,.$$

Therefore the equation for the superficial degree can be written as

$$D = [\tilde{\Gamma}^{(n)}] - \sum_k [\lambda_{0\ k}] \cdot V_k \,. \tag{11.3}$$

Now the meaning of this equation must be obvious. For $\Lambda >> |p_i|$ the dominating contribution of the diagram with V_k vertices λ_k is

$$\tilde{\Gamma}^{(n)} \sim \left[\prod_k \lambda_{0\ k}^{V_k} \right] \Lambda^D \,, \tag{11.4}$$

and our equation simply describes the balance of dimensions.

It is clear from the Eq.(11.3) that the mass dimensions of the coupling constants $\lambda_{0\ k}$ must play the key role in the analysis of the perturbative divergences. Suppose some coupling $\lambda_{0\ k}$ in the action have strictly negative mass dimension, $[\lambda_{0\ k}] < 0$. Then there are divergent contributions to $\tilde{\Gamma}^{(n)}$ with any n which come from the diagrams with sufficiently large V_k. In other words such theory has infinitely many primitive divergences which of course cannot be absorbed by any finite number of counterterms. Quantum field theories of this type are called (perturbatively) *unrenormalizable*. Overall consistency of unrenormalizable theories is questionable (but not impossible). From pure pragmatic point of view, the necessity to introduce infinitely many counterterms brings in also infinitely many free parameters, and predictive power of such theories is limited.

If mass dimensions of all coupling constants in the action are non-negative, the equation (11.3) shows that there is only finite number of primitively divergent proper vertices (this is true if $d > 2$, the case $d = 2$ is exceptional and must be studied separately); these divergences then can be absorbed by finitely many renormalization constants. For this reason such theories are called (perturbatively) *renormalizable*. Moreover, if all $\lambda_{0\ k}$ have strictly positive mass dimensions there is only finite number of divergent diagrams. The theories of this last kind are often referred to as *super-renormalizable*, in which case the term "renormalizable" is reserved for the theories with infinitely many divergent diagrams but finitely many primitive divergences; in view of (11.3) this requires that at least one of the coupling constants is dimensionless. We will see later when studying renormalization group that super-renormalizable scalar theories (with the polynomial $V(\varphi_0)$ bounded from below) are consistent field theories. Overall consistency of a renormalizable theories require more subtle analysis, but at least they make sense perturbatively.

One observes from (11.2) that for $d > 6$ there are no (interacting) renormalizable scalar theories. For $6 \geq d > 2$ renormalizable field theories are

$$\varphi^3 \qquad\qquad \text{in} \quad d = 6 \quad \text{dimensions} \,,$$

$$\varphi^3, \varphi^4 \qquad \text{in} \quad d = 4 \quad \text{dimensions},$$
$$\varphi^3, \varphi^4, \varphi^5, \varphi^6 \qquad \text{in} \quad d = 3 \quad \text{dimensions}.$$

It is not difficult to show that any polynomial in φ is super-renormalizable in $d = 2$ dimensions.

I want to stress here that nonexistence of perturbatively renormalizable field theories in high dimensionalities does not imply that consistent field theories are limited to low space-time dimensions; there may exist perfectly consistent field theories which are just "too far" from free field theory to admit meaningful perturbative interpretation. I hope to discuss this possibility in greater details later.

Systematics of Renormalization

If the theory under consideration is renormalizable, like φ^4 theory in 4 dimensions, one can develop renormalizable perturbation theory. According to the program outlined above one has to start with the action in its "bare" form, implicitly equipped with some "regularization", or "cutoff", i.e. with some modification of the theory at the momenta $\geq \Lambda$ which makes all integrals in perturbation theory convergent. There are many possible implementations of the cutoff which can be more or less convenient depending on circumstances. Let us discuss some more frequently used ones.

Regularization methods

Lattice regularization: We already discussed this regularization in relation with the definition of the functional integral. In this approach continuous space R^d is replaced by d-dimensional, say hypercubic, lattice with some lattice spacing Δ which plays the role of inverse cutoff parameter Λ^{-1},

$$x \to x_n = \Delta \sum_{\mu=1,\cdots,d} e_\mu n_\mu ,$$

where e_μ is the unit vector in the direction μ and $n_1, \cdots, n_d$ are integers. The lattice action is obtained by replacing derivatives by finite differences,

$$\mathcal{A}_{lat} = \Delta^d \sum_{x \in \Delta Z^d} \left[\frac{1}{2} \sum_{\mu=1,\cdots,d} \left(\frac{\varphi_0(x + \Delta e_\mu) - \varphi_0(x)}{\Delta} \right)^2 + \frac{m_0^2}{2} \varphi_0^2(x) + V(\varphi_0(x)) \right] \quad (11.5)$$

and the integration measure in the functional integral is taken as

$$[D\varphi] \to \prod_{x \in \Delta Z^d} d\varphi_0(x) \,.$$

Advantage of this method is that it makes sense nonperturbatively. The lattice theory (11.5) is perfectly meaningful in classical statistical mechanics. At the same time perturbative calculations in this approach are relatively complicated. In addition, continuous space symmetries are broken, and one only expects them to restore after the cutoff is removed.

Exercise: Develop the momentum-space Feynman rules in the above lattice theory with $V(\varphi) = \frac{\lambda_0}{4!}\,\varphi^4$. In particular, find explicit form of the propagator $\tilde{D}(p)$, and compare it with the finite Δ expression in the Problem 5. Why does this regularization makes all diagrams finite?

Proper-time regularization: We have seen before that the momentum-space propagator $\tilde{D}(p)$ admits Schwinger's proper-time representation

$$\tilde{D}(p) = \int_0^\infty ds\, e^{-s(p^2+m^2)} \,,$$

where s is interpreted as "renormalized length" of the path of relativistic particle in the Euclidean space-time. One can exclude the path which are "too short" by replacing $\tilde{D}(p) \to \tilde{D}_\Lambda(p)$.

$$\tilde{D}_\Lambda(p) = \int_{1/\Lambda^2}^\infty ds\, e^{-s(p^2+m^2)} \,.$$

Note that this is particular case of the regularized propagator we considered before,

$$\frac{1}{p^2+m^2}\,\Phi(p^2/\Lambda^2) \,. \tag{11.6}$$

$\Phi(x) = e^{-x}$.

Pauli-Villars regularization is another version of (11.6), with

$$\Phi_{PV}(p^2/\Lambda^2) = \frac{\Lambda^2 - m^2}{p^2+\Lambda^2} \,.$$

In this case

$$\tilde{D}(p) \to \frac{1}{p^2+m^2}\,\Phi(p^2/\Lambda^2) = \frac{1}{p^2+m^2} - \frac{1}{p^2+\Lambda^2}\,.$$

Dimensional regularization: This is the most technically advanced (although perhaps the least physically transparent) regularization method. Its advantage is that it usually preserves important symmetries (in particular the gauge symmetry in the gauge theories) and significantly simplifies calculations, and for these reasons this method is used today in perturbative calculations more frequently then others. The idea is this. We have already observed that lowering the space dimension generally improves the large-momentum convergence of Feynman diagrams. Suppose that for d sufficiently low the momentum integral associated with given diagram is convergent without any additional cutoff. Suppose that in addition we have managed to calculate this integral as an analytic functions of d. Then we can analytically continue the result to the "physical value" $d = 4$; the 4-dimensional divergences manifest themselves as singularities (poles) in the variable d at $d = 4$. On more formal level one can define the dimensional continuation of momentum integrals by three conditions (obviously valid for convergent integrals):

$$(i) \qquad \int d^d k\, F(k+p) = \int d^d k\, F(k) \qquad \text{translation}$$

$$(ii) \qquad \int d^d k\, F(C\,k) = |C|^{-d} \int d^d k\, F(k) \qquad \text{dilatations}$$

$$(iii) \qquad \int d^d k\, d^{d'} k'\, F(k) G(k') = \int d^d k\, F(k) \int d^{d'} k'\, G(k') \qquad \text{factorization}$$

Let me illustrate this approach by simple example. Consider the integral

$$\int \frac{d^d k}{(2\pi)^d}\,\frac{1}{(k^2+m^2)^2}\,. \tag{11.7}$$

This integral converges for $d < 4$. For any $d < 4$ it can be transformed as follows

$$\int_0^\infty s ds \int \frac{d^d k}{(2\pi)^d}\, e^{-s(k^2+m^2)} = \int_0^\infty s ds\, (4\pi s)^{-\frac{d}{2}}\, e^{-m^2 s}\,. \tag{11.8}$$

where the properties (iii) and (ii) were used in evaluating the k-integral. Using the definition of the Euler's Gamma function

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1}\, e^{-t}$$

we obtain for $d < 4$

$$\frac{(m^2)^{\frac{d}{2}-2}}{(4\pi)^{\frac{d}{2}}}\,\Gamma\left(2-\frac{d}{2}\right)\,. \tag{11.9}$$

As is known the Gamma function $\Gamma(z)$ is analytic - actually, meromorphic (Google the term!) - function of z with poles at $z = 0, -1, -2, \cdots$. In particular, the above expression has pole at $d = 4$, i.e. exactly where the integral diverges logarithmically. For $d = 4 - \epsilon$, $\epsilon \to 0$

$$\Gamma\left(2-\frac{d}{2}\right) = \frac{2}{\epsilon} - \gamma + O(\epsilon)$$

where $\gamma = 0.5772..$ is the Euler's constant.

It is interesting to explain on this example the interplay between dimensional continuation and cutoff regularization. If we regularize the propagators in (11.7), say by proper-time regularization (which amounts to replacing $\int_0^\infty ds \to \int_{1/\Lambda^2}^\infty ds$ in (11.8)) the integral (11.7) becomes regular function of d for all d including $d = 4$. In the neighborhood of $d = 4$ one would have instead of (11.8)

$$\frac{(m^2)^{\frac{d}{2}-2}}{(4\pi)^{\frac{d}{2}}}\int_{m^2/\Lambda^2}^\infty dt\, t^{1-\frac{d}{2}}\, e^{-t}\,.$$

The integral here can be transformed using integration by parts,

$$\int_{m^2/\Lambda^2}^\infty dt\, t^{1-\frac{d}{2}}\, e^{-t} = \frac{2}{4-d}\left(t^{2-\frac{d}{2}}\, e^{-t}\Big|_{\frac{m^2}{\Lambda^2}}^\infty + \int_{m^2/\Lambda^2}^\infty dt\, t^{2-\frac{d}{2}}\, e^{-t}\right).$$

For $\epsilon = 4 - d << 1$ and $m^2 << \Lambda^2$ the cutoff integral becomes

$$\frac{1}{(4\pi)^{\frac{d}{2}}}\,\frac{2}{4-d}\left((m^2)^{\frac{d}{2}-2} - (\Lambda^2)^{\frac{d}{2}-2}\right).$$

If at fixed $d < 4$ we send the cutoff momentum Λ to infinity we obtain the "dimensionally continued" expression with the pole at $d = 4$. On the other hand, if at fixed

Λ we take the limit $d \to 4$ we get a finite result with $\log \Lambda^2$ replacing the pole in $4 - d$.

With this understanding one can often use dimensional continuation to do the calculations efficiently and then interpret the results in terms of cutoff regularization. Instead of giving the bare parameters m_0^2, λ_0 and Z dependence on the cutoff momentum Λ, within the framework of dimensional regularization one adjusts their dependence on $\epsilon = 4 - d$, so that that the correlation functions of renormalized fields φ have finite limit $d \to 4$.

Some generalizations of the integral (11.7) will be useful in future calculations. For instance,

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m^2)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty ds\, s^{\alpha - 1} \int \frac{d^d k}{(2\pi)^d} e^{-s(k^2 + m^2)} = \frac{(m^2)^{\frac{d}{2} - \alpha}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)}, \tag{11.10}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 + m^2)^\alpha} = \int \frac{d^d k}{(2\pi)^d} \left[\frac{1}{(k^2 + m^2)^{\alpha - 1}} - \frac{m^2}{(k^2 + m^2)^\alpha} \right] = \frac{(m^2)^{\frac{d}{2} - \alpha + 1}}{(4\pi)^{\frac{d}{2}}} \frac{d}{2} \frac{\Gamma(\alpha - \frac{d}{2} - 1)}{\Gamma(\alpha)}. \tag{11.11}$$

In deriving (11.11) the defining property of the Γ function $\Gamma(z + 1) = z\,\Gamma(z)$ was used.

12 Lecture 12

Renormalizations in φ^4 theory (cont'd)

Let us recall the basic idea of the (perturbative) renormalization program. To develop renormalized perturbation theory, one starts with the action

$$\mathcal{A} = \int d^d x \left[\frac{1}{2} (\partial\varphi)^2 - \frac{m^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 + \text{counterterms} \right],$$

expressed in terms of the renormalized quantities. I have written it in the d-dimensional form having in mind dimensional regularization. This action contain counterterms

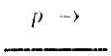
$$\text{counterterms} = \frac{\delta Z}{2} (\partial\varphi)^2 - \frac{\delta m^2}{2} \varphi^2 + \frac{\delta\lambda}{4!} \varphi^4 .$$

We are planning to treat the theory perturbatively, with

$$\mathcal{A}_I = \int_x \left[\frac{\lambda}{4!} \varphi^4 + \text{counterterms} \right]$$

taken as the perturbation.

The counterterms bring in additional vertices, so the diagrams will contain the following elements (momentum δ-functions are implicit)


$= \dfrac{1}{p^2 + m^2}$.


$= -\lambda$.


$= -(p^2 \delta Z + \delta m^2)$,


$= -\delta\lambda$.

The counterterms δZ, δm^2 , $\delta\lambda$ are assumed to be power series in λ, i.e.

$$\delta Z = z^{(1)} \lambda + z^{(2)} \lambda^2 + \cdots .$$

$$\delta m^2 = b^{(1)}\,\lambda + b^{(2)}\,\lambda^2 + \cdots\,,$$
$$\delta\lambda = a^{(1)}\,\lambda + a^{(2)}\,\lambda^2 + \cdots\,,$$

with the coefficients dependent on Λ, or, in dimensional regularization, on

$$\epsilon = 4 - d\,.$$

in such a way that all diagram contributions to the correlation functions of renormalized fields have finite limit $\lambda \to \infty$, or, respectively, $\epsilon \to 0$, order by order in λ. This is general strategy of renormalized perturbation theory. Specific calculations depend on the choice of the so called "Renormalization Scheme".

Renormalization Schemes

Obviously, just satisfying this requirement - that the counterterms cancell all singular contributions of ordinary diagrams - is not sufficient to fix the counterterms uniquely. For instance, the renormalized field φ always can be renormalized further by multiplying it by a finite (i.e. regular at $\epsilon = 0$) constant

$$\varphi = Z_{\text{finite}}^{\frac{1}{2}}\,\tilde{\varphi}\,;$$

similar finite renormalizations of m^2 and λ are possible. Obviously, such renormalizations do not spoil finiteness of renormalized perturbation theory. We see that there is a whole class of renormalizable perturbation theories, related one to another by finite renormalizations. These different renormalized perturbation theories are called the *Renormalization Schemes*. Of course all these renormalized perturbation theories describe the same field theory, and yield the correlation functions with identical physical content. What is different, besides finite renormalizations of the correlation functions due to the field renormalizations, is *parametrization* of the correlation functions. If φ, m^2, λ and $\tilde{\varphi}$, $\tilde{m}^2$, $\tilde{\lambda}$ are renormalized quantities in two different renormalization schemes, the corresponding correlation functions are related as

$$\Gamma^{(n)}(p_i|m^2,\lambda) = \left[Z_{\text{finite}}(m^2,\lambda)\right]^{-\frac{n}{2}}\,\Gamma^{(n)}(p_i|\tilde{m}^2(m^2,\lambda),\tilde{\lambda}(m^2,\lambda))\,.$$

One can think of the parameters (m^2,λ) as coordinates in the "space of φ^4 theories". Different renormalization schemes correspond to different coordinate systems in that space.

A renormalization scheme is usually defined by certain *Normalization conditions*.

Normalization conditions

These are conditions that specify the choice of the renormalized parameters m^2, λ, as well as the overall normalization of φ. This is usually done by relating the renormalized parameters to the physical quantities (this is not so in the so called "minimal subtraction scheme" which has more technical definition; we will discuss it in some details later). In practice, the schemes where these relations are simple are more convenient.

One such normalization condition we have discussed aleady when we defined the renormalized mass parameter m^2 as the physical mass sqared. Namely, assuming that $\tilde{\Gamma}^{(2)}(p^2)$ has zero at some negative value of p^2 we have defined $-m^2$ as the position of this zero. Note that this condition does not depend on the overall normalization of φ. The later must be fixed by a separate condition. One can fix for instance the slope of $\tilde{\Gamma}^{(2)}(p^2)$ at this zero $p^2 = -m^2$; a convenient normalization condition is to set this slope equal 1, i.e.

$$\tilde{\Gamma}^{(2)}(p^2) = p^2 + m^2 - O((p^2+m^2)^2) \qquad \text{as} \qquad p^2 + m^2 \to 0\,. \tag{12.1}$$

Equivalently, this condition means that the two-point correlation function

$$\tilde{W}(p) = \frac{1}{p^2 - m^2} - O(1) \qquad \text{as} \qquad p^2 + m^2 \to 0 \tag{12.2}$$

has a pole at $p^2 = -m^2$ with the residue one. We also need to fix normalization of λ. This usually is done by imposing certain condition on $\tilde{\Gamma}^{(4)}$. The condition often taken in high-energy theory calculations is

$$\tilde{\Gamma}^{(4)}(p,p,-p,-p)\Big|_{p^2=-m^2} = \lambda \tag{12.3}$$

(the quantity in the left-hand side is related to $2 \to 2$ forward scattering amplitude at zero energy). The equations (12.2) and (12.3) above specify one possible renormalization scheme (I will refer to it as the scheme SI).

Of course there are other schemes. Another scheme which is more frequently used in statistical mechanics context is specified by the normalization conditions

$$\tilde{\Gamma}^{(2)}(p^2) = \tilde{m}^2 + p^2 + O(p^4)\,, \tag{12.4}$$

$$\tilde{\Gamma}^{(4)}(0,0.0,0) = \tilde{\lambda}\,. \tag{12.5}$$

Here I have used the notations $\tilde{m}^2$ and $\tilde{\lambda}$ for the associated renormalized parameters to stress that these parameters are different from those defined in the scheme (SI) (in particular $\tilde{m}^2$ does not coinside with the physical mass), i.e. it defines different "coordinates". Henceforth this schime is referred to as SII.

Let us see how this program works in few leading orders in the coupling constant. The following calculation is equivalent to the one in PS, §10.2, 10.5. There it is done using the normalization condition (SI). For the sake of diversity I will instead adopt (SII). To facilitate the comparison I will use notations $\tilde{m}^2$ and $\tilde{\lambda}$ for the renormalized parameters. Also, to simplify notations I will write $\Gamma^{(n)}$ for the *momentum space* proper vertices (omitting the tilde).

Renormalization at the leading (one loop) order

Consider first the two-point function,

$$\Gamma^{(2)}(p^2) = p^2 + \tilde{m}^2 + \Sigma(p^2)\,.$$

The normalization condition (12.4) states

$$\Sigma(0) = 0: \qquad \Sigma'(0) = 0\,. \tag{12.6}$$

In the leading order in $\tilde{\lambda}$ there are two diagrams contributing to Σ,

$$-\Big(\quad \underset{(a)}{\text{---}\bigcirc\text{---}} \quad + \quad \underset{(b)}{\overset{(1)}{\text{---}\otimes\text{---}}} \quad\Big)\,,$$

where the diagram (b) comes from the $\sim \tilde{\lambda}$ terms in δZ and $\delta\tilde{m}^2$,

$$\overset{(1)}{\text{---}\otimes\text{---}} \quad = \quad -\tilde{\lambda}\,(z^{(1)}\,k^2 + b^{(1)})\;;$$

The coefficients $z^{(1)}$ and $b^{(1)}$ are to be adjusted to satisfy (12.6).

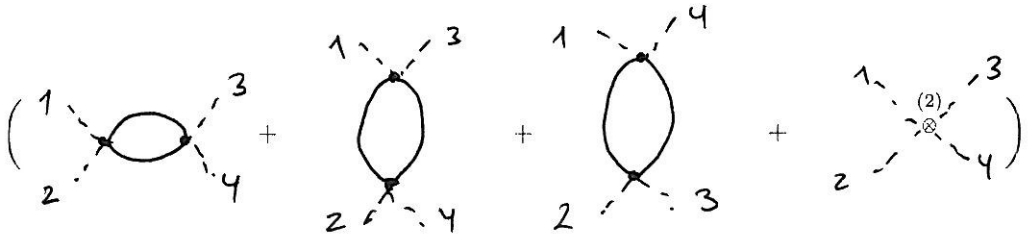
We already have considered the diagram (a). It is a constant independent on p^2. Therefore to satisfy (12.6) we have to choose

$$z^{(1)} = 0\;; \qquad b^{(1)} = \;(a)\; = -\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2+\tilde{m}^2} = -\frac{(\tilde{m}^2)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\,\Gamma(1-d/2)\,,$$

where the dimensional regularization was used to evaluate the integral. Note that at this order there is a trivial cancelation between the diagrams (a) and (b). Because of this, in considering higher-order diagrams we can in fact set $b^{(1)} = 0$ if we at the same time decide to ignore the "bubble" contributions (a) in all propagators.

Consider now the leading perturbative corrections to the four-point function $\Gamma^{(4)}$. At the tree level $\Gamma^{(4)} = \tilde{\lambda}$ and the normalization condition (12.5) is satisfied at this order if we set $a^{(1)} = 0$, i.e. $\delta\tilde{\lambda} = O(\tilde{\lambda}^2)$. Nontrivial contributions appear at the order $\tilde{\lambda}^2$; the corresponding diagrams are

$$\Gamma^{(1)}(p_1, \cdots, p_4) = \quad \tilde{\lambda} \quad -$$



(a) (b) (c) (d)

where again the diagram (d) represents the contribution of the counterterm $\delta\lambda$ in the order $\tilde{\lambda}^2$,

$$= \quad -\tilde{\lambda}^2\, a^{(2)}\,. \tag{12.7}$$

Each of the diagrams (a-c) can be expressed through the integral

$$I(p^2) = \int \frac{d^d k}{(2\pi)^d}\, \frac{1}{(k^2 - \tilde{m}^2)((p+k)^2 - \tilde{m}^2)}\,, \tag{12.8}$$

namely

$$(a) + (b) + (c) = \frac{\tilde{\lambda}^2}{2}\left(I(p_{12}^2) + I(p_{13}^2) + I(p_{14}^2)\right),$$

where $p_{12} = p_1 + p_2$, $p_{13} = p_1 + p_3$, $p_{14} = p_1 + p_4$.

The normalization condition (12.5) requires that

$$\Big[(a)+(b)+(c)+(d)\Big]_{p_i=0}=0\,,$$

which fixes the value of the counterterm constant in (12.7).

$$a^{(2)}=\frac{3}{2}\,I(0)\,.$$

With this we obtain

$$\Gamma^{(4)}(p_1,\cdots,p_4)=\tilde{\lambda}-\frac{\tilde{\lambda}^2}{2}\left(I_r(p_{12}^2)+I_r(p_{13}^2)+I_r(p_{14}^2)\right)+O(\tilde{\lambda}^3)\,, \tag{12.9}$$

where

$$I_r(p^2)=I(p^2)-I(0)\,.$$

Our goal now is to show that while the integral (12.8) is singular at $d=4$ the combination appearing in (12.9) has finite limit at $d=4$.

The integrand in (12.8) can be transformed using the so called *Feynman parametrization*. The identity[5]

$$\frac{1}{A\,B}=\int_0^1\frac{du}{(uA-(1-u)B)^2}$$

which can be checked by explicit evaluation of the integral in the r.h.s., allows one to write the integral (12.8) as

$$I(p^2)=\int_0^1 du\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{(\tilde{m}^2+k^2+2u\,(kp)+u\,p^2)^2}\,. \tag{12.11}$$

The combination in the denominator can be put in the form

$$\tilde{m}^2+k^2-2u\,(kp)-u\,p^2=\tilde{m}^2+u(1-u)\,p^2+(k-u\,p)^2\,.$$

[5]More general identity

$$\frac{1}{A^\alpha}\frac{1}{B^\beta}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\frac{dw\,w^{\alpha-1}(1-w)^{\beta-1}}{[wA+(1-w)B]^{\alpha+\beta}} \tag{12.10}$$

is often useful. To verify it, change the integration variable to $w=Bz/(Bz+A(1-z))$ to reduce the integral to the Euler beta integral

$$B(\alpha,\beta):=\int_0^1 dz\,z^{\alpha-1}(1-z)^{\beta-1}=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,.$$

and by applying the translation property (i) of the dimensionaly continued integrals we have

$$I(p^2) = \int_0^1 du \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + u(1-u)\,p^2 + \tilde{m}^2)^2} \,.$$

The momentum integral has the form (11.10) with $\alpha = 2$ (see Lecture 11), and hence

$$I(p^2) = \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 du \,[\tilde{m}^2 + u(1-u)\,p^2]^{\frac{d}{2}-2} \,.$$

We see that due to the factor $\Gamma(2 - d/2) = 2/\epsilon - \gamma + O(\epsilon)$ this expression has pole at $d = 4$. However the combination

$$I_r(p^2) = I(p^2) - I(0) = \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 du \left([\tilde{m}^2 + u(1-u)\,p^2]^{\frac{d}{2}-2} - [\tilde{m}^2]^{\frac{d}{2}-2} \right)$$

has finite value at $d = 4$ as the result of compensation between the pole of the gamma function and zero of the integrand. One finds

$$I_r(p^2)\Big|_{d=4} = -\frac{1}{(4\pi)^2} \int_0^1 du \,\log\left(1 + u(1-u)\,\frac{p^2}{\tilde{m}^2}\right) =$$

$$-\frac{1}{(16\pi)^2} \left[\sqrt{\frac{p^2 + 4\tilde{m}^2}{p^2}} \,\log \frac{\sqrt{p^2 + 4\tilde{m}^2} + \sqrt{p^2}}{\sqrt{p^2 + 4\tilde{m}^2} - \sqrt{p^2}} - 2 \right] .$$

I leave evaluating the last explicit expression as an **Exercise**. We see that by giving the counterters coefficient $a^{(2)}$ certain dependence on d which is singular at $d = 4$,

$$a^{(2)} = \frac{3}{2}\, I(0) = \frac{3}{2}\, \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\, (\tilde{m}^2)^{\frac{d}{2}-2} \,, \tag{12.12}$$

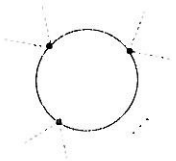
we compensated the singularities of the diagrams (a), (b), (c) above.

Exercise: Compaire the above calculation with the one in §10.2 of PS. Find the relation (to the order λ^2) between our renormalized parameters $\tilde{m}^2, \tilde{\lambda}$ and parameters m^2, λ defined there, which correspond to the scheme (SI).

It is instructive to study renormalization at the next, two loop order. The two loop contributions correspond to the terms $\sim \lambda^2$ in $\Gamma^{(2)}$ and λ^3 in $\Gamma^{(4)}$. Analysis of the two loop terms in $\Gamma^{(2)}$ can be found in PS Sect.10.5. I repeat it in the Appendix below, along with the brief discussion of the two-loop divergences in $\Gamma^{(2)}$.

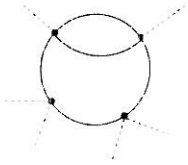
To complete our analysis, we have to check that no new counterterms are required to compensate for divergences in higher proper vertices $\Gamma^{(6)}$, $\Gamma^{(8)}$, etc. At one-loop level the higher vertices are finite without any counterterms since the diagrams

$-\Gamma^{(2n)}_{1-\text{loop}}$ =



are convergent at $d = 4$. The two-loop contributions to these vertices involve divergent diagrams, with the divergences coming from the 2- or 4-leg subdiagrams. For example, $\Gamma^{(6)}_{2-\text{loop}}$ receives contribution from the diagram

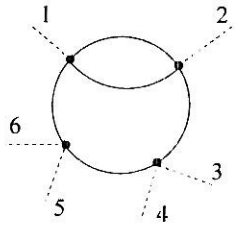
$-\Gamma^{(6)}$ = $\cdots$ +

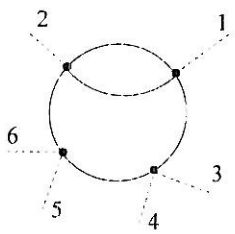


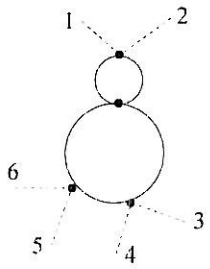
which is singular as $1/\epsilon$. However, it is possible to check that the one-loop counterterms

$$\frac{\lambda\, b^{(1)}(\epsilon)}{2}\,\varphi^2 + \frac{\lambda^2\, a^{(2)}(\epsilon)}{4!}\,\varphi^4$$

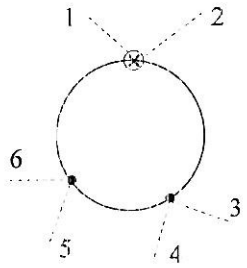
are sufficient to cancel all such divergences. For instance, the above divergent diagram comes along with two other divergent diagrams







The sum of the singular part of these three diagrams is canceled by the diagram



involving the one-loop counterterm vertex.

Similar situation persists to higher loops. Suppose we have renormalized the theory (i.e. determined all the counterterms as the functions of ϵ) up to $L-1$-loops. Then contributions of all L-loop diagrams, with all $L-1$-loop counterterms, to the effective action $\Gamma[\phi]$ can be written as

$$\Gamma_L^{\rm div}[\phi] + \Gamma_L^{\rm finite}[\phi]\,,$$

where $\Gamma_L^{\rm div}$ has the form

$$-\tilde{\lambda}^L \int d^dx \left[\frac{Z^{(L)}(\epsilon)}{2}(\partial\phi)^2 - \frac{B^{(L)}(\epsilon)}{2}\phi^2 + \frac{\tilde{\lambda}\,A^{(L)}(\epsilon)}{4!}\phi^4\right],$$

with coefficients having Lth order pole in ϵ.

$$Z^{(L)} = \sum_{k=1}^{L-1}\frac{Z_{-k}^{(L)}}{\epsilon^k}\,; \qquad B^{(L)} = \sum_{k=1}^{L}\frac{B_{-k}^{(L)}}{\epsilon^k}\,; \qquad A^{(L)} = \sum_{k=1}^{L}\frac{A_{-k}^{(L)}}{\epsilon^k}\,, \tag{12.13}$$

and $\Gamma^{\rm finite}$ is finite at $\epsilon = 0$. It is clear then that the divergent part can be compensated for if one adds to the action the L-loop counterterm

$$\mathcal{A}^{(L)} = \tilde{\lambda}^L \int d^dx \left[\frac{z^{(L)}(\epsilon)}{2}(\partial\varphi)^2 + \frac{b^{(L)}(\epsilon)}{2}\varphi^2 + \frac{\tilde{\lambda}\,a^{(L)}(\epsilon)}{4!}\varphi^4\right],$$

with

$$z^{(L)}(\epsilon) = Z^{(L)}(\epsilon) + Z_0^{(L)}\,; \qquad b^{(L)}(\epsilon) = B^{(L)}(\epsilon) + B_0^{(L)}\,; \qquad a^{(L)}(\epsilon) = A^{(L)}(\epsilon) + A_0^{(L)}\,,$$

where $Z_0^{(L)}, B_0^{(L)}, A_0^{(L)}$ are finite numbers.

The freedom in the choice of these finite numbers represents the freedom in the choice of the renormalization scheme. In given scheme they are determined

by the normalization conditions. like (12.2),(12.3)) or (12.4),(12.5), which usually directly relate the parameters λ and m^2 to physical quantities. The form (12.13) of the divergent part of the effective action suggests another class of renormalization schemes. In so called *Minimal Subtraction Scheme* one simply sets $Z_0^{(L)} = 0, B_0^{(L)} = 0, A_0^{(L)} = 0$. In other words, in this scheme one chooses the L-loop counterterm to be exactly equal to the divergent part of the effective action renormalized to $L-1$ loops, as defined above, i.e.

$$\mathcal{A}^{(L)}[\varphi] = \Gamma_L^{\rm div}[\varphi]\,.$$

In this scheme the parameters m^2_{MS} and λ_{MS} entering the the action

$$\mathcal{A} = \int d^dx \left[\frac{1}{2}(\partial\varphi)^2 + \frac{m^2_{MS}}{2}\varphi^2 + \frac{\lambda_{MS}}{4!}\varphi^4 + \text{Counterterms in MS Scheme}\right],$$

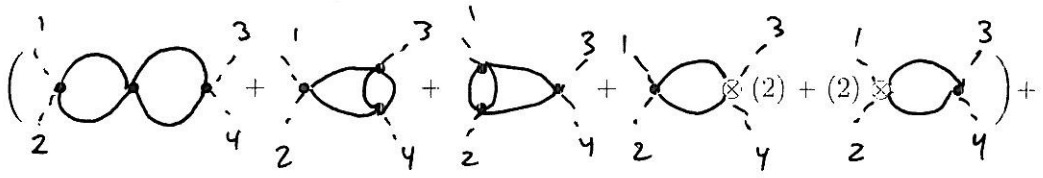
are not apriori related to physical mass and scattering amplitude. This relation has to be calculated order by order in perturbation theory.

Full proof that such procedure works to all orders is rather involved (it can be found e.g. in *J.Zinn-Justin, Quantum Field Theory and Critical Phenomena*, or in original papers quoted on p.338 of PS). According to these general theorems similar renormalization procedure works to all orders in any field theory whose perturbation theory has only finitely many primitive divergences. We have seen that this is true if and only if the mass dimensions of all couplings are non-negative.

Appendix: Two loop analysis of φ^4 (PS Sect.10.6)

Here we carry out two-loop renormalization analysis in the φ^4 theory. Let us now start with the $\tilde{\lambda}^3$ contributions to $\Gamma^{(4)}$. I will still assume the renormalization scheme SII. Also, I will disregard the $b^{(1)}$ term in δm^2 counterterm and the "trivial bubble diagram"; we saw that these cancel each other exactly. Then we have 16 diagrams contributing to $\Gamma^{(4)}$ in this order:

$$-\Gamma^{(4)}(p_1,\cdots,p_4) \;=\; \cdots \;+$$



(a) (b) (c) (d) (e)

$$\Big(\text{10 diagrams with permutations of the external legs}\Big) \quad +$$

$$\Big(\quad \overset{(3)}{\otimes} \quad \Big) \quad - \quad \ldots$$

The diagrams (d) and (e) (and 4 similar diagrams with permutations of the external legs) involve the counterterm vertex $-\tilde{\lambda}^2\, a^{(2)}$ with the coefficient $a^{(2)}$ which was already determined in the one-loop analysis, see Eq.(12.14) above.

The last diagram shown above represents the contribution of the term $\tilde{\lambda}^3$ in $\delta\lambda$ counterterm. We must see that it is indeed possible to adjust this single coefficient $a^{(3)}$ in such a way that all divergences (i.e. all singularities at $d = 4$) cancel. This last contribution is momentum independent constant. For our renormalization program to be successful to this order we must prove that all momentum-dependent divergences cancell out among the first 15 diagrams above.

Consider 5 diagrams (a-e) above. It is useful to divide them into three groups as follows

$I)$ $\quad + \frac{1}{3} \quad + \frac{1}{3}$

$II)$ $\quad + \frac{2}{3}$

$III)$ $\quad + \frac{2}{3}$

Let us show that all momentum dependent divergences cancell separately within each group.

Consider first the group I. The first diagram there is easily calculated to be

$$-\frac{\tilde{\lambda}^3}{4}\left[I(p_{12}^2)\right]^2 ,$$

the 4 in the denominator being the symmetry factor, $p_{12} = p_1 + p_2$, and again

$$I(p^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+\tilde{m}^2)((p+k)^2+\tilde{m}^2)} .$$

Now, each of the other two diagrams in this group contribute

$$\frac{1}{3}\ \frac{\tilde{\lambda}^3}{2}\, a^{(2)}\, I(p_{12}^2) ,$$

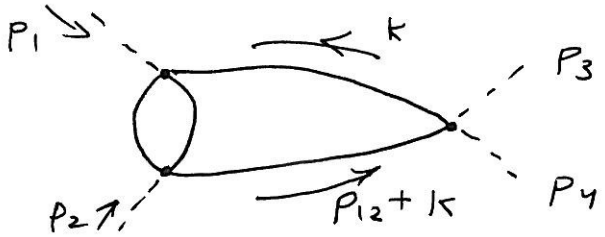
so that the sum of all three diagrams in this group is

Group I =

$$-\frac{\tilde{\lambda}^3}{4}\left[I(p_{12}^2)\right]^2 + \frac{2}{3}\ \frac{\tilde{\lambda}^3}{2}\ \frac{3}{2} I(0)\ \ I(p_{12}^2) = -\frac{\tilde{\lambda}^3}{4}\left[I(p_{12}^2) - I(0)\right]^2 + \frac{\tilde{\lambda}^3}{4}\left[I(0)\right]^2$$

where the expression $a^{(2)} = \frac{3}{2} I(0)$, Eq.(12.14), is used. We know from the one-loop analysis that the difference $I_r(p^2) = I(p^2) - I(0)$ is finite at $d = 4$ and therefore the divergent parts of the diagrams in the group I indeed sum up to a momentum independent constant (the last term in the expression above). Note that this divergent constant has second order pole at $d = 4$, i.e. it behaves as $1/\epsilon^2$ as $\epsilon \to 0$.

Consider now the group II. It contains the diagram



$$-\frac{\tilde{\lambda}^3}{2}\int \frac{d^d k}{(2\pi)^d}\ \frac{1}{k^2+\tilde{m}^2}\ \frac{1}{(p_{12}+k)^2+\tilde{m}^2}\ I((k+p_1)^2) ,$$

where $I(p^2)$ is the integral (10.2), which we already found equal to

$$I(p^2) = \frac{\Gamma(2-d/2)}{(4\pi)^{\frac{d}{2}}}\int_0^1 du \left[\tilde{m}^2 + u(1-u)\,p^2\right]^{\frac{d}{2}-2} .$$

Using this formula and combining the propagators with the help of Feynman parametrization we obtain for the above diagram

$$-\frac{\tilde{\lambda}^3}{2}\,\frac{\Gamma(2-d/2)}{(4\pi)^{\frac{d}{2}}}\;\times$$

$$\int_0^1 du \int_0^1 dv \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{(k^2+2v\,(kp_{12})+v\,p_{12}^2+\tilde{m}^2)^2}\,\frac{1}{(\tilde{m}^2+u(1-u)\,(k+p_1)^2)^{2-\frac{d}{2}}}\,.$$

The k integral here can be evaluated by using the transformation generalizing the Feynman parametrization. Namely, consider the identity

$$\frac{1}{A^\alpha\,B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \frac{dw\,w^{\alpha-1}\,(1-w)^{\beta-1}}{[w\,A+(1-w)\,B]^{\alpha+\beta}}\,. \tag{12.14}$$

It can be derived by explicit evaluation of the integral in the right-hand side: projective transformation of the integration variable $w = Bz/(Bz+A(1-z))$ brings it to the form of the Beta-integral

$$B(\alpha,\beta) = \int_0^1 dz\,z^{\alpha-1}\,(1-z)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,.$$

(details can be found in § 10.5 of PS). Using this identity one can write the above integral as

$$-\frac{\tilde{\lambda}^3}{2}\,\frac{\Gamma(2-d/2)}{(4\pi)^{\frac{d}{2}}}\;\times\;\int_0^1 du\,dv\,dw$$

$$\int \frac{d^dk}{(2\pi)^d}\,\frac{w^{1-\frac{d}{2}}\,(1-w)}{[w\,(\tilde{m}^2+u(1-u)\,(k+p_1)^2)+(1-w)\,(\tilde{m}^2+k^2+2v\,(kp_{12})+v\,p_{12}^2)]^{4-\frac{d}{2}}}\,.$$

By appropriate shift of the d-momentum integration variable the denominator can be brought to the form

$$A\,q^2+P^2-\tilde{m}^2$$

where

$$A = (1-w)-wu(1-u)\,,\qquad A|_{w=0}=1\,,$$

and P is complicated function of p_{12}, p_1 and the parameters u, v, w. We only need to know that

$$P^2\Big|_{w=0} = v(1-v)\, p_{12}^2 .$$

In this representation the integral over q is readily evaluated,

$$-\frac{\tilde{\lambda}^3}{2}\,\frac{\Gamma(4-d)}{(4\pi)^d}\int_0^1 du\, dv\, dw\, \frac{w^{1-\frac{d}{2}}(1-w)}{A^{\frac{d}{2}}}\,(\tilde{m}^2+P^2)^{d-4} .$$

Evidently this expression is singular at $d=4$. First, there is a pole at $d=4$ in the factor

$$\Gamma(4-d) = \Gamma(\epsilon) = \frac{1}{\epsilon} - O(1)$$

in front. In addition, at $d=4$ the integral over w diverges at $w \to 0$ because of the factor $w^{1-\frac{d}{2}}$ in the integrand; as the result the integral over w has additional pole at $d=4$, i.e. the whole diagram again has a second order pole at this point.

To isolate the singular contributions let us write the integral over w as

$$\int_0^1 dw\, w^{1-\frac{d}{2}}\, f(w) = f(0)\int_0^1 dw\, w^{1-\frac{d}{2}} + \int_0^1 dw\, w^{1-\frac{d}{2}}\,[f(w)-f(0)] .$$

The second piece is

$$-\frac{\tilde{\lambda}^3}{2}\,\frac{\Gamma(4-d)}{(4\pi)^d}\int_0^1 du\, dv\, dw\; w^{1-\frac{d}{2}}\left[\frac{1-w}{A^{\frac{d}{2}}}\,(\tilde{m}^2+P^2(w))^{d-4} - (\tilde{m}^2+P^2(w=0))^{d-4}\right] .$$

Here the integral over u, v, w is convergent and brings in no additional singularity at $d=4$. Therefore the above expression has only simple pole at $\epsilon = 0$. To evaluate the residue at this simple pole one can set $d=4$ everywhere except in $\Gamma(4-d)$. It is clear then that this residue does not depend on external momenta, i.e. this part of the above diagram has the form

$$\frac{C_0}{\epsilon} + C_1(p_1, \cdots, p_4) ,$$

where C_0 does not depend on p_i, and the C_1 is finite at $\epsilon = 0$.

Let us take a look at the other piece, proportional to $f(0)$. It is

$$f(0)\int_0^1 dw\, w^{1-\frac{d}{2}} = \frac{2}{\epsilon}\, f(0) = -\frac{2}{\epsilon}\,\frac{\tilde{\lambda}^3}{2}\,\frac{\Gamma(\epsilon)}{(4\pi)^d}\int_0^1 du\, dv\; (\tilde{m}^2 + v(1-v)\, p_{12}^2)^{-\epsilon}$$

The u integral here is trivial. Now, for $\epsilon \to 0$ the integral can be written as

$$\int_0^1 dv \left(1 - \epsilon \log(\tilde{m}^2 + v(1-v)\, p_{12}^2) + O(\epsilon^2)\right),$$

and therefore this piece of the diagram is

$$-\frac{\tilde{\lambda}^3}{(4\pi)^4}\,\frac{1}{\epsilon}\left(\frac{1}{\epsilon} - \gamma + \log 4\pi - \int_0^1 dv\, \log[\tilde{m}^2 + v(1-v)\, p_{12}^2]\right) + \text{finite terms}.$$

We have to add here the second diagram in the group II, which is

$$\frac{2}{3}\,\frac{\tilde{\lambda}^3}{2}\, a^{(2)}\, I(p_{12}^2) = \frac{\tilde{\lambda}^3}{2}\, I(0)\, I(p_{12}^2),$$

where the expression (10.4) for the coefficient $a^{(2)}$ is used. Recall that

$$I(p^2) = \frac{\Gamma(2 - d/2)}{(4\pi)^{\frac{d}{2}}} \int_0^1 du\, [\tilde{m}^2 - u(1-u)\, p^2]^{\frac{d}{2}-2} =$$

$$\frac{1}{(4\pi)^{\frac{d}{2}}}\left(\frac{2}{\epsilon} - \gamma - \int_0^1 du\, \log(\tilde{m}^2 + u(1-u)\, p^2) + O(\epsilon)\right).$$

Therefore the contribution of this diagram is

$$\frac{\tilde{\lambda}^3}{(4\pi)^4}\,\frac{1}{\epsilon}\left(\frac{2}{\epsilon} - 2\gamma + 2\log 4\pi - \log\tilde{m}^2 - \int_0^1 du\, \log[\tilde{m}^2 + u(1-u)\, p_{12}^2]\right) + \text{finite terms}.$$

Adding the two diagrams of the group II we see that the p_{12}- dependent singular terms cancell out,

$$\text{Group II} = \frac{\tilde{\lambda}^3}{(4\pi)^4}\,\frac{1}{\epsilon}\left(\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\tilde{m}^2\right) + \text{finite terms}.$$

The group III can be analyzed similarly. We find that all singular parts of the first 15 diagrams are pole terms with the coefficients independent of external momenta,

$$15 \text{ diagrams} = \tilde{\lambda}^3\left(\frac{c_{-2}}{\epsilon^2} + \frac{c_{-1}}{\epsilon} + F(p_1, \cdots, p_4)\right), \tag{10.5}$$

where c_{-2}, c_{-1} do not depend on p_i, and F is finite at $d = 4$. Therefore the counterterm coefficient $a^{(3)}$ appearing in the 16th diagram always can be appropriately

adjusted to compensate the singular terms. In fact, according to our normalization condition (SIIb) ($\Gamma^{(4)}(0,0,0,0) = \tilde{\lambda}$), this coefficient is determined as

$$a^{(3)} = \frac{c_{-2}}{\epsilon^2} + \frac{c_{-1}}{\epsilon} + F(0, \cdots, 0)$$

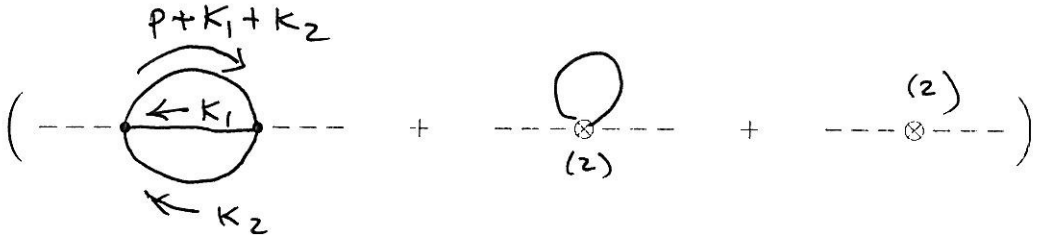
and hence

$$\text{All 16 diagrams} = \tilde{\lambda}^3 \left(F(p_1, \cdots, p_4) - F(0, \cdots, 0) \right).$$

Exercise Verify these two-loop calculations. Evaluate c_{-2} and c_{-1}.

Let us briefly consider the $\tilde{\lambda}^2$ contributions to $\Gamma^{(2)}$. These are given by three diagrams

$$\Gamma^{(2)}(p^2) = p^2 - \tilde{m}^2 -$$



(a) (b) (c)

The diagram (b) involves the counterterm vertex $-\tilde{\lambda}^2 a^{(2)}$, and the diagram (c) contains the $\tilde{\lambda}^2$ part of the counterterms δZ and δm^2,

$$\overset{(2)}{- - - \otimes - - -} \quad = \quad -\tilde{\lambda}^2 (z^{(2)} p^2 + b^{(2)}).$$

Again, the coefficients here must be determined from the normalization condition (SIIa).

Let us denote $\Sigma_a(p^2)$ the contribution of the diagram (a) to the mass operator. One can write it as

$$\Sigma_a(p^2) = \Sigma_a(0) + p^2 \Sigma_a'(0) + \Sigma_r(p^2),$$

where by the definition

$$\Sigma_r(p^2) = O(p^4) \qquad \text{as} \qquad p^2 \to 0\,.$$

Let us analyse the diagram (a) and show that $\Sigma_r(p^2)$ has finite value at $d = 4$.

The contribution of the diagram (a) is given by the integral

$$\Sigma_a(p^2) = -\frac{\tilde{\lambda}^2}{3!}\int \frac{d^dk_1\,d^dk_2}{(2\pi)^{2d}}\frac{1}{(k_1^2+\tilde{m}^2)(k_2^2+\tilde{m}^2)((p+k_1+k_2)^2-\tilde{m}^2)}\,.$$

One of the k-integrations, say over k_2, can be done exactly the way we did one-loop integrals for $\Gamma^{(4)}$:

$$\int \frac{d^dk_2}{(2\pi)^d}\frac{1}{(k_2^2+\tilde{m}^2)((p+k_1+k_2)^2+\tilde{m}^2)} = I((p+k_1)^2) =$$

$$\frac{\Gamma(2-d/2)}{(4\pi)^{\frac{d}{2}}}\int_0^1 du\left[\tilde{m}^2+u(1-u)\,(p+k_1)^2\right]^{\frac{d}{2}-2}.$$

We find

$$\Sigma_a(p^2) = -\frac{\tilde{\lambda}^2}{3!}\frac{\Gamma(2-d/2)}{(4\pi)^{\frac{d}{2}}}\int_0^1 du\int\frac{d^dk_1}{(2\pi)^d}\frac{1}{(u(1-u)(p+k_1)^2+\tilde{m}^2)^{2-d/2}(k_1^2+\tilde{m}^2)}\,.$$

The remaining integral over k_1 can be done using (12.14) which allows one to transform the integral as

$$\Sigma_a(p^2) = -\frac{\tilde{\lambda}^2}{3!}\frac{\Gamma(2-d/2)}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(3-d/2)}{\Gamma(2-d/2)\Gamma(1)}\int_0^1 du\int_0^1 dw\,J(u,w)\,,$$

$$J(u,w) = \int\frac{d^dk_1}{(2\pi)^d}\frac{w^{1-\frac{d}{2}}}{[\tilde{m}^2+(1-w)\,k_1^2+wu(1-u)\,(k_1+p)^2]^{3-\frac{d}{2}}}\,.$$

The integral over k_1 can be evaluated after the shift of this momentum variable

$$q = k_1 - \frac{wu(1-u)}{A}\,p\,,$$

where again

$$A = (1-w)+wu(1-u)\,.$$

One obtains

$$J(u,w) = \int \frac{d^d q}{(2\pi)^d} \frac{w^{1-d/2}}{\left(A\, q^2 + w(1-w)u(1-u)\, p^2/A + \tilde{m}^2 \right)^{3-\frac{d}{2}}} .$$

With appropriate rescaling of q one can bring this integral to the form (11.10), and obtain

$$J(u,w) = \frac{A^{3-\frac{3d}{2}}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(3-d)}{\Gamma(3-d/2)} w^{1-d/2} \left[w(1-w)u(1-u)\, p^2 + A\, \tilde{m}^2 \right]^{d-3} .$$

Combining these results we get

$$\Sigma_a(p^2) = -\frac{\tilde{\lambda}^2}{3!} \frac{\Gamma(3-d)}{(4\pi)^d} \int_0^1 dw\, du\; w^{1-\frac{d}{2}}\, A^{3-\frac{3d}{2}} \left[u(1-u)w(1-w)\, p^2 + A\, \tilde{m}^2 \right]^{d-3} .$$

This quantity has second order pole in ϵ, one singular factor coming from the Gamma function

$$\Gamma(3-d) = -\frac{1}{\epsilon} + O(1) ,$$

and another appearing due to the divergence of the integral over w at $w \to 0$. However, subtracting $\Sigma_a(0)$ and $p^2\, \Sigma_a'(0)$ one obtains

$$\Sigma_r(p^2) = -\frac{\tilde{\lambda}^2}{3!} \frac{\Gamma(3-d)}{(4\pi)^d} (\tilde{m}^2)^{d-3} \int_0^1 du\, dw\; w^{1-\frac{d}{2}}\, A^{-\frac{d}{2}}\, H(u,w) .$$

where

$$H(u.w) = (1 + p^2\, B)^{d-3} - 1 - (d-3)\, p^2\, B ,$$

and

$$B = \frac{u(1-u)w(1-w)}{\tilde{m}^2\, A} .$$

Note that $H(u, w = 0) = 0$ and therefore the w integration does not produce singularity in this expression. Moreover, one can see that $H = O(\epsilon)$,

$$H(u.w) = \epsilon \left[p^2\, B - (1 + p^2\, B)\, \log(1 + p^2\, B) \right] + O(\epsilon^2) ,$$

and therefore Σ_r has finite value at $d = 4$,

$$\Sigma_r(p^2)\Big|_{d=4} = \frac{\tilde{\lambda}^2}{3!} \frac{\tilde{m}^2}{(4\pi)^4} \int_0^1 \frac{du\, dw}{w\, A} \left[p^2\, B - (1 + p^2\, B)\, \log(1 + p^2\, B) \right]$$

At the same time the subtracted terms $\Sigma_a(0)$ and $\Sigma'_a(0)$ are singular at $d = 4$. The contributions of two diagrams (a) and (b) to Σ then has the form

$$\Sigma_a(p^2) + \Sigma_b(p^2) = \Sigma_r(p^2) + \tilde{\lambda}^2 \left(M(\epsilon) + p^2 N(\epsilon) \right),$$

where the constants $M(\epsilon)$ and $N(\epsilon)$ have poles in ϵ,

$$M(\epsilon) = \frac{M_{-2}}{\epsilon^2} + \frac{M_{-1}}{\epsilon} + O(1); \qquad N(\epsilon) = \frac{N_{-1}}{\epsilon} + O(1). \tag{12.15}$$

The diagram (c) brings the contribution

$$\Sigma_c(p^2) = \tilde{\lambda}^2 \left(p^2 z^{(2)} + b^{(2)} \right),$$

and our normalization condition

$$\Sigma(0) = \Sigma'(0) = 0$$

requires the choice

$$b^{(2)} = -M(\epsilon); \qquad z^{(2)} = -N(\epsilon)$$

of the counterterm coefficients at this order. For the proper vertex we then have simply

$$\Gamma^{(2)}(p^2) = p^2 + \tilde{m}^2 + \Sigma_r(p^2) + O(\tilde{\lambda}^3).$$

13 Lecture 13

Renormalization of composite fields in φ^4 theory.

In many cases one may be interested in correlation functions involving composite fields, such as powers of $\varphi^n(x)$, or energy-momentum tensor. Such correlation functions can be computed perturbatively, as the power series in the coupling constant λ. However new renormalizations, in addition to the basic renormalizations of the parameters m^2, λ and the field φ, have to be made to make such correlation functions finite. Let us demonstrate this for the simplest case of the composite field

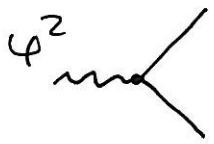
$$\varphi^2(x)$$

in the φ^4 theory.

Consider the 3-point connected correlation function

$$\left\langle \frac{1}{2}\,\varphi^2(x)\,\varphi(x_1)\,\varphi(x_2) \right\rangle_c$$

where $\varphi(x)$ stands for the renormalized field, $\varphi = Z^{-1/2}\varphi_0$, $Z = 1 + \delta Z$, with Z chosen according to some scheme, say SII. Diagrams can be used to describe perturbative contributions to this correlation function if one introduces additional "external" vertex



$= 1$

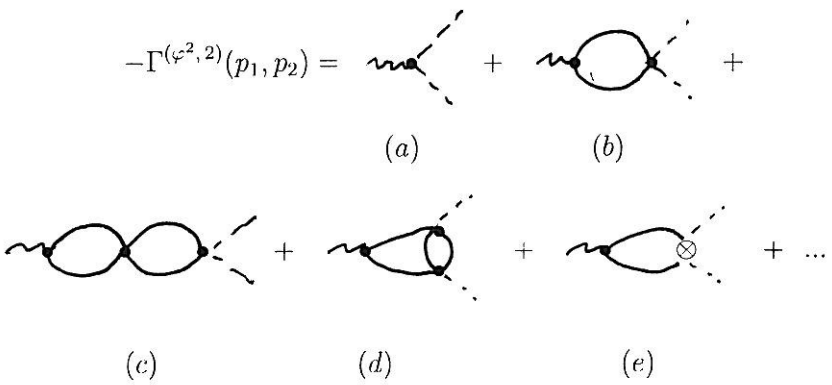
associated with the insertion $\frac{1}{2}\varphi^2$. As before, it is useful to switch to the momentum representation, and introduce the associated amputated vertex

$$-\Gamma^{(\varphi^2,2)}(p_1,p_2) = \frac{W^{(\varphi^2,2)}(p_1,p_2)}{W(p_1)W(p_2)}\,, \tag{13.1}$$

where

$$(2\pi)^4\delta^{(4)}(q-p_1-p_2)\,W^{(\varphi^2,2)}(p_1,p_2) = \text{f.t.}\,\left\langle \frac{1}{2}\,\varphi^2(x)\,\varphi(x_1)\varphi(x_2) \right\rangle_c\,,$$

with q, p_1, p_2 being the momenta corresponding to x, x_1, x_2, respectively. It is given by the sum of all one-particle irreducible diagrams



Most of these diagrams are singular at $d = 4$, and a quick look reveals that these singularities can not be eliminated by renormalizations of the parameters m^2, λ, and the field φ. This is already seen in the order $\sim \lambda$. All the counterterm vertices that we have introduced already show up at the order λ^2 (diagram (e)) and higher. But the diagram (b), which is $\sim \lambda$, is singular. Indeed its contribution is

$$-\Gamma^{(\varphi^2,2)}(p_1,p_2) = 1 - \frac{\lambda}{2} I(p_{12}^2) + O(\lambda^2)$$

where $I(p^2)$ is the same integral (12.11) which we encountered previously. We know that

$$I(p_{12}^2) = I(0) + I_r(p_{12}^2)\,,$$

where $I_r(p^2)$ has finite limit at $d = 4$, while $I(0)$ diverges as $1/\epsilon$. In fact this shows what we need to do. We write

$$-\Gamma^{(\varphi^2,2)}(p_1,p_2) = \left(1 - \frac{\lambda}{2} I(0) + O(\lambda^2)\right)\left(1 - \frac{\lambda}{2} I_r(p_{12}^2) + O(\lambda^2)\right) + O(\lambda^2)\,.$$

Therefore, at least to the leading order, we can handle the problem by defining the renormalized field $[\varphi^2]_R$,

$$\varphi^2 = Z_{\varphi^2}^{-1}\,[\varphi^2]_R\,, \tag{13.2}$$

where

$$Z_{\varphi^2}^{-1} = 1 - \frac{\lambda}{2} I(0) + O(\lambda^2)\,. \qquad I(0) = \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\,(m^2)^{d/2-1}\,.$$

Then

$$\Gamma^{(\varphi^2,2)} = Z_{\varphi^2}^{-1}\ \Gamma^{(\varphi_R^2,2)}\,,$$

where the vertex function of the renormalized field $[\varphi^2]_R$ is finite, to the order λ.

It is convenient to invert the above relation,

$$[\varphi^2]_R(x) = Z_{\varphi^2}\ \varphi^2 = Z_{\varphi^2}\, Z^{-1}\, \varphi_0^2\,. \tag{13.3}$$

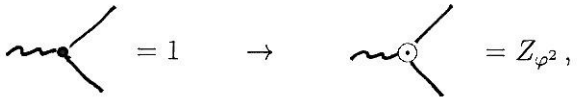
(Note that in the ratio (13.1) defining $\Gamma^{(\varphi^2,2)}(p_1,p_2)$ the field renormalization factor Z cancels out.) As with other renormalization parameters, we assume that Z_{φ^2} expands in power series in λ,

$$Z_{\varphi^2} = z_{\varphi^2}^{(0)} + \lambda\, z_{\varphi^2}^{(1)} + \lambda^2\, z_{\varphi^2}^{(2)} + \ldots$$

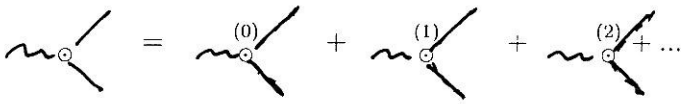
In the above calculation we have set $z_{\varphi^2}^{(0)} = 1$; with this we have

$$z_{\varphi^2}^{(1)} = \frac{1}{2}\, I(0) + \text{finite}$$

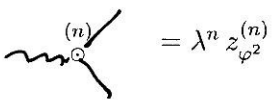
By inserting the renormalized field $[\varphi^2]_R(x)$ instead of $\varphi^2(x)$ we replace the φ^2 vertex with



In the perturbation theory (i.e. expansion in λ), this new vertex is represented as the series in λ,



where



In computing the correlation functions with the insertion of $[\varphi^2]_R$ one has to take into account these new field renormalization vertices along with the counterterm

vertices associated with the counterterm part of the action. The field renormalization constants $z^{(n)}_{\varphi^2}$ are to be adjusted, order by order in the perturbation theory, to cancel the divergent parts of the diagrams.

As in the case of the renormalizations of the parameters of the action, there is freedom in of finite renormalizations

$$[\varphi^2]_R \to Z_{\text{finite}}\,[\varphi^2]_R\,.$$

This ambiguity can be fixed by imposing some normalization condition. For instance, one can fix the normalization of the field $[\varphi^2]_R$ by

$$-\Gamma^{([\varphi^2]_R,2)}(0,0) = 1\,,$$

where $-\Gamma^{([\varphi^2]_R,2)}(p_1,p_2)$ is the momentum-space amputated correlation function

$$-\Gamma^{([\varphi^2]_R,2)}(p_1,p_2)\,W(p_1)\,W(p_2) = \text{f.t.}\,\left\langle \frac{1}{2}[\varphi^2]_R(x)\,\varphi(x_1)\varphi(x_2)\right\rangle_c$$

Under this condition we have

$$z^{(0)}_{\varphi^2} = 1\,,\qquad z^{(1)}_{\varphi^2} = \frac{1}{2}\,I(0)\,.$$

Renormalizations of higher composite fields follow the same idea but are more complicated. In particular, it generally involves the phenomenon of "operator mixing" - renormalizations of higher composite fields may require forming linear combinations of them, with renormalization factors becoming matrices. Brief discussion of simplest case of this phenomenon - mixing of the fields φ^3 and $\Delta\varphi$ - are given in the Appendix. More systematically this phenomenon is discussed in the context of *Renormalization Group.*

Perturbative Renormalization Group

Finite renormalizations

Renormalized perturbation theory allows one to obtain the correlation functions of φ^4 theory as power series in the renormalized coupling parameter λ with well defined finite coefficients. This concerns the correlation functions of of the field φ as well as the ones involving appropriately renormalized composite fields. This is achieved by adjusting the counterterm coefficients δm^2, $\delta\lambda$, δZ in the action (and, additionally, the renormalization factors in the composite fields) as the functions

of regularization parameter ($\epsilon = 4 - d$ in dimensional regularization, or Λ if some cutoff regularization is assumed). Renormalized correlation functions appearing in this way depend on two parameters, λ and m^2, named the "renormalized coupling" and the "renormalized mass parameter". Precise relation of these parameters to physical quantities, as well as the field normalizations, are specified (explicitly or implicitly) by the *renormalization scheme.*

By doing calculations in two different renormalization schemes we are not solving two different field theories; rather, we are describing the same theory in terms of different parameters. One can think of two-dimensional (because two parameters, m^2 and λ, are involved) "manifold of φ^4 theories", with the parameters m^2 and λ playing the role of (local) coordinates on this manifold (the latter is often called the "theory space").

Each point of this manifold represents a field theory, i.e. a collection of all correlation functions, defined up to finite field renormalizations

$$\langle \varphi(x_1) \cdots \varphi_n(x_n) \rangle \ \sim Z_{\text{finite}}^{\frac{n}{2}} \langle \varphi(x_1) \cdots \varphi_n(x_n) \rangle \,.$$

To simplify things one can consider the ratios like

$$R(x_1, x_2; x_3, \cdots, x_n) = \langle \varphi(x_1)\varphi(x_2) \cdots \varphi(x_{2n}) \rangle / \langle \varphi(x_1)\varphi(x_2) \rangle^n$$

which do not depend on the normalizations of the fields. Different renormalization schemes correspond to different choices of the coordinate system on the above manifold. Ones the coordinate system (i.e. renormalization scheme) is fixed, that is the points are somehow "labeled" by two parameters m^2 and λ, the above ratios become functions of these parameters,

$$R(x_i | m^2, \lambda) \,.$$

Relation between the parameters (m^2, λ) and $(\tilde{m}^2, \tilde{\lambda})$, associated with two different renormalization schemes, is just a coordinate transformation,

$$m^2 \ \to \ \tilde{m}^2(m^2, \lambda), \quad \lambda \ \to \ \tilde{\lambda}(m^2, \lambda) \,.$$

At $d = 4$ this relation has the form (by dimensional analysis)

$$\tilde{m}^2 = m^2 \, f(\lambda) \,, \qquad \tilde{\lambda} = g(\lambda) \,, \tag{13.4}$$

where the functions f, g can in principle be computed order by order in λ,

$$f(\lambda) = f_0 + \lambda \, f_1 + \lambda^2 \, f_2 + \cdots \,,$$

$$g(\lambda) = \lambda + \lambda^2 g_2 + \lambda^3 g_3 + \cdots ,$$

the coefficients f_k, g_k being finite numbers. The correlation functions (or rather the above ratios R) computed in the two schemes are related as

$$\tilde{R}(x_i|m^2 f(\lambda), g(\lambda)) = R(x_i|m^2, \lambda) ,$$

so that these are the same correlation functions expressed in terms different parameters, that is, coordinates in the space of φ^4 theories.

Short distance problem

Renormalized perturbation theory, no matter what scheme is assumed, yields the correlation functions as formal power series in the coupling parameter λ. Although in each given order these series give finite and well behaved correlation functions, the problem of overall *non-perturbative* consistency of the theory is a separate question. There is an issue of convergence of the perturbative expansion. It is possible to show that these series in φ^4 have in fact zero radius of convergence, and at the best they can be understood as asymptotic series. If so, there is an important question if these series can be understood as asymptotic expansions of some non-perturbative correlation functions which are compatible with requirements of *locality* and with *quantum mechanical interpretation* (the property usually called *positivity*). Although one can show that these properties - locality and positivity - are preserved at the perturbative level, i.e. in each given order in λ, it is conceivable that they can break down after summing the series up.

We have seen that in Euclidean version of quantum field theory locality implies in particular that the correlation function

$$\langle \varphi(x_1)\varphi(x_2)\cdots\varphi(x_{2n})\rangle ,$$

viewed as the function of, say, x_1, has only point-like singularities at $x_1 = x_2, x_3, \cdots, x_{2n}$. (We have observed this in the free Klein-Gordon theory, but arguments we have applied in that analysis depend only on general properties of the energy-momentum spectrum, and are generalized straightforwardly to general interacting theories; I hope to come back to this point later.) This is because upon continuation to complex coordinates and ultimately to the Minkowski space-time these singularities translate into the singularities at the light cones associated with the points $x_2, \cdots, x_{2n}$; we have seen that these singularities are directly related to commutators of the field operators. Locality states that the commutators of Heisenberg field operators must vanish outside the light cone,

$$[\hat{\varphi}(\mathbf{x}, t) , \hat{\varphi}(0, 0)] = 0 \qquad \text{for} \qquad t^2 < \mathbf{x}^2 .$$

This property of locality is preserved in the renormalized perturbation theory at each order in λ but there is no guarantee it will not break down in full non-perturbative theory. It is very well conceivable that upon summing the perturbative series up singularities at finite space-like separations may be generated, leading to violations of the local commutativity.

The following example demonstrates this possibility. Consider the power series

$$\sum_{n=0}^{\infty} (-\lambda)^n \left(\log\, m|x_1 - x_2| \right)^n ,$$

which models a perturbative series for some correlation function. At each given order the above expression has point-like singularity at $x_1 = x_2$. However its sum is

$$\frac{1}{1 + \lambda \log\, m|x_1 - x_2|} .$$

This function is singular at

$$|x_1 - x_2| = x_0 \equiv m^{-1}\, e^{-\frac{1}{\lambda}} .$$

As the result, if one continues to the Minkowski space-time the light cone "fattens" and causality is broken at short scales $\simeq x_0$ [6]

[6] One can take different attitudes toward such possibility. One pragmatic attitude is that we do not know what describes short-distance physics anyway. After all, at short distances quantum gravity must play important role, perhaps along with extra dimensions, superstrings, D-branes, and who knows what. And therefore we do not care about possible inconsistencies at these short distances. With this attitude quantum field theory is viewed as a way of producing (renormalized) perturbative expansions in the coupling parameter. Such expansions of course are useful only if the coupling parameter is small. In quantum electrodynamics (QED) where the analog of λ is the fine structure constant which is small indeed, this is valid point of view. And if the interaction is not small in any sense, according to this philosophy one would conclude that quantum field theory is not useful at all.

In the past, at certain stage of development, such was prevailing attitude. Discovery of *Asymptotic Freedom* in nonabelian gauge theories, along with better understanding of *Renormalization Group* has changed that. On the face of it, the discovery of the asymptotic freedom was pure perturbative result, obtained by computing few diagrams, but through understanding the Renormalization Group one can see that it actually guaranteed full nonperturbative consistency of quantum nonabelian gauge theories. More generally, renormalization allows one to understand quantum field theory in terms of criticality, or vice versa, to the extent that from theoretical point of view quantum field theory and theory of critical phenomena are much one and the same thing. Moreover, main idea of the Wilson's renormalization group has proved very useful in theoretical physics well beyond quantum field theory. This is why I hope to devote significant part of this course to it.

The problem of the short-distance consistency of a quantum field theory requires analysis of its behavior under changes of the length scale. The latter is generally understood in terms of *Renormalization Group*. Here I will briefly discuss the basics of this approach, in its perturbative version, and applied to φ^4 theory. The analysis is substantially simpler in the case of *massless theory*.

Massless φ^4

As a preparation, let us discuss what happens with our renormalized perturbation if one sets the renormalized mass parameter m^2 to zero. Although precise meaning of m^2 depends on the renormalization scheme, the condition

$$m^2 = 0$$

is *scheme independent*, i.e. ones it is satisfied in one scheme, it so does in any other scheme, as the Eq.(13.4) suggests. There are at least two reasons why this case is interesting.

- Potential problem with locality can occur at short scales, much shorter then m^{-1}. The mass term is not important at such scales, and by analyzing this case one can understand the general situation.

- We will see that in applications to critical phenomena the case $m^2 = 0$ describes physics at exactly critical point, and therefore it is of central interest.

Taking a look at our previous perturbative results we discover that it is not possible to just set $m^2 = 0$ there, since nearly all quantities we have computed are singular at this point. Consider for instance the 4-point vertex,

$$\Gamma^{(4)}(p_1, \cdots, p_4) = \lambda - \frac{\lambda^2}{2}\left(I_r(p_{12}^2) + I_r(p_{13}^2) + I_r(p_{14}^2)\right) + O(\lambda^3)\,,$$

where again

$$I_r(p^2) = -\frac{1}{4\pi}\int_0^1 du\, \log\left(1 + u(1-u)\,\frac{p^2}{m^2}\right).$$

Obviously, there is a singularity at $m^2 = 0$. It can signal one of two things. Either the theory itself becomes sick at this point, or our coordinate system does not extend smoothly to this case. I want to argue that $m^2 = 0$ singularities of the perturbative correlation functions originate from a coordinate singularity[7].

[7]Think of some a function $f(x, y)$ written in new coordinates (x', y'),

$$x = x'\,, \qquad y = y' + y'^2\,\log x'^2\,.$$

Indeed, a particular coordinate system is determined by the normalization conditions. For example, in our scheme SII we have assumed

$$\Gamma^{(2)}(p^2)\Big|_{p^2=0} = m^2\,, \quad \text{and} \quad \frac{d\Gamma^{(2)}(p^2)}{dp^2}\Big|_{p^2=0} = 1\,. \tag{13.5}$$

At $m^2 = 0$ the first of these conditions remains useful, just stating that $\Gamma^{(2)}(p^2)$ has zero at $p^2 = 0$, which can be considered as a scheme-independent definition of the "massless theory", but at $m^2 = 0$ this is no longer a simple zero, being replaced by more complicated singular point. We will see in a moment that at $m^2 = 0$ $\Gamma^{(2)}$ contains terms like $p^2 \log p^2$ and therefore the second of the conditions (13.5) cannot be imposed. The coordinate system associated with our scheme is singular at $m^2 = 0$.

It is useful to have a scheme which admits smooth limit $m^2 \to 0$. Such scheme must involve dimensional parameter different from m^2. Let us denote it μ^2. Since μ can be chosen arbitrarily, we will have families of schemes, parameterized by auxiliary scale μ. Let us take one such μ-dependent scheme, defined by the normalization conditions

$$\frac{d\Gamma^{(2)}(p^2)}{dp^2}\Big|_{p^2=\mu^2} = 1\,, \tag{13.6}$$

and

$$\Gamma^{(4)}(p_1, \cdots, p_4)\Big|_{p_i=\mu\, e_i} = \lambda_{(\mu)}\,, \tag{13.7}$$

where e_i is some set of four fixed 4-vectors, which satisfy $e_1+e_2+e_3+e_4 = 0$, but have no other linear dependence. To be specific, we can choose symmetric arrangement,

$$e_i^2 = 3 \quad \text{and} \quad (e_i\, e_j) = -1 \quad \text{for} \quad i \neq j\,.$$

I write $\lambda_{(\mu)}$ for the renormalized coupling constant as the reminder about the scale μ associated with this scheme. It is conventional to refer to this auxiliary scale as the "normalization scale". It is perfectly legitimate to use such scheme in general case, but we will apply it to massless theory, which is singled out by the condition

$$\Gamma^{(2)}(0) = 0\,. \tag{13.8}$$

We can now develop renormalized perturbation theory for massless φ^4 theory taking exact same steps we did before. We start with the action in its bare form,

$$\mathcal{A} = \int d^d x \left[\frac{1}{2} \left(\partial \varphi_0 \right)^2 + \frac{m_0^2}{2} \varphi_0^2 + \frac{\lambda_0}{4!} \varphi_0^4 \right],$$

with some bare mass parameter (yet to be adjusted), and write it as

$$\mathcal{A} = \int d^d x \left[\frac{1}{2} \left(\partial \varphi \right)^2 + \frac{\lambda_{(\mu)}}{4!} \varphi^4 + \text{counterterms} \right],$$

where

$$\varphi = Z_{(\mu)}^{-1/2} \varphi_0 \,,$$

and the counterterms are

$$\text{counterterms} = \frac{\delta Z_{(\mu)}}{2} \left(\partial \varphi \right)^2 + \frac{\delta m_{(\mu)}^2}{2} \varphi^2 + \frac{\delta \lambda_{(\mu)}}{4!} \varphi^4 \,,$$

with

$$\begin{aligned} \delta Z_{(\mu)} &= Z_{(\mu)} - 1 \,, \\ \delta m_{(\mu)}^2 &= m_0^2 \, Z_{(\mu)} \,, \\ \delta \lambda_{(\mu)} &= \lambda_0 \, Z_{(\mu)}^2 - \lambda_{(\mu)} \,. \end{aligned}$$

Again, we assume that all the counterterm coefficients are power series in the renormalized coupling $\lambda_{(\mu)}$,

$$\begin{aligned} \delta Z_{(\mu)} &= z_{(\mu)}^{(2)} \lambda_{(\mu)}^2 + \cdots \,. \\ \delta \lambda_{(\mu)} &= a_{(\mu)}^{(2)} \lambda_{(\mu)}^2 + \cdots \,, \\ \delta m_{(\mu)}^2 &= b_{(\mu)}^{(1)} \lambda_{(\mu)} + \cdots \,, \end{aligned}$$

with the coefficients determined from the normalization conditions (13.6), (13.7),(13.8).

A remark is in order. We have put two parameters, m_0^2 and λ_0, in the bare action. Renormalized massless theory depends on a single parameter, the renormalized coupling $\lambda_{(\mu)}$, only. The normalization scale μ is not a parameter of the theory, in that it is an attribute of the renormalization scheme, rather then the theory itself. We have imposed three normalization conditions Eqs.(13.6)-(13.8). While Eq.(13.7) defines the renormalized coupling $\lambda_{(\mu)}$ and Eq.(13.6) fixes normalization of φ, the Eq.(13.8) states that the theory is massless, and thus it imposes restriction on the

theory itself. In principle, this last condition can be translated into certain relation between the bare parameters m_0^2 and λ_0. The bare theory is implicitly equipped with some regularization, and precise form of such relation depends on the regularization method. For instance, if some cutoff regularization is implied, with the cutoff momentum Λ, at $d = 4$ the relation has general form

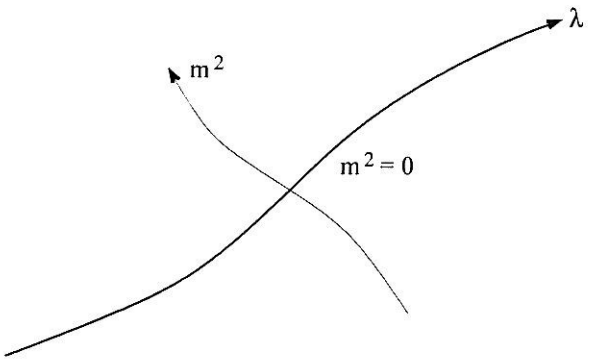
$$m_0^2 = \Lambda^2 C(\lambda_0) . \tag{13.9}$$

Precise shape of $C(\lambda_0)$ heavily depends on specific implementation of the cutoff. Subset of φ^4 theories selected by this condition corresponds to critical domain in the theory of second order phase transitions. In dimensional regularization there is no cutoff parameter Λ, and simple dimensional analysis shows that this relation must take simple form

$$m_0^2 = 0 \qquad \text{(dimensional regularization)} .$$

This means in particular that $\delta m^2 = 0$, and we do not have to bother with the mass renormalization at all. This is one of the technical advantages of the dimensional regularization technique.

Again, thinking in terms of the "two-dimensional manifold of φ^4 theories", the massless φ^4 theories correspond to certain one-dimensional "critical" submanifold (critical curve, in our case),



In statistical mechanics, the "bare action" models microscopic interaction, and the parameters m_0^2 and λ_0, as well as the cutoff energy Λ, are related to temperature and the microscopic physical characteristics of the matter under consideration (e.g. molecular interactions in a gas/liquid). When these parameters cross the "critical subbmanifold" the macroscopic system undergoes a second order phase transition.

Renormalization as the scale transformation. Callan-Symanzik equation

Although in general φ^4 theory the coordinates (m^2, λ) depend on the renormalization scheme, the "critical submanifold" $m^2 = 0$ is scheme independent. It is because the condition (13.8), being a physical statement, is scheme independent. But even within the critical curve there still is a freedom in the choice of the coordinate λ along . Within the above class of schemes (13.6)-(13.7) this choice is controlled by the choice of the normalization scale μ. If μ and $\tilde{\mu}$ are two different values of the normalization scale, the associated renormalized couplings constants

$$\lambda = \lambda_{(\mu)} \qquad \text{and} \qquad \tilde{\lambda} = \lambda_{(\tilde{\mu})}$$

are related as

$$\tilde{\lambda} = g(\lambda) = g(\lambda|\, \tilde{\mu}/\mu)\,,$$

where at $d = 4$, by dimensional counting, the function g can depend on λ and the dimensionless ratio $\tilde{\mu}/\mu$. And if $\tilde{\mu}$ is infinitesimally close to μ, i.e. $\tilde{\mu} = \mu + \delta\mu$, then

$$\tilde{\lambda} = \lambda + \frac{\delta\mu}{\mu}\,\beta(\lambda)\ ,$$

where

$$\beta(\lambda) = \left.\frac{dg(\lambda|\xi)}{d\xi}\right|_{\xi=1}$$

is a function of λ alone.

Of course the correlation functions are essentially independent on the renormalization scheme, in the present context on the choice of the scale μ. More precisely, the ratios of the correlation functions, e.g.

$$R(x_i|\lambda_{(\mu)}, \mu) = \frac{\langle\varphi(x_1)\cdots\varphi(x_{2n})\rangle_{(\mu)}}{\langle\varphi\varphi\rangle^n_{(\mu)}} \tag{13.10}$$

which are insensitive to the normalization of φ, do not change under substitutions

$$\mu \to \tilde{\mu}, \quad \lambda_{(\mu)} \to \lambda_{(\tilde{\mu})}\ .$$

Here the index μ in $\langle\cdots\rangle_{(\mu)}$ signifies that the correlation function is calculated in the scheme with the normalization scale μ. This independence can be expressed in the form of partial differential equation

$$\left(\mu\,\frac{\partial}{\partial\mu} + \beta(\lambda)\,\frac{\partial}{\partial\lambda}\right) R(x_i|\lambda, \mu) = 0\ ,$$

where now λ and μ are viewed as two independent variables.

These equations can be extended to the correlation functions themselves if one recalls that a change in renormalization scheme in general induces a finite change in normalization of φ,

$$\varphi_{(\tilde{\mu})} = Z_{\text{finite}}^{-\frac{1}{2}}(\lambda \,|\, \tilde{\mu}/\mu\,)\,\varphi_{(\mu)} \ ,$$

where again at $d = 4$ the constant Z_{finite} can depend on λ and dimensionless ratio $\tilde{\mu}/\mu$. For an infinitesimal change $\tilde{\mu} = \mu + \delta\mu$

$$\varphi_{(\mu+\delta\mu)} = \left(1 - \frac{\delta\mu}{\mu}\,\gamma(\lambda)\right)\varphi_{(\mu)} \ ,$$

with

$$\gamma(\lambda) = -\left.\frac{dZ_{\text{finite}}^{-\frac{1}{2}}(\lambda,\xi)}{d\xi}\right|_{\xi=1} .$$

Taking into account this transformation of φ we obtain the so-called *Callan - Symanzik equation*:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\,\gamma(\lambda)\right)\langle\varphi(x_1)\cdots\varphi(x_n)\rangle_{(\mu)} = 0 \qquad (13.11)$$

where the correlation function is viewed as depending on two parameters, λ and μ.

At the first glance the Callan-Symanzik equation contains strictly zero physical information. Indeed, in order to make a meaningful renormalized perturbation theory we have introduced an auxiliary scale μ having in mind that nothing physical can depend on this arbitrary scale, and then the equation (13.11) just states that yes indeed nothing physical depends on μ. This is absolutely true if no information about the functions $\beta(\lambda)$ and $\gamma(\lambda)$ is added. However, if something is known about those functions the Callan-Symanzik equation becomes useful.

To see this let us first notice that by dimensional counting any correlation function of the $d = 4$ massless theory can be written as

$$\langle\varphi(x_1)\cdots\varphi(x_n)\rangle_{(\mu)} = \mu^n\, C^{(n)}(\mu x_1, \cdots, \mu x_n|\lambda) \ ,$$

where $C^{(n)}$ is dimensionless function of dimensionless variables μx_i and λ. Therefore the Callan-Symanzik equation (13.11) can be written as

$$\left(\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i} + n\,(1+\gamma(\lambda)) + \beta(\lambda)\frac{\partial}{\partial\lambda}\right)\langle\varphi(x_1)\cdots\varphi(x_n)\rangle = 0 \ , \qquad (13.12)$$

where the coordinate derivative operators have the following exact meaning

$$x_i \frac{\partial}{\partial x_i} = \sum_{\mu=1}^{4} x_i^\mu \frac{\partial}{\partial x_i^\mu} \, .$$

Note that the differential operator $\sum_i x_i \partial/\partial x_i$ describes infinitesimal dilation of the coordinates x_i and therefore the equation (13.12) is the statement that such dilation can be essentially compensated (up to the field renormalization) by appropriate change of the coupling parameter λ. More precisely, an infinitesimal dilation of the coordinates is equivalent to certain infinitesimal shift of λ plus an infinitesimal renormalization of φ,

$$x^\mu \to (1 - \delta L)\, x^\mu \qquad \Leftrightarrow \qquad \lambda \to \lambda + \beta(\lambda)\, \delta L \, , \quad \varphi \to \varphi + (1 + \gamma(\lambda))\, \varphi\, \delta L \, .$$

Let $\lambda(L)$ be a solution of the following differential equation and initial condition

$$L \frac{d}{dL} \lambda(L) = \beta(\lambda(L)) \, , \qquad \lambda(1) = \lambda \, . \tag{13.13}$$

Then the Callan-Symanzik equation states that

$$Z^{\frac{n}{2}}(L)\, \langle \varphi(Lx_1) \cdots \varphi(Lx_n) \rangle_{\lambda(L)} = \langle \varphi(x_1) \cdots \varphi(x_n) \rangle_\lambda \, , \tag{13.14}$$

where the index λ now shows the value of renormalized coupling constant, and $Z(L)$ solves the linear differential equation

$$L \frac{d}{dL} Z^{\frac{1}{2}}(L) = \Big(1 + \gamma(\lambda(L)) \Big)\, Z^{\frac{1}{2}}(L) \, , \qquad Z^{\frac{1}{2}}(1) = 1 \, . \tag{13.15}$$

It is assumed that the correlation functions in (13.14) are computed within the same scheme, with the same normalization scale μ, but with different values, $\lambda(L)$ and λ, of the renormalized coupling parameter. Note that μ does not appear in (13.14), i.e. in this form the Callan-Symanzik equation is not a statement about scheme dependence but rather determines how the theory behaves when one goes from one scale to another.

Appendix: More on renormalizations of composite fields

Simple multiplicative renormalization of the field φ^3, i.e. the form $[\varphi^3]_{\rm R} = Z_3(\epsilon)\, \varphi^3$, is not sufficient for removing divergences in all correlation functions. Instead, one has to look for the renormalized field in the form of the linear combination

$$\frac{1}{3!} [\varphi^3]_{\rm R} = \frac{Z_3}{3!} \varphi^3 + Z_{3,1}\, \varphi - Z_{3,1''}\, \partial_\mu^2 \varphi \, . \tag{13.16}$$

The reason for adding the term $\sim \partial^2\varphi$ will become clear in a moment. As usual, the coefficients Z are assumed to be certain power series in the renormalized coupling constant,

$$Z_3 = 1 + \lambda\, z_3^{(1)} + \cdots ,$$

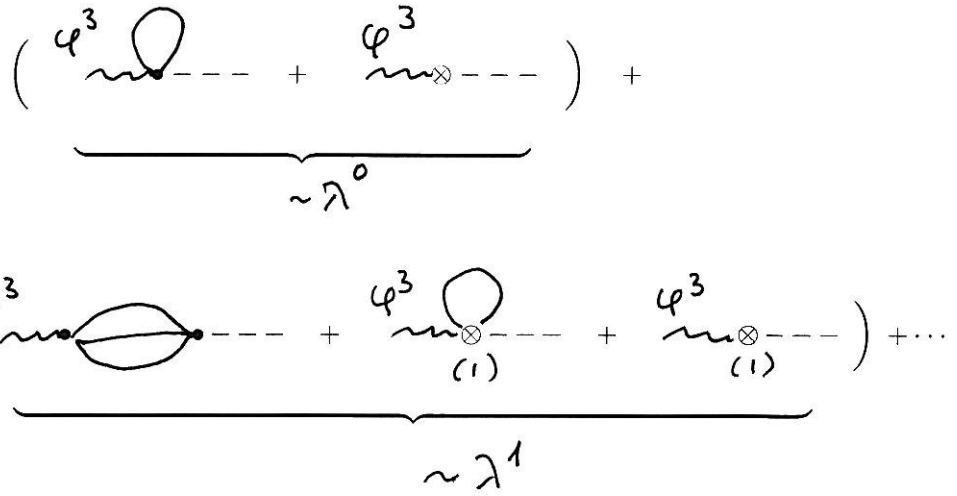
$$Z_{3,1} = z_{3,1}^{(0)} + \lambda\, z_{3,1}^{(1)} + \cdots ,$$

$$Z_{3,1''} = \lambda\, z_{3,1''}^{(1)} + \cdots .$$

Therefore, insertion of the renormalized field $\frac{1}{3!}[\varphi^3]_{\mathrm{R}}$ gives rise to series of vertices,

and

Then, for the vertex $\Gamma^{(\varphi^3_{\mathrm{R}},1)}$ we have the following series of diagrams



We already know from the free theory that

$$z_{3,1}^{(0)} = -\frac{1}{2}\,D(0) + \text{finite} = -\frac{1}{2}\int \frac{d^dk/(2\pi)^d}{k^2+m^2} + \text{finite}\,. \tag{13.17}$$

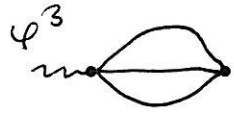
The finite parts here, and in higher order counterterms are arbitrary in that they are not constrained by general requirement of locality. After all, finite renormalization of $[\varphi^3]_{\rm R}$, or adding finite portions of (already renormalized) fields φ and $\partial^2\varphi$ produces equally good local field that one can take for renormalized φ^3. One can fix this ambiguity by adopting minimal subtraction scheme, or by imposing some normalization conditions, say

$$-\Gamma^{\varphi^3_{\rm R},1}(p^2) = O(p^4)\,, \tag{13.18}$$

in addition to the one we have taken already,

$$-\Gamma^{\varphi^3_{\rm R},3}(0,0,0) = 1\,.$$

With this normalization condition, the finite term in (13.17) must be zero, so that the two diagrams $\sim \lambda^0$ cancel exactly. At the order λ^1, the first of the above diagrams is



$$= -\frac{\lambda}{3!}\,\Pi(p^2)\,,$$

where $\Pi(p^2)$ is already familiar integral

$$\Pi(p^2) = \int \frac{d^dk_1\; d^dk_2}{(2\pi)^{2d}}\,\frac{1}{(k_1^2+m^2)(k_2^2+m^2)((p+k_1+k_2)^2+m^2)}\,.$$

We have found that

$$\Pi_r(p^2) = \Pi(p^2) - \Pi(0) - p^2\,\Pi'(0)$$

is finite, and the above normalization condition is satisfied by the choice

$$z_{3,1}^{(1)} = \frac{1}{3!}\,\Pi(0) - \frac{z_3^{(1)}}{2}\,D(0)\,, \qquad z_{3,1''}^{(1)} = \frac{1}{3!}\,\Pi'(0)\,,$$

where $z_3^{(1)} = \frac{3}{2}\,I(0)$ is the first-order counterterm associated with Z_3. Under this choice all singular parts cancel, and the above three diagrams $\sim \lambda^1$ sum to finite quantity $-\frac{\lambda}{3!}\,\Pi_r(p^2)$.

We have seen few lectures ago that the fields in the correlation functions satisfy the equation of motion in the "weak sense",

$$(m_0^2 - \partial_\mu^2)\varphi_0(x) + \frac{\lambda_0}{3!}\,\varphi_0^3(x) \simeq 0\,, \tag{13.19}$$

the "weak" equation $F(x) \simeq 0$ means that

$$\langle F(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = 0 \qquad \text{for} \quad x \neq x_1, \cdots x_n\,,$$

with any local fields $\mathcal{O}_i$. The above "weak" equation was obtained by formal manipulations with the functional integral, namely by performing the change of variables

$$\varphi_0(x) \to \varphi_0(x) + \epsilon(x)$$

in the functional integral written in its "bare" form. It involves therefore the bare parameters and the bare field φ_0. As it stands it makes sense only if some regularization is present. One can repeat the same manipulation with the functional integral expressed through the renormalized field and renormalized parameters. This way one obtains

$$(m^2 + \delta m^2)\varphi(x) - (1 - \delta Z)\,\partial_\mu^2\varphi(x) + \frac{\lambda + \delta\lambda}{3!}\,\varphi^3(x) \simeq 0\,,$$

which of course is just the same "bare" equation (13.19) expressed through renormalized parameters and renormalized φ. It contains the singular counterterm coefficients as well as the field φ^3 which, before renormalization, is also singular. Again, it makes sense only in regularized theory. However now we can use (13.16) to exclude the field φ^3 in favor of the renormalized field $[\varphi^3]_R$. One then obtains the equation

$$m^2\,B_m^2\,\varphi(x) - \partial_\mu^2\,\varphi(x) + \frac{\lambda}{3!}\,B_\lambda\,[\varphi^3]_R(x) \simeq 0\,, \tag{13.20}$$

where

$$B_{m^2} = \frac{1}{m^2}\,\frac{(m^2 + \delta m^2)\,Z_3 - (\lambda + \delta\lambda)\,Z_{3,1}}{(1 + \delta Z)\,Z_3 - (\lambda + \delta\lambda)\,Z_{3,1''}}\,,$$

$$B_\lambda = \frac{1 + \delta\lambda/\lambda}{(1 + \delta Z)\,Z_3 - (\lambda + \delta\lambda)\,Z_{3,1''}}\,.$$

The equation (13.20) states certain relation between the correlation functions of fully renormalized fields which remain finite when the regularization is removed (i.e. when ϵ is set equal 0). This suggests that the coefficients B_{m^2} and B_λ, which appear as

complicated combination of singular counterterms, are in fact finite. This finiteness is direct consequence of the existence of renormalized perturbation theory. Let us check it in the leading order in λ. As $\delta\lambda/\lambda = a^{(2)}\,\lambda + O(\lambda^2)$, $Z_3 = 1 + z_3^{(1)}\,\lambda + O(\lambda^2)$ and $\delta Z = O(\lambda^2)$ we find for example

$$B_\lambda = 1 + \left(a^{(2)} - z_3^{(1)}\right)\lambda + O(\lambda^2)\,.$$

It is not difficult to check that the difference $a^{(2)} - z_3^{(1)}$ is finite in any renormalization scheme.

Note that canonical mass dimension of φ^3 at $d = 4$ is

$$[\varphi_0^3] = 3\,,$$

and

$$[\varphi] = 1\,, \qquad [\partial^3\varphi] = 3\,.$$

The renormalized field $[\varphi^3]_{\rm R}$ is a linear combination of the bare fields of canonical dimension 3 or lower, which have the same symmetries as φ^3. It is not difficult to check that there are no other composite fields, besides φ_0, $\partial^2\varphi_0$, and φ_0^3 itself, which i) have canonical dimension ≤ 3, ii) transform as scalars under coordinate rotations, and iii) change sign under obvious symmetry of the φ^4 theory: $\varphi \to -\varphi$. This is a general feature of renormalization of composite fields. In order to eliminate divergences, it is sufficient to look for renormalized fields in the forms of linear combinations of the bare fields of the same or lower canonical dimensions, and of the same symmetries. In doing so one usually assumes that the fields of lower dimensions are already renormalized, and so renormalizations of composite fields goes step by step, from lower dimensions up.

To illustrate this further, let us briefly discuss renormalization of the fields of canonical dimension 4. There are two new fields of this dimension,

$$\mathcal{O}_I = \frac{1}{2}\,(\partial\varphi)^2\,, \qquad \mathcal{O}_{II} = \frac{1}{4!}\,\varphi^4\,. \tag{13.21}$$

Along with these we must consider two other fields

$$\mathcal{O}_{III} = \frac{1}{2}\,\partial^2\,\varphi^2\,, \qquad \mathcal{O}_{IV} = \frac{1}{2}\,\varphi^2\,,$$

because the first of them also has canonical dimension 4, and canonical dimension of the second is 2 which is lower then 4, and both have the same $\varphi \to -\varphi$ symmetry.

Of course renormalization of these last two fields is completely determined by the renormalization of the field φ^2 which we already considered. At the same time renormalized fields corresponding to (13.6) must be obtained as linear combinations of the bare fields $\mathcal{O}_{A,\,0}$. $A = I, II, III, IV$, i.e.

$$\mathcal{O}_{A,\,R} = \sum_B \left[Z^{-1}\right]_{AB} \mathcal{O}_{B,\,0}\,.$$

where Z is 4×4 matrix

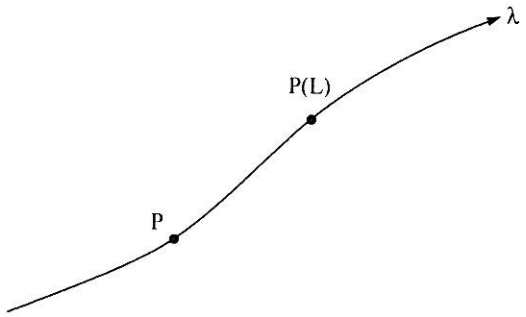
$$Z = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}.$$

Here $\mathbf{C} = \mathrm{diag}\,(Z_{\varphi^2}, Z_{\varphi^2})$, and 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ must be found from suitable normalization conditions order by order in λ. Note that this matrix has triangular (in terms of dimensions) form: fields of higher dimensions do not mix into the fields of lower dimensions. Renormalizations of composite fields of yet higher canonical dimensions, and renormalizations of tensor fields, follow the same pattern.

14 Lecture 14

Renormalization group flow

We can give this equation the following geometric interpretation. The coupling constants λ and $\lambda(L)$ are coordinates representing two points, P and $P(L)$, of the critical submanifold.



The equation (14.5) then states that the two field theories P and $P(L)$ are related by the scale transformation

$$x \to L\,x \tag{14.1}$$

(the field renormalizations obviously do not affect the physical content of the theory). The equation

$$L\frac{d}{dL}\lambda(L) = \beta(\lambda(L)) \tag{14.2}$$

describes, in given coordinate system, the "flow" in the space of field theories generated by the scale transformations. It is called the *renormalization group flow*, and the equation (14.2) is usually referred to as the *renormalization group equation*. The quantity $\lambda(L)$ is often referred to as "running", or "scale-dependent" coupling constant. The fact that the scale transformations (14.1) generates a "flow" is not limited to the massless theory; we will see shortly that the Callan-Symanzik equation generalizes to the case $m^2 \neq 0$.

We have seen that the functions $\beta(\lambda)$ and $\gamma(\lambda)$ contain most important information about the scale dependence of the massless theory. Unfortunately, these functions are not known exactly. However we can compute them perturbativly, order by order in the coupling parameter λ.

There are many ways to do this. The simplest is just to compute the correlation functions perturbatively and then determine the functions β and γ from the Callan-Symanzik equation (13.11) itself. Let us first note that the Callan-Symanzik equation in the form (13.11) applies to the momentum-space correlation functions as well as to coordinate-space ones (the Fourier transform relating the two depends neither on λ nor on μ). Moreover, simple calculation shows that this equation can be written as the equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}-n\,\gamma(\lambda)\right)\Gamma^{(n)}(p_1,\cdots,p_n)=0 \tag{14.3}$$

involving the proper vertices.

To the order λ^2 the proper vertices $\Gamma^{(2)}$ and $\Gamma^{(4)}$, when computed in the scheme (13.7)-(13.6), are (recall your solution of the Problem 6)

$$\Gamma^{(2)}(p^2)=p^2+\frac{1}{12}\,\frac{\lambda^2}{(4\pi)^4}\,p^2\left(\log(\mu^2/p^2)+1\right)+O(\lambda^3)\ . \tag{14.4}$$

$$\Gamma^{(4)}(p_1..,p_4)=\lambda+\frac{\lambda^2}{2\,(4\pi)^2}\left[\log\left(\frac{p_{12}^2}{4\mu^2}\right)+\log\left(\frac{p_{13}^2}{4\mu^2}\right)+\log\left(\frac{p_{14}^2}{4\mu^2}\right)\right]+O(\lambda^3) \tag{14.5}$$

where $p_{12}^2=(p_1+p_2)^2$, etc. Let us first plug (14.4) into (14.3). The result is

$$\frac{2}{12}\,\frac{\lambda^2}{(4\pi)^4}\,p^2+\beta(\lambda)\,\frac{2}{12}\,\frac{\lambda}{(4\pi)^4}\,p^2\left[\log(\mu^2/p^2)+1\right]-2\gamma(\lambda)\,p^2+O(\lambda^3)=0\,.$$

where the first term came from the $\mu\partial/\partial\mu$ term in (14.3). The second term contains $p^2\,\log\mu^2/p^2$: this dependence cannot be canceled by two other terms exposed in the above equation. The only way to satisfy this equation in the order λ^2 is to assume that $\beta(\lambda)\sim\lambda^2$ so that the second term has to be attributed to the $O(\lambda^3)$ corrections. We then find

$$\gamma(\lambda)=\frac{1}{12}\,\frac{\lambda^2}{(4\pi)^4}+O(\lambda^3)\,. \tag{14.6}$$

Similarly, substitute the above perturbative result for $\Gamma^{(4)}$ into (14.3). We have

$$-3\,\frac{\lambda^2}{(4\pi)^2}+\beta(\lambda)-4\,\lambda\,\gamma(\lambda)+O(\lambda^3)=0\ .$$

We already know that $\gamma(\lambda) \sim \lambda^2$ and so the term involving this function is $O(\lambda^3)$. Therefore

$$\beta(\lambda) = 3\,\frac{\lambda^2}{(4\pi)^2} + O(\lambda^3)\ . \tag{14.7}$$

Note that the leading term (14.7) of the β-function is *positive*. This is the feature of φ^4 theory. The flow equation (13.13) then shows that for small λ the scale-dependent coupling constant $\lambda(L)$ *increases* with L. The equation (13.14) shows that larger L correspond to *shorter* length scales (this is clear if one rewrites (13.14) in equivalent form

$$\langle\varphi(x_1)\cdots\varphi(x_n)\rangle_{\lambda(L)} \sim \langle\varphi(x_1/L)\cdots\varphi(x_n/L)\rangle_{\lambda}$$

showing that the correlation functions computed with $\lambda(L)$ are equivalent to the correlation functions computed with λ, with the separations "shrunken" by the scale factor L).

If one ignores the higher order corrections in (14.7) the flow equation (13.13) is easily integrated,

$$\lambda(L) = \frac{\lambda}{1 - 3\,\frac{\lambda}{(4\pi)^2}\,\log L}\ . \tag{14.8}$$

At sufficiently large L (i.e. sufficiently short distances) the function $\lambda(L)$ diverges. This signals potential trouble with locality which we discussed earlier. Of course when $\lambda(L)$ becomes large one cannot ignore the higher-order corrections in (14.7); in fact in order to understand short-distance behavior of φ^4 theory we need to know the shape of the β-function at large λ. We conclude that the question of consistency of φ^4 theory cannot be settled on the perturbative level.

In nonabelian gauge theories the leading term of the β-function turns out to be *negative*, meaning that the running coupling constant *decreases* at short distances (instead it increases at large distances); as a result, although nonabelian gauge theory usually cannot be solved by means of perturbation theory, the question of its *consistency* as a field theory can be positively answered on the basis of perturbative calculations. We will discuss this phenomenon known as *asymptotic freedom* in greater details later.

Mass perturbation

Let us refine our techniques further. So far we have considered the renormalization group flow in massless φ^4 theory. We would like to generalize this picture to the

massive case. The most natural way to do this is to consider the massive φ^4 theory as the massless theory perturbed by adding the term $\frac{m^2}{2}\varphi^2$ to the action density. We already know that the field φ^2 requires renormalization. Let us discuss how this field renormalizes in our μ-dependent scheme.

In the massless theory the normalization condition $\Gamma^{(\varphi^2_R,2)}(0,0)=1$ we have used before has to be replaced by some nonsingular μ dependent condition. One can take for example

$$\Gamma^{(\varphi^2_R,2)}(p_1,p_2)\Big|_{\substack{p_i^2=2\mu^2\\(p_1p_2)=-\mu^2}} = 1\,. \tag{14.9}$$

The renormalized field field $[\varphi^2]_{(\mu)}$ defined by this condition differs from the bare field φ_0^2 by the renormalization constant[8],

$$[\varphi^2]_{(\mu)} = \left(Z^{\varphi^2}_{(\mu)}\right)^{-1}\varphi_0^2\,.$$

Simple calculation similar to that discussed in the previous lecture shows that in the leading order

$$Z_{\varphi^2}(\mu) = 1 - \frac{1}{2}\,\frac{\lambda}{(4\pi)^{\frac{d}{2}}}\,(2\mu^2)^{\frac{d}{2}-2}\,\frac{\Gamma(2-d/2)\Gamma^2(d/2-1)}{\Gamma(d-2)} + O(\lambda^2) \tag{14.10}$$

Although $Z^{\varphi^2}_{(\mu)}$ is divergent at $d=4$, the ratio $Z^{\varphi^2}_{(\tilde{\mu})}/Z^{\varphi^2}_{(\mu)}$ is expected to be finite. An infinitesimal change of the renormalization scale $\mu \to \mu+\delta\mu$ then leads to infinitesimal renormalization

$$[\varphi^2]_{(\mu+\delta\mu)} = \left(1-\frac{\delta\mu}{\mu}\,\gamma_{\varphi^2}(\lambda)\right)[\varphi^2]_{(\mu)}\,,$$

where

$$\gamma_{\varphi^2}(\lambda) = \left(Z^{\varphi^2}_{(\mu)}\right)^{-1}\mu\frac{d}{d\mu}\,Z^{\varphi^2}_{(\mu)} \tag{14.11}$$

is finite function of λ. From (14.10) we obtain

$$\gamma_{\varphi^2}(\lambda) = \frac{\lambda}{(4\pi)^2} + O(\lambda^2)\,. \tag{14.12}$$

[8]Note that this definition of the φ^2 renormalization constant differs from that introduced in Lecture 13, Eq.(13.3). There, we have defined the constant Z_{φ^2} as the coefficient between $[\varphi^2]_R$ and φ^2. The new Z^{φ^2} differs from Z_{φ^2} in Lecture 13 by the factor of Z.

Repeating our previous calculations we find that a correlation function with N insertions of $[\varphi^2]_{(\mu)}$ and n insertions of φ, satisfies the modified Callan-Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+n\,\gamma(\lambda)+N\,\gamma_{\varphi^2}(\lambda)\right)\langle\varphi(x_1)\cdots\varphi(x_n)\,[\varphi^2]_{(\mu)}(y_1)\cdots[\varphi^2]_{(\mu)}(y_N)\rangle_{(\mu)}=0\,.$$

Let us now introduce the mass term as

$$\mathcal{A}_{M^2}=\mathcal{A}_0+\int d^dx\,\frac{M^2}{2}\,[\varphi^2]_{(\mu)}\;,$$

where $\mathcal{A}_0$ is the massless action

$$\mathcal{A}_0=\int d^dx\left[\frac{1}{2}\,(\partial\varphi)^2+\frac{\lambda}{4!}\,\varphi^4+\frac{\delta Z_{(\mu)}}{2}\,(\partial\varphi)^2+\frac{\delta\lambda_{(\mu)}}{4!}\,\varphi^4\right].$$

Here $\delta Z_{(\mu)}$ and $\delta\lambda_{(\mu)}$ are counterterms computed in the massless theory, within the above μ-dependent renormalization scheme (and I assumed dimensional regularization and therefore the δm^2 counterterm is not included). Recalling that

$$[\varphi^2]_{(\mu)}=\frac{Z_{(\mu)}}{Z^{\varphi^2}_{(\mu)}}\,\varphi^2\;,\qquad Z_{(\mu)}=1+\delta Z_{(\mu)}\;,$$

the massive action can be written as

$$\mathcal{A}_{M^2}=\int d^dx\left[\frac{1}{2}\,(\partial\varphi)^2+\frac{M^2}{2}\,\varphi^2+\frac{\lambda}{4!}\,\varphi^4+\frac{\delta Z_{(\mu)}}{2}\,(\partial\varphi)^2+\frac{\delta M^2_{(\mu)}}{2}\,\varphi^2+\frac{\delta\lambda_{(\mu)}}{4!}\,\varphi^4\right],$$

where

$$\delta M^2_{(\mu)}=M^2\left(\frac{Z_{(\mu)}}{Z^{\varphi^2}_{(\mu)}}-1\right).\tag{14.13}$$

This action can be used now to develop a renormalized perturbation theory in the massive case. Note that the above definition of the mass parameter corresponds to specific renormalization scheme in which

- The counterterms $\delta Z_{(\mu)}$ and $\delta\lambda_{(\mu)}$ are borrowed from the massless theory; in particular these coefficients do not depend on M^2.

• The counterterm $\delta M^2_{(\mu)}$ is expressed by (14.13) through the φ^2-renormalization constant $Z^{\varphi^2}_{(\mu)}$; again, this constant has to be computed upfront in the massless theory, and $\delta M^2_{(\mu)}$ depends on M^2 linearly.

Although it is possible to show that the counterterms thus defined indeed compensate for the divergences in the renormalized perturbation theory, no explicit normalization conditions are imposed in the massive theory. Relation between M^2, λ and physical quantities in this scheme is not specified in advance but must be computed order by order in the perturbation theory. In this sense this renormalization scheme is similar to the "minimal subtraction" scheme.

Note that in this scheme the correlation functions depend on two mass parameters, M and our auxiliary scale μ, although we know only one is needed. This is the price we pay for defining specific coordinate system (M^2, λ) which is nonsingular on the critical submanifold $M^2 = 0$. It is not difficult to check that correlation functions computed within this scheme admit smooth limit $M^2 = 0$; in fact it is clear in advance: at $M^2 = 0$ this perturbation theory simply reduces to the massless renormalized perturbation theory.

Like in the massless case, nothing except for normalizations depends on the auxiliary scale μ. Under the infinitesimal change

$$\mu \to \mu + \delta\mu$$

the renormalized field $[\varphi^2]_{(\mu)}$ transforms as

$$[\varphi^2]_{(\mu+\delta\mu)} = \left(1 - \frac{\delta\mu}{\mu}\,\gamma_{\varphi^2}(\lambda)\right)[\varphi^2]_{(\mu)}\,.$$

This brings in certain change in the action, which however is compensated if we simultaneously change the mass parameter,

$$M^2 \to \left(1 + \frac{\delta\mu}{\mu}\,\gamma_{\varphi^2}(\lambda)\right) M^2\,.$$

Invariance of the action w.r.t. the above simultaneous change in μ and M^2 leads to the equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\,\gamma(\lambda) + \gamma_{\varphi^2}(\lambda)\,M^2\frac{\partial}{\partial M^2}\right)\langle\varphi(x_1)\cdots\varphi(x_n)\rangle = 0\,.$$

This equation shows that in the massive case the scale transformations induce the "flow" in the two-dimensional space of φ^4 theories,

$$L\frac{d}{dL}\,\lambda = \beta(\lambda)\,, \qquad L\frac{d}{dL}\,M^2 = \gamma_{\varphi^2}(\lambda)\,M^2\,. \tag{14.14}$$

Note that $\beta(\lambda)$ and $\gamma(\lambda)$ here are exactly the same as in the massless theory, while $\gamma_{\varphi^2}(\lambda)$ is also determined from the massless theory via Eq.(14.11).

Spin

So far we have limited attention to scalar field theories, where the "fundamental field" (=integration variable in the functional integral) was taken to be a scalar under Lorentz transformations. More realistic field theories are based on fields with non-trivial transformation properties. Such are the

$$\text{Electromagnetic field :} \qquad F_{\mu\nu}(x) = -F_{\nu\mu}(x)$$

$$\text{Dirac (electron-positron) field :} \qquad \psi_\omega(x), \bar{\psi}^\omega(x)\,, \quad \omega = 1,2,3,4\,.$$

More generally, fields are classified according to irreducible representations of the Lorentz Group.

Representations of the Lorentz Group

The Lorentz group includes both spatial rotations and Lorentz boosts. Formally, it consists of all linear transformations of the space-time coordinates $x^\mu = (x^0, x^1, x^2, x^3) = (t, \mathbf{x})$

$$x^\mu \to \tilde{x}^\mu = \Lambda^\mu_\nu\, x^\nu \quad \text{or simply} \quad x \to \tilde{x} = \Lambda x$$

which preserve Minkowski pseudo metric

$$s^2(x,x') = g_{\mu\nu}(x-x')^\mu (x-x')^\nu = (\mathbf{x}-\mathbf{x})^2 - (t-t')^2$$

Here

$$g_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$$

is the Minkowski pseudo-metric[9]. In mathematics, this group is referred as $O(3,1)$.

Scalar field transforms the most simple way: The value $\varphi(x)$ depends only on the point in space-time, but not on the coordinate system. In other words, when coordinates are changed by any Lorentz transform $x \to \tilde{x} = \Lambda x$, we have

$$\varphi(x) \to \tilde{\varphi}(\tilde{x}) = \varphi(\Lambda^{-1}\tilde{x})$$

However one can consider fields with more complicated transformation properties. For example, a vector field $V^\mu(x)$ should transform as

$$V^\mu(x) \to \tilde{V}^\mu(\tilde{x}) = \Lambda^\mu{}_\nu\; V^\nu(\Lambda^{-1}\tilde{x})$$

[9]In PS notations, $g_{\mu\nu}$ has opposite sign, i.e. $g_{\mu\nu}$ there denotes diag(+1,-1,-1,-1)

For co-variant components V_μ of this field

$$V_\mu(x) \to \tilde{V}_\mu(\tilde{x}) = (\Lambda^{-1})^\nu_\mu \, V_\nu(\Lambda^{-1}\tilde{x})$$

Another example is the rank-2 tensor field

$$B^{\mu\nu}(x) \to \tilde{B}^{\mu\nu}(\tilde{x}) = \Lambda^\mu{}_\rho \, \Lambda^\nu{}_\sigma \, B^{\rho\sigma}(\Lambda^{-1}\tilde{x})$$

In general case we can consider multicomponent fields $\Phi_I(x)$ with the transformation law

$$\Phi_I(x) \to \tilde{\Phi}_I(\tilde{x}) = M_{IJ}(\Lambda) \; \Phi_J(\Lambda^{-1}\tilde{x}) \; .$$

The Lorentz transformations form a group, i.,e. for two transformations Λ, Λ' the product $\Lambda'\Lambda$ is also a Lorentz transformation. Then $M(\Lambda)$ has to be a *matrix representation* of the Lorentz group

$$M(\Lambda'\,\Lambda) = M(\Lambda')\, M(\Lambda) \; .$$

Once we have a matrix representation of the group a question arises. Is it possible to break apart the field into smaller sets that do not mix under Lorentz transformation? How do we make this decomposition into *irreducible representations* of the Lorentz group?

Exercise: Does $B^{\mu\nu}$ form an irreducible representation? If not, what irreducibles it decomposes to? What are the dimensions of those irreducible representations?[10]

Complex group $O(4,\mathbb{C})$

To understand better the structure of the group as well as its representation, it is useful to start with complexified group. To that end, let us promote the space-time coordinates to the complex variables

$$x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x}) \in \mathbb{R}^4 \quad \to \quad z^\mu = (z^0, z^1, z^2, z^3) = (z^0, \mathbf{z}) \in \mathbb{C}^4 \, .$$

[10]The solution is obtained via the decomposition

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{4} \, g^{\mu\nu} \, T(x) \; ,$$

where $A^{\mu\nu}$ is antisymmetric ($A^{\mu\nu} = -A^{\nu\mu}$) and $S^{\mu\nu}$ is symmetric ($S^{\mu\nu} = S^{\nu\mu}$) and traceless $g_{\mu\nu}S^{\mu\nu} = 0$). It is not difficult to see that $A^{\mu\nu}$, $S^{\mu\nu}$ and T do not mix each other under Lorentz transformations. Is it possible to further break apart these fields into still smaller sets that do not mix under Lorentz transformation?

We already have done something of this spirit when dealing with the paths integral. Then we complexified the time variable $t = x^0$; since in Lorentz-invariant theory the time comes on almost equal footing with the spatial coordinates, this complexification comes as a natural step. In fact, it will be convenient to relabel the variables as follows. We introduce

$$z^a = (z^1, z^2, z^3, z^4) = (z^1, z^2, z^3, iz^0)$$

with z^1, z^2, z^3 are the same as before, but $z^4 = iz^0$ (Since our coordinates z^a are already complex numbers, appearance of i here is legitimate.) Then the Minkowski interval takes the (complexified) Euclidean form

$$s^2(z, z') = \sum_a (z^a - z'^a)^2 = \delta_{ab}(z^a - z'^a)(z^b - z'^b)$$

which generally takes complex values. The group of linear transformations which preserves this form is the complex orthogonal group $O(4, \mathbb{C})$, i.e. the group of complex orthogonal 4×4 matrices Ω such that

$$\Omega\Omega^t = I\,.$$

The group $O(4, \mathbb{C})$ includes several real groups ("real sections") as subgroups. The most important are (1) Real orthogonal group $O(4, \mathbb{R})$ (rotations of 4D Euclidean space, by taking all z^a real), and (2) Lorentz group $O(1, 3)$, discussed above (by setting all z^0, z^1, z^2, z^3 real).

Relation to $SL(2, \mathbb{C})$

$O(4, \mathbb{C})$ is locally isomorphic to the direct product

$$O(4, \mathbb{C}) \sim SL(2, \mathbb{C}) \times SL(2, \mathbb{C})\,,$$

where $SL(2, \mathbb{C})$ is the group of 2×2 complex matrices with det $= 1$. To see this, for any vector $z^a = (\mathbf{z}, z^4) \in \mathbb{C}^4$ consider 2×2 matrix

$$Z_+(z) = z_4 + i\boldsymbol{\sigma}\mathbf{z}\,,$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices. Here $z_4 = z^4$, as a part of standard Euclidean $z_a = \delta_{ab}\, z^b$. We can also define

$$Z_-(z) = z_4 - i\boldsymbol{\sigma}\mathbf{z}\,,$$

which satisfies identically (direct check)

$$Z_+(z)Z_-(z) = z_4^2 + \mathbf{z}^2 = \sum_{a=1}^{4} (z^a)^2 \;:=\; (z)^2$$

For $(z^2) \neq 0$ we have

$$Z_- = \frac{1}{(z)^2}\, Z_+^{-1}$$

It is also straightforward to check that

$$\det Z_+(z) = \det Z_-(z) = (z)^2$$

Next, consider linear transformations

$$z \;\rightarrow\; \tilde{z}$$

defined via

$$Z_+(\tilde{z}) = A\, Z_+(z)\, B^{-1}\,,$$

or equivalently

$$Z_-(\tilde{z}) = B\, Z_-(z)\, A^{-1}\,,$$

where A and B can be arbitrary complex 2×2 matrices such that

$$\det A = \det B = 1$$

It is straightforward to check that such transformations indeed preserve the complex Euclidean form $s^2(0, z)$, i.e.

$$(z)^2 = (\tilde{z})^2$$

The pairs (A,B) of such matrices form the direct product group

$$(A, B) \;\in\; SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$$

so that indeed $O(4, \mathbb{C})$ is essentially this direct product.

I say essentially because there is a subtle differences. First, $O(4, \mathbb{C})$ contains two disjoint components $O(4, \mathbb{C}, \pm)$ defined by the sign of the determinant

$$\Omega\, \Omega^{\mathrm{t}} = 1 \qquad \det \Omega = \pm 1\,.$$

On the other hand, $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ is connected. In fact

$$SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \;\sim\; SO(4, \mathbb{C}) \equiv O(4, \mathbb{C}, +)$$

which is the connected component of unit element. On the other hand any element of $O(4,\mathbb{C},-)$ can be obtained by combining some transformation from $O(4,\mathbb{C},+)$ with "spatial inversion", or "parity operation"

$$\mathbb{P}: \quad z=(\mathbf{z},z_4) \;\leftrightarrow\; \bar{z}=(-\mathbf{z},z_4)$$

Note that

$$\mathbb{P}: Z_+(z) \quad \leftrightarrow \quad Z_-(z)$$

so that the "parity operation" interchanges the roles of the "left" and the "right" factors in $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$.

Second, the pairs (A,B) and $(-A,-B)$, which are different elements of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$, realize the same transformation of z. Therefore, in fact

$$SO(4,\mathbb{C})=SL(2,\mathbb{C})\times SL(2,\mathbb{C})/\mathbb{Z}_2$$

while $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ provides two-fold cover of $SO(4,\mathbb{C})$.

Real subgroups

Some real subgroups of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ are worth mentioning here.

First, the "diagonal" subgroup $SL(2,\mathbb{C})_{\rm diag}$ is obtained by taking the pairs (A,B) with $B=A$, i.e. (A,A). Obviously,

$$Z_+(\tilde{z})=A\,Z_+(z)\,A^{-1}$$

leaves z_4 unchanged ($\tilde{z}_4=z_4$), and this diagonal subgroup $SL(2,\mathbb{C})$ realizes the proper orthogonal rotations of the complex 3-vector $\mathbf{z}$. More precisely, $SL(2,\mathbb{C})_{\rm diag}$ provides double cover of $SO(3,\mathbb{C})$. If, in addition, A is unitary, $A^\dagger A=I$, so that elements $A\in SU(2)$ realize proper rotations of real 3-vector $\mathbf{z}$, the group alternatively named as $SO(3,\mathbb{R})$.

If A and B are independent matrices but both are unitary, we have $SU(2)\times SU(2)$. In this case

$$Z_+(\tilde{z})=A\,Z_+(z)\,B^\dagger$$

brings real vector z^a to real $\tilde{z}^a$. This is because with real z the matrix $Z_+(z)$ (as well as $Z_-(z)$) is unitary, and vice versa. Therefore $SU(2)\times SU(2)$ is a double cover of the the group $SO(4,\mathbb{R})$ - proper rotations of real 4-vector z^a.

Finally, choosing $B^{-1}=A^\dagger$, i.e considering the pairs

$$(A,(A^{-1})^\dagger)$$

one easily checks that

$$Z_+(\tilde{z}) = A\, Z_+(z)\, A^\dagger$$

brings anti-hermitian Z_+ to anti-hermitian $\tilde{Z}_+$,

$$Z_+^\dagger = -Z_+ \leftrightarrow \tilde{Z}_+^\dagger = -\tilde{Z}_+$$

This corresponds to z^a of the form

$$z^a = (\mathbf{z}, z_4) = (\mathbf{x}, ix^0)$$

with real x^μ, $\mu = 0, 1, 2, 3$. This subgroup corresponds to the proper component of the Lorentz group

$$SO(1.3) = SL(2, \mathbb{C})/Z_2\,.$$

15 Lecture 15

Spin (cont'd)

Representations of $SL(2,\mathbb{C})$ and $O(4,\mathbb{C})$

Irreducible Matrix Representations of $SL(2,\mathbb{C})$ are well known. The basic "fundamental representation" is a complex 2-vector $\xi_\alpha \in \mathbb{C}^2$, with $SL(2,\mathbb{C})$ acting as

$$R^\beta_\alpha(A) = A^\beta_\alpha \, \xi_\beta$$

with any $A \in SL(2,\mathbb{C})$ (complex 2×2 matrix with $\det = 1$). One also can consider "contra-variant" vectors η^α, which transform as

$$R^\alpha_\beta(A)\eta^\beta = \tilde{A}^\alpha_\beta \, \eta^\beta$$

where

$$\tilde{A} = (A^{-1})^{\mathrm{t}} \, .$$

However, The contra-variant representation is equivalent to co-variant one . This is because the quantity

$$\eta_\alpha = \varepsilon_{\alpha\beta} \, \eta^\beta \, , \qquad \varepsilon_{\alpha\beta} = i(\sigma_2)_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$$

transforms as co-variant vector.

General matrix representation of $SL(2,\mathbb{C})$ is a symmetric tensor

$$\xi^{(s)}_{\alpha_1 \, \alpha_2 \, ... \, \alpha_{2s}}$$

of rank $2s$, with

$$s = 0, 1/2, 1, 3/2, ...$$

which transforms as

$$\xi^{(s)} \;\to\; \underbrace{A \otimes A \otimes ... \otimes A}_{2s \text{ times}} \, \xi^{(s)}$$

This has $2s+1$ independent components, and referred to as the "spin s representation". Only symmetric tensors are irreducible, because

$$\varepsilon_{\alpha\beta}\eta^{\alpha\beta}$$

is invariant (scalar) under ant $SL(2,\mathbb{C})$ transformations.

In $O(4,\mathbb{C},+) = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ we have two factors $SL(2,\mathbb{C})$. Correspondingly, generic matrix representation is a tensor product of the representations of the factors. Such representation is characterized by two spins

$$(s,\dot{s}) \qquad s,\dot{s} = 0, 1/2, 1, 3/2, ...$$

so that it is $(2s+1)(2\dot{s}+1)$ component object

$$\left(\xi^{(s,\dot{s})}\right)^{\{\dot{\alpha}_1\dot{\alpha}_2...\dot{\alpha}_{2\dot{s}}\}}_{\{\alpha_1\alpha_2...\alpha_{2s}\}}$$

which transforms as

$$\xi^{(s,\dot{s})} \;\to\; \underbrace{A\otimes A\otimes ...\otimes A}_{2s \text{ times}}\;\underbrace{B\otimes B\otimes ...\otimes B}_{2\dot{s} \text{ times}}\,\xi^{(s,\dot{s})}$$

Here "dot" indicates that the corresponding spin $\dot{s}$ or index $\dot{\alpha}$ is an attribute of the second ("right") factor in $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$.

We see that an (irreducible) field Φ may be characterized by the "spins" $(s,\dot{s})$

$$\left(\Phi^{(s,\dot{s})}\right)^{\{\dot{\alpha}_1\dot{\alpha}_2...\dot{\alpha}_{2\dot{s}}\}}_{\{\alpha_1\alpha_2...\alpha_{2s}\}}(x)\,, \qquad (2s+1)(2\dot{s}+1) \quad \text{components}\,.$$

Thus, $\Phi^{(0,0)}(x)$ is a scalar, while $\Phi^{(1/2,1/2)}(x)$ is a 4-vector. Indeed, we can write

$$\Phi_\alpha{}^{\dot{\alpha}} = V_4\;\delta_\alpha{}^{\dot{\alpha}} + i\mathbf{V}\,\boldsymbol{\sigma}_\alpha^{\dot{\alpha}}\,,$$

where $\Phi_\alpha^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\,\Phi_{\alpha\dot{\beta}}$, and

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

is inverse of $\varepsilon_{\alpha\beta}$. One can write, equivalently

$$\Phi_\alpha{}^{\dot{\alpha}}(x) = i\,V^\mu(x)\,(\sigma_\mu)_\alpha^{\dot{\alpha}}$$

where $\mu = 0,1,2,3$ and $V_4 = iV^0$, and

$$\sigma_\mu = (1, \boldsymbol{\sigma})$$

It is not difficult to check that V^μ transforms as a vector (compare to the transformation of $Z_+(z)$).

More generally. the representations with integer $s+\dot{s}$ are equivalent to irreducible tensor representations $T_{\mu\nu...}$, with certain symmetry properties.

On the other hand, when $s+\dot{s}$ is a half-integer, no equivalent tensor representation exists. This is clear from the fact that under the $SL(2)\times SL(2)$ transformation with $(A,B) = (-1,-1)$ such representation changes sign, while $O(4)$ tensors unsensitive to that transformation.

Spinors and bispinors

The simplest cases are $(1/2, 0)$ and $(0, 1/2)$. In both cases we deal with the *spinors* ξ_α and $\eta^{\dot\alpha}$, respectively. These transform as

$$(A, B) \ : \qquad \begin{cases} \xi_\alpha \to \ A^\beta_\alpha \, \xi_\beta \\ \eta^{\dot\alpha} \to B^{\dot\alpha}_{\dot\beta} \, \eta^{\dot\beta} \end{cases}$$

Note that if $(A, B) = (-1, -1)$ the spinors change sign (this is true for all representations with half-integer $s + \dot s$). Since (A, B) and $(-A, -B)$ correspond to the same element of $SO(4)$, the spinors are called the "double-valued" representations of $SO(4)$. All said applies to all "half-integer spin" $(s + \dot s \in 1/2 + \mathbb{Z}_+)$ representations.

If we take just the spinor, there is no way in which the inversion $\mathbb{P}$, which is legitimate part of Lorentz group, can be realized. Recall that the inversion interchanges the factors in $SL(2) \times SL(2)$. Therefore we need to add the "right" component $\eta^{\dot\alpha}$, thus forming what is called 4-component "bispinor" $(\xi_\alpha, \eta^{\dot\alpha})$. The inversion acts by interchanging

$$\mathbb{P} \ : \qquad (\xi_\alpha, \eta^{\dot\alpha}) \ \leftrightarrow \ \ (\eta_{\dot\alpha}, \xi^\alpha)$$

where, as usual, $\eta_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}$, and similarly for ξ^α. The four component bispinor is usually organized as

$$\psi_\omega(x) = \begin{pmatrix} \xi_\alpha \\ \eta^{\dot\alpha} \end{pmatrix}$$

where ω runs 4 values $1, 2, 3, 4$

$$\omega = \underbrace{1, 2}_{\alpha = 1, 2}, \ \underbrace{3, 4}_{\dot\alpha = 1, 2}$$

Dirac Field

Important example of bispinor field is the free fermion field of the mass m. Define two matrix differential operators ($x_4 = ix^0$)

$$D_+ \equiv (D_+)_{\alpha\dot\alpha} = \frac{\partial}{\partial x_4} + i\boldsymbol{\sigma}\,\boldsymbol{\nabla} = -i\left(\frac{\partial}{\partial x^0} - \boldsymbol{\sigma}\,\boldsymbol{\nabla}\right)$$

$$D_- \equiv (D_-)^{\dot\alpha\alpha} = \frac{\partial}{\partial x_4} - i\boldsymbol{\sigma}\,\boldsymbol{\nabla} = -i\left(\frac{\partial}{\partial x^0} + \boldsymbol{\sigma}\,\boldsymbol{\nabla}\right)$$

(these play the role similar to $Z_\pm$, with $z^a \to \partial/\partial z^a$). One can check that $D_\pm$ transform correspondingly, as

$$D_+ \;\to\; AD_+B^{-1}\,, \qquad D_- \;\to\; BD_-A^{-1}$$

and

$$D_+D_- = \frac{\partial^2}{(\partial x_4)^2} + \boldsymbol{\nabla}^2 = -\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla}^2 := -\Box$$

where $\Box$ denotes d'Alembert operator $\partial^2/\partial x_\mu \partial x^\mu$. It follows that the equations

$$\begin{aligned} (D_-)^{\dot\alpha\alpha}\,\xi_\alpha &= -m\,\eta^{\dot\alpha} \\ (D_+)^{\alpha\dot\alpha}\,\eta^{\dot\alpha} &= -m\,\xi_\alpha \end{aligned} \tag{15.1}$$

often abbreviated as

$$\left\{\begin{aligned} D_-\xi &= -m\,\eta \\ D_+\eta &= -m\,\xi \end{aligned}\right\}$$

are $O(4.\mathbb{C})$ invariant. Eq.(15.1) are known as the *Dirac equations*. It is straightforward to check that each of the spinor components ξ and η solves the KG equation

$$(m^2 + \Box)\xi = (m^2 + \Box)\eta = 0\,.$$

More convenient notations use 4-component form of the bi-spinor,

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}\,,$$

in which the Dirac equation (15.1), i.e.

$$\begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + m \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0$$

can be written as

$$(i\gamma^\mu\partial_\mu + m)\,\psi(x) = 0\,, \tag{15.2}$$

where

$$\gamma^\mu : \qquad \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\,, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$$

are 4×4 *Dirac gamma matrices*; the entries represent 2×2 blocks. It is straightforward to check that γ^μ satisfy

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu,\gamma^\nu + \gamma^\nu,\gamma^\mu = -2\,g^{\mu\nu}\,.$$

Integer vs half-integer spins

For generic representations $(s, \dot{s})$ the field

$$\Phi^{(s,\dot{s})}_{[\alpha_1,....,\alpha_{2s}][\dot{\alpha}_1,....,\dot{\alpha}_{2\dot{s}}]}$$

transforms as

$$\Phi^{(s,\dot{s})}(x) \;\rightarrow\; \tilde{\Phi}^{(s,\dot{s})} = \mathcal{A}^{(s)} \otimes \mathcal{B}^{(\dot{s})}\, \Phi^{(s,\dot{s})}(\Lambda^{-1}x)\,,$$

where

$$\mathcal{A}^{(s)} = \underbrace{A \otimes A \otimes \cdots \otimes A}_{2s}$$
$$\mathcal{B}^{(\dot{s})} = \underbrace{B \otimes B \otimes \cdots \otimes B}_{2\dot{s}}$$

and

$$\Lambda_\mu{}^\nu = \mathrm{tr}\,(\sigma_\mu\, A\, \sigma^\nu\, B)\;;$$

Here again $\sigma^\mu = (1, \boldsymbol{\sigma})$.

The numbers $(s, \dot{s})$ are $SL(2)$ spins, which take integer or half-integer values

$$s, \dot{s} = 0, 1/2, 1, 3/2, ...$$

There is important difference between the fields with integer and half-integer values of $s+\dot{s}$ (which we refer to, for shortness, as the fields of integer and half-integer spin). One aspect was already mentioned - the half-integer spin fields are "double-valued": the sign of such fields are not uniquely fixed. This is seen in the way they transform under $SL(2) \otimes SL(2)$: the transformation with $(A, B) = (-I, -I)$, which maps to unit element of $O(4, \mathbb{C})$, changes the signs of half-integer spin fields. If the group is viewed as continuous manifold, the element $(-I, -I)$ corresponds to 360^o rotation in $O(4)$. Generally, transformations (A, B) and $(-A, -B)$ result in different signs of half-integer fields.

There is another important implication of the sign ambiguity which shows up when one considers half-integer spin fields in quantum field theory. In the presence of half-integer spin fields the notion of locality requires significant modification.

Locality and half-integer spins

As we discussed in the context of scalar field theory, the locality can be expressed by the relation

$$[\Phi_1(t, \mathbf{x}), \Phi_2(t', \mathbf{x}')] = 0 \qquad \text{for} \quad |\mathbf{x} - \mathbf{x}'| > |t - t'| \tag{15.3}$$

Here Φ_1, Φ_2 are Heisenberg field operators

$$\Phi(t,\mathbf{x}) = e^{iHt-i\mathbf{P}\mathbf{x}}\,\Phi(0,0)\,e^{-iHt+i\mathbf{P}\mathbf{x}}$$

It turns out that if both fields Φ_1 and Φ_2 correspond to double-valued representations (we say "carry half-integer spins") the relation (15.3) is inconsistent with quantum mechanics and relativistic invariance.

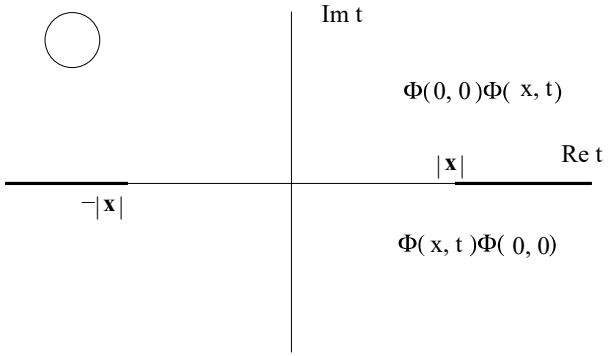
Let us demonstrate this inconsistency in a very general framework. Assume that Φ_1 and Φ_2 are two fields obeying local commutativity in the form (15.3). Consider the vacuum-vacuum matrix element

$$G_{12}(\mathbf{x},t) = \langle 0 \mid \Phi_1(\mathbf{x},t)\Phi_2(\mathbf{0},0) \mid 0\rangle\,. \tag{15.4}$$

One can show, using the condition of positivity of the energy spectrum, that this matrix element extends to a function of complex t analytic in lower half plane[11]. Indeed, the intermediate state decomposition

$$\sum_n \langle 0 \mid \Phi_1(0,\mathbf{0}) \mid n\rangle\, e^{-iE_nt+i\mathbf{P}_n\mathbf{x}}\, \langle n \mid \Phi_2(\mathbf{0},0) \mid 0\rangle$$

makes this property explicit.



Similarly the matrix element

$$G_{21}(-t,-\mathbf{x}) = \langle 0 \mid \Phi_2(0,\mathbf{0})\Phi_1(t,\mathbf{x}) \mid 0\rangle = \langle 0 \mid \Phi_2(-t,-\mathbf{x})\Phi_1(0,\mathbf{0}) \mid 0\rangle$$

[11] We have shown this previously in discussing the KG field theory. The arguments depend only on the spectral positivity and thus apply to general theory.

admits analytic continuation into the upper half plane of complex t. And because the segment

$$-|\mathbf{x}| < t < |\mathbf{x}|$$

of the real axis corresponds to space-like separations, these two functions coincide at this segment, and hence the above two functions represent the upper- and lower-half plane values of a single function analytic on the whole t-plane except for the branch cuts from $-\infty$ to $-|\mathbf{x}|$ and from $|\mathbf{x}|$ to $+\infty$. For that function, the analytic relation

$$G_{12}(t,\mathbf{x}) = G_{21}(-t,-\mathbf{x}) \tag{15.5}$$

must hold everywhere except for the brunch cuts.

Now, consider (complexified) Lorentz transformation with particular choice of A, B, namely

$$(A,B) = (-I, I)\,,$$

so that

$$Z_+ \;\to\; \tilde{Z}_+ \;=\; AZ_+B^{-1} \;=\; -\,Z_+\,.$$

This represents

$$(t,\mathbf{x}) \;\to\; (\tilde{t},\tilde{\mathbf{x}}) \;=\; (-t,-\mathbf{x})\,,$$

i.e. the combination of 180^o rotations around (12) and (34) planes. Then

$$\begin{aligned}\Phi_1 \;&\to\; \mathcal{A}^{(s_1)}\mathcal{B}^{(\dot{s}_1)}\,\Phi_1 = (-)^{2s_1}\,\Phi_1\\ \Phi_2 \;&\to\; \mathcal{A}^{(s_2)}\mathcal{B}^{(\dot{s}_2)}\,\Phi_2 = (-)^{2s_2}\,\Phi_2\,,\end{aligned}$$

and

$$G_{12}(t,\mathbf{x}) = (-)^{2(s_1+s_2)}\,G_{12}(-t,-\mathbf{x})\,.$$

Alternatively, we could consider $(A,B) = (I,-I)$, obtaining the same relation with s_1+s_2 replaced by $\dot{s}_1+\dot{s}_2$. Consistency requires that either

$$(-)^{2(s_1+s_2+\dot{s}_1+\dot{s}_2)} = +1$$

or $G_{12}=0$. In other words, either both $s_1+\dot{s}_1$ and $s_2+\dot{s}_2$ are integers, or both of them are half-integers - otherwise the matrix element (15.4) vanishes. Thus, G_{12} can be nonzero only if the fields in the product both carry integer spins, or both have half-integer spins.

But there is a less trivial conclusion. If $G_{12}\neq 0$, we have

$$G_{12}(t,\mathbf{x}) = \pm\,G_{12}(-t,-\mathbf{x})\,,$$

where + or − appears when both Φ_1 and Φ_2 have integer or half integer spins, respectively.

Assume the latter. Then

$$G_{12}(t,\mathbf{x}) = -\,G_{12}(-t,-\mathbf{x})\,, \tag{15.6}$$

Combining this with (15.5) we have

$$G_{12}(t,\mathbf{x}) = -G_{21}(t,\mathbf{x})\,, \tag{15.7}$$

or

$$\langle 0 \mid \Phi_1(t,\mathbf{x})\Phi_2(t',\mathbf{x}') \mid 0\rangle + \langle 0 \mid \Phi_2(t,\mathbf{x})\Phi_1(t',\mathbf{x}') \mid 0\rangle = 0$$

where trivial renaming of the space-time coordinates was made. To see implications of this equation, assume that Φ_1 and Φ_2 are hermitian conjugate of each other, i.e. $\Phi_1 = \Phi, \Phi_2 = \Phi^\dagger$, and consider two states

$$\begin{aligned} \mid \Psi_1\rangle &= \int f(t,\mathbf{x})\,\Phi^\dagger(t,\mathbf{x}) \mid 0\rangle\,, \\ \mid \Psi_2\rangle &= \int f(-t,-\mathbf{x})\,\Phi(t,\mathbf{x}) \mid 0\rangle\,. \end{aligned}$$

Compute

$$\begin{aligned} &\langle \Psi_1 \mid \Psi_1\rangle + \langle \Psi_2 \mid \Psi_2\rangle = \\ &\int d^4x\, d^4x' \left\{ f^*(x) f(x') \langle 0 \mid \Phi(x')\Phi^\dagger(x) \mid 0\rangle + f(-x) f^*(-x') \langle 0 \mid \Phi^\dagger(x')\Phi(x) \mid 0\rangle \right\} \end{aligned}$$

Changing $x \to -x, x' \to -x'$ in the second integral, and using

$$\langle 0 \mid \Phi(x)\Phi^\dagger(x') + \Phi^\dagger(x)\Phi(x') \mid 0\rangle = 0$$

we find

$$\langle \Psi_1 \mid \Psi_1\rangle + \langle \Psi_2 \mid \Psi_2\rangle = 0\,,$$

in contradiction with positivity of norms.

Note that the trouble is due to the minus sign in

$$G_{12}(t,\mathbf{x}) = -G_{21}(t,\mathbf{x})\,.$$

If we did the same computation with integer-spin fields Φ_1 and Φ_2, the 180^o rotation would not bring minus, and then the troublesome minus in the above equation did

not appear. The minus there is direct consequence of relativistic invariance, it is unavoidable attribute of fields which transform for the double-valued representations. The only way out is to "fix" the sign in the equation (15.5), i.e. to change it to minus as well. That means postulating instead that for half-integer spins

$$G_{12}(t,\mathbf{x}) = -G_{21}(-t,-\mathbf{x})\,. \tag{15.8}$$

How this could be arranged? Recall that in our analysis above the equation (15.5). i.e.

$$G_{12}(t,\mathbf{x}) = G_{21}(-t,-\mathbf{x})$$

was consequence of the assumption that

$$\langle 0 \mid \Phi_1(t,\mathbf{x})\Phi_2(0,\mathbf{0}) \mid 0\rangle = \langle 0 \mid \Phi_2(0,\mathbf{0})\Phi_1(t,\mathbf{x}) \mid 0\rangle$$

for $-|\mathbf{x}| < t < |\mathbf{x}|$, i.e. outside the light cone. To arrange the "minus" there one has to assume instead that for half-integer spin fields Φ_1 and Φ_2

$$\langle 0 \mid \Phi_1(t,\mathbf{x})\Phi_2(0,\mathbf{0}) \mid 0\rangle = -\langle 0 \mid \Phi_2(0,\mathbf{0})\Phi_1(t,\mathbf{x}) \mid 0\rangle$$

for $(t,\mathbf{x})$ outside the light cone of $(0,\mathbf{0})$. This in turn implies that for half-integer spin Φ_1, Φ_2 we should write

$$\{\Phi_1(x), \Phi_2(x')\} = 0 \qquad \text{for} \quad (x-x') \quad \text{space-like}\,,$$

with anti-commutator replacing commutator.

The fields which obey such anti-commutation relations (instead of the commutation relations) are said to obey fermi statistics, and are called the *Fermi Fields*, as opposed to *Bose Fields* which satisfy local commutativity relations in usual form, with commutators. Existence of local fields which obey anti-commutation relations instead of commutation relations can not be inferred from any arguments based on canonical quantization of a classical theory. Importance of the anti-commutator structures in was one of remarkable discoveries made by the founders of quantum field theory at the early stages of its development.

The above analysis proves that there is universal relation between spin of a field and the type of statistics it obeys:

Integer spin	$\Leftrightarrow$	Bose field
Half-Integer spin	$\Leftrightarrow$	Fermi field

This general statement is known as the *Spin-Statistics Theorem*: The only way to express locality consistent with relativistic invariance and positivity of norms in the space of states[12] is

$$\Phi_1)(x)\Phi_2(x') = (-)^{\sigma(12)}\,\Phi_2(x')\Phi_1(x) \qquad \text{for} \quad (x-x') - \text{space-like}\,,$$

where

$$\sigma(12) = \begin{cases} 1 & \text{if} \quad s_1 + \dot{s}_1 \in \mathbb{Z} + 1/2 \;\; \text{and} \;\; s_2 + \dot{s}_2 \in \mathbb{Z} + 1/2 \\ 0 & \text{in all other cases} \end{cases}$$

[12]Positivity is an important part of the statement. The motion of QFT admits extensions in which the condition of positivity; in such theories spin-statistics relation may be broken.

16 Lecture 16

Bose and Fermi fields

We say that field $\Phi^{(s,\dot{s})}(x)$ which transforms as $(s,\dot{s})$ representation of Lorents group is said to have integer (half-integer) spin if $s+\dot{s}$ is integer (half-integer). As we observed the last time, there is a fundamental difference between these two type of fields in the form of the condition of locality. Let

$$\varphi_I(x) \quad \text{integer-spin fields}\,, \qquad \psi_K(x) \quad \text{half-integer spin fields}$$

Then, as we have demonstrated, the usual local commutativity condition must be generalized as

$$[\varphi_I(x),\varphi_J(x')] = [\varphi_I(x),\psi_K(x')] = 0 \qquad \text{for} \qquad x-x' \quad - \quad \text{space-like}$$

but

$$\{\psi_K(x),\psi_L(x')\} = 0 \qquad \text{for} \quad x-x' \quad - \quad \text{space-like}$$

where $\{\ ,\ \}$ is the anti-commutator. $\{A.B\} = AB+BA$. As usual (and no matter is relativistic invariance is present or not), the fields which obey locality condition in the anti-commutator form are referred to as Fermi fields, as opposed to the Bose fields which commute when localized in casually independent regions of space-time.

The above universal relation between values of spin of a local field. and its commutativity conditions is known as the *spin-statistics theorem*. Recall that its proof requires assumptions of (a) relativistic invariance (b) positivity of the energy spectrum (for analyticity), and positivity of norms in te space of states (which is not technical - if this condition does not hold. the statement is not valid!)

How to incorporate the presence of fermi fields with the path integral formulation of QFT? Recall that in the paths integral approach the "fundamental fields" plays role of integration variables[13]. But if the integration variables are just c-numbers, that the associated field operators will always commute at space-like separations. Indeed, in the imaginary-time paths integral, representing a correlation function

$$\langle ...\varphi_I(x_1)\,\varphi_J(x_2)...\rangle = \int D[\Phi(x)]\,...\,\varphi_I(x_1)\varphi_J(x_2)\,...\exp\{-\mathcal{A}[\Phi]\} = \langle ...\varphi_J(x_2)\,\varphi_I(x_1)...\rangle$$

where $x=(x_1,x_2,x_3,x_4)$, there is no difference in which order the insertions $\varphi(x_i)$ are put in the integrand. In that sence the fields associated with c-number integra-

[13]This, of course, is a tautology. "Findamental Fields" is just the general name of integration variables in the paths integral.

tion variables always commute[14]. In order to incorporate anticommutativity at the space-like separations. one is forced to assume that at least a part of the integration variables are not c-numbers, but anticommuting variables. If such field variables $\psi(x)$ satisfy

$$\psi(x)\psi(x') = -\psi(x')\psi(x)\,,$$

then we would have for the correlations

$$\int D[....\psi(x)]\,...\,\psi(x_1)\psi(x_2)\,...\exp\{-\mathcal{A}\} = -\int D[...,\psi(x)]\,...\,\psi(x_2)\psi(x_1)\,...\exp\{-\mathcal{A}\}$$

leading to the anti-commutativity of the field operators $\psi(x)$ at space-like separations. Fortunately, mathematical theory of such variables, including integrations, is known under the name *Grassmann variables*; we will briefly review it in the next section.

Here I just want to mention another possibility of constructing fermi fields in paths integral formulation, without introducing the Grassmann variable. In order to satisfy the local commutativity conditions a field $O(x)$ does not necessarily have to be local function of the fundamental fields. With certain non-local functionals $O(x,[\Phi])$ the notion of the order in which such insertions are placed in the integrant is often non-trivial. Thus, in principle, specially designed non-local functionals can nonetheless obey local commutativity or anti-commutativity. Explicit construction exists in 1+1 dimensions, and known as the *Bosonization of Fermions*.

Grassmanian algebra and Grassmanian integration (PS Ch 9.5)

Consider a collection of n formal variables $\{\xi_i\} = \{\xi_1, \xi_2, ..., \xi_n\}$, which can be thought of as the coordinates in n-dimensional space. Assume that, unlike usual coordinates, these variables anticommute

$$\xi_i\xi_j = -\xi_j\xi_i\,.$$

In particular

$$\xi_i^2 = 0\,.$$

[14]This does not contradict to non-zero commutators of the associated real-time Heisenberg operators. Recall that the notion of commutator appears upon analytic continuation to the real axis in the compex-t plane, and is related to the difference of the values on the upper and lower edges of the branch cuts extending into the internal parts of the light cone. In the "lacunae" $-|x| < t < |x|$ the different order products of operators then coincide, in accord with local **commutativity**.

One can define functions of these variables $f(\xi_i)$. Usual functions of c-number variables (with some assumption of analyticity) is represented as Taylor series

$$f(x_i) = \sum_{N=0}^{\infty} x_{i_1} x_{i_2} ... x_{i_n}\, a_{[i_1 i_2 ... i_N]}$$

with the coefficients $a_{[i_1 ... i_N]}$ expressed trough the partial derivatives of $f(x)$ at $x = 0$. In the anti-commuting case any function $f(\xi_i)$ is *defined* through the power series

$$f(\xi_i) = \sum_{N=0}^{n} \xi_{i_1} ... \xi_{i_N}\, a_{\{i_1 ... i_N\}}$$

where now coefficients $a_{\{i_1 ... i_N\}}$ are completely antisymmetric in the indices. The series obviously truncates at $N > n$, so for finitely many variables any function is a polynomial. For example, a function of one variable is determined by two parameters

$$f(\xi) = f_0 + \xi\, f_1$$

Derivative $\partial/\partial\xi$, by definition, is

$$\frac{\partial f(\xi)}{\partial \xi} = f_1\,,$$

or, in multi-dimensional case

$$\frac{\partial f(\xi)}{\partial \xi_i} = \sum_{N=0}^{n} \sum_{k=1}^{N} \delta^i_{i_k}\, (-)^k\, \xi_1 ... \cancel{\xi_{i_k}} ... \xi_N\, a_{\{i_1 ... i_N\}}$$

Equivalently. if one first writes $f(\xi_i)$ in the form

$$f(\xi_1 ... \xi_n) = A(\xi_1 ... \cancel{\xi_i} ... \xi_n) + \xi_i\, B(\xi_1 ... \cancel{\xi_i} ... \xi_n)$$

then

$$\frac{\partial f}{\partial \xi_i} = B(\xi_1, ..., \cancel{\xi_i}, ..., \xi_N)$$

Note that by this convention we first bring ξ_i to the left, therefore this derivative is called "left derivative". Of course, one can define the "right derivative", with similar properties. We will use only left derivatives. It is easy to check that

$$\frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_j} = -\frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_i}\,,$$

which can be written as

$$\left\{\frac{\partial}{\partial\xi_i},\,\frac{\partial}{\partial\xi_j}\right\}=0$$

$(A,B:=AB+BA)$, and

$$\left\{\frac{\partial}{\partial\xi_i},\,\xi_j\right\}=\delta_i^j\,.$$

The Grassmann numbers will appear as the variables in the functional integral, and we need integration rules. For a single variable ξ the rules are[15]

$$\int d\xi=0\,,\qquad \int d\xi\;\xi=1\,.$$

For a collection of variables $\{\xi_i\}$ this rule can be written as

$$\int d\xi^i\,\xi_j=\delta_j^i\,,$$

and one assumes that

$$\{\xi_i,\,d\xi^j\}=0\,,\qquad \{d\xi^i,\,d\xi^j\}=0\,.$$

Note that I have ascribed an upper index to the "differential" dx^i; the reason will be explained in a moment.

Multiple integrals are understood as repeated ones, for example

$$\int d\xi^2 d\xi^1\,(a+b_1\xi_1+b_2\xi_2+c\xi_1\xi_2)=c\int d\xi^1\,\xi_1\int d\xi^2\,\xi_2=c\,.$$

For arbitrary function

$$f(\xi_1,...,\xi_n)=f_0+\xi_i\,f^i+...+\xi_1\xi_2...\xi_n\,F$$

$$\int d\xi^n...d\xi^1\,f(\xi_1,...,\xi_n)=F\int\xi_1\xi_2...\xi_n\;d\xi^n...d\xi^1=F\,.$$

Consider linear transformation of the variables ξ_i,

$$\xi_i\;\rightarrow\;\eta_i=\sum_j L_i^j\,\xi_j\,,$$

[15]The rules are enforced by postulations the properties (i) $\int d\xi\,(f(\xi)+\tilde f(\xi))=\int d\xi f(\xi)+\int d\xi\,\tilde f(\xi)$, and (ii) $\int d\xi\,(\partial/\partial\xi)f(\xi)=0$ for any $f(\xi)$.

where L is a c-number invertible matrix. Our integration rules

$$\int d\xi^i \, \xi_j = \delta^i_j$$

will be invariant under such transformations if we assume that

$$d\xi^i \;\rightarrow\; d\eta_i = \sum_j \left[(L^{-1})^t\right]^i_j \, d\xi^j \, .$$

That was the reason we ascribed the upper index to the "differentials". We have to assume that. unlike usual c-numbers, the differentials $d\xi^i$ transform contra-variantly.

Take the case of two grassmann variables ξ_1, ξ_2, and consider particular linear transformation

$$(\xi_1, \xi_2) \;\rightarrow\; (\eta_1, \eta_2) = (\xi_1 + i\xi_2, \xi_1 - i\xi_2) \, .$$

If we think of ξ_1, ξ_2 as "real" variables[16], then η_1, η_2 look like complex conjugate, and one may use the notations

$$(\eta_1, \eta_2) = (\eta, \bar{\eta})$$

Let me emphasize that this is purely semantic, as grassmann variables are not numbers and usual intuition about real and complex numbers does not always apply. For instance, as

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 & +i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

we have

$$\begin{pmatrix} d\eta^1 \\ d\eta^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix} \begin{pmatrix} d\xi^1 \\ d\xi^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} d\xi^1 - id\xi^2 \\ d\xi^1 + id\xi^2 \end{pmatrix} ,$$

that is

$$d\eta = \frac{1}{2}(d\xi^1 - id\xi^2) \, , \qquad d\bar{\eta} = \frac{1}{2}(d\xi^1 - id\xi^2) \, .$$

With this rule we find

$$\int d\eta \, \eta = 1 \, , \qquad \int d\bar{\eta} \, \bar{\eta} = 1 \, ,$$

but

$$\int d\eta \, \bar{\eta} = \int d\bar{\eta} \, \eta = 0 \, ,$$

[16]One defines an anti-linear involution - "complex conjugation" - acting on grassmann algebra; "real" are elements invariant w.r.t the complex conjugation.

so that η and $\bar{\eta}$ should be thought of as the independent grassmann variables.

The idea how to introduce fermions in the functional integral is to assume that in the expression

$$\langle \Phi_1(x_1)...\Phi_N(x_N)\rangle = Z^{-1}\int [D\phi]\ (\Phi_1(\phi((x_1))...\Phi_N(\phi(x_N)))\ e^{-\mathcal{A}[\phi]}$$

some or all of the integration variables $\phi(x) = \{\phi_A(x)\}$ are the grassmann variables, and we understand this integral in terms of the gressmann integration.

Let us start with free field theory involving fermions. In the scalar case, the functional integral associated with free field theory was gaussian integral, i.e. an infinite-dimensional version of

$$\int dx_1...dx_n\ \exp\left\{-\frac{1}{2}\sum_{i,j} x_i S_{ij} x_j\right\}$$

where x_i was c-number variables, and S_{ij} formed a positive symmetric matrix S. We know that it is

$$Const\ \ (\det(S))^{-1/2}\ , \qquad Const = (2\pi)^{n/2}\,.$$

One way to obtain this result is to apply linear transformation

$$x_i = B_i^j\, y_j$$

such that $BSB^t = I$ (we assume that S has no negative or zero eigenvalues), which reduces the integral to

$$\det(B)\int dy_1...dy_n\ \exp\left\{-\frac{1}{2}\sum_i y_i^2\right\}\ .$$

Since $\det B = (\det S)^{-1/2}$, we obtain the desired result.

Let us apply similar analysis to evaluate the integral

$$\int d\xi_n...d\xi_1\ \exp\left\{-\frac{1}{2}\sum_{i,j}\xi_i A_{ij}\xi_j\right\}\ , \tag{16.1}$$

where now $\{\xi_1,...,\xi_n\}$ are anti-commuting grassmann variables, and A_{ij} form an anti-symmetric matrix. we can not diagonalize an anti-symmetric matrix, but for $N = 2M$ it is possible to find B such that

$$B\,A\,B^t = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where the blocks are $M \times M$. Then, if

$$\xi_i = \sum_j B_i^j \, \eta_j$$

the above gaussian integral over grassmann variables reduces to

$$(\det B)^{-1} \prod_{i=1}^{M} \left(\int d\bar{\eta}_i d\eta_i \, e^{-\bar{\eta}_i \eta_i} \right)$$

with $\bar{\eta}_i := \eta_{i+M} (i = 1, ..., M)$. Since

$$\int d\bar{\eta} d\eta e^{-\bar{\eta}\eta} = \int d\bar{\eta} d\eta (1 + \eta\bar{\eta}) = 1$$

we find[17]

$$(16.1) \;\; = \;\; (\det A)^{1/2} \; .$$

The difference with the evaluation of the gaussian integral over c-variables is in contra-variant transformation of the differentials. Note that unlike the c-variables case, where the matrix S had to be positive in order to make integral convergent, in the grassmann integral the sign of A is not restricted

Exercise: Using the translation property of the grassmann integral

$$\int d\xi^n ... d\xi^1 f(\xi_1, ..., \xi_n) = \int d\xi^n ... d\xi^1 f(\xi_1 + \alpha_1, ..., \xi_n + \alpha_n)$$

where $\alpha_1, ..., \alpha_n$ are grassmann numbers, $\{\alpha_i, \alpha_j\} = \{\alpha_i, \xi_j\} = 0$, show that

$$\int d\xi_n ... d\xi_1 \, \exp\left\{ -\frac{1}{2} \sum_{i,j} \xi_i A_{ij} \xi_j + \sum_i \alpha_i \xi_i \right\} = (\det A)^{1/2} \, \exp\left\{ \frac{1}{2} \sum_{i,j} \alpha_i (A^{-1})_{ij} \alpha_j \right\}$$

Paths integral for Dirac Field

The Dirac field transforms as $\xi^{(1/2,0)} \oplus \xi^{(0,1/2)}$ representation. It has four components

$$\psi(x) = \begin{pmatrix} \xi_\alpha \\ \eta^{\dot{\alpha}} \end{pmatrix} \; .$$

[17] For any real antisymmetric matrix A the determinant is non-negative; $\det(A) = (\mathrm{pf}(A))^2$, where pf denotes the *Pfaffian*.

We need to construct a quadratic action invariant with respect to $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. There are several ways to construct a scalar out of the Dirac field. The most important is the case of massive "charged" field. The last property means that the theory must have additional $U(1)$ invariance. The $U(1)$ acts on $\psi(x)$ by phase rotations

$$\psi(x) \;\to\; e^{i\alpha}\,\psi(x)\,;$$

In order to make an invariant, we need additional field $\bar\psi(x)$ that transforms as

$$\bar\psi(x) \;\to\; e^{-i\alpha}\,\bar\psi(x)\,.$$

It is sometimes said that $\bar\psi$ is "complex conjugate" of ψ. This is consistent with the above phase rotation rules, but generally is not much more then semantics. Recall again that Grassmann variables are not numbers, and notions of being real or complex, or complex conjugate, do not apply. It is correct to regard $\psi(x)$ and $\bar\psi(x)$ as independent fields. The Lorentz representation content of $\bar\psi(x)$ is identical to $\psi(x)$, i.e. $(1/2,0)\oplus(0,1/2)$, but the basis is usually taken as

$$\bar\psi(x) = \left(\bar\eta^\alpha(x), \bar\xi_{\dot\alpha}(x)\right)\,,$$

where the components can be understood as

$$\bar\eta^\alpha = \varepsilon^{\alpha\beta}\,\bar\eta_\beta\,, \qquad \bar\xi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\,\bar\xi^{\dot\beta}\,.$$

Correspondingly, the four spinors transform as

$$\begin{aligned} \xi \;&\to\; A\xi\,, \qquad \eta \;\to\; B\eta\,, \\ \bar\eta \;&\to\; \bar\eta\,A^{-1}\,, \qquad \bar\xi \;\to\; \bar\xi\,B^{-1} \end{aligned}$$

where $\bar\eta^\alpha = (\bar\eta^1, \bar\eta^2)$ and $\bar\xi_{\dot\alpha} = (\bar\xi_1, \bar\xi_2)$ are regarded as 2-rows. Recall that the operators D_+ and D_- defined in the previous lecture[18] transform as

$$\begin{aligned} D_+ \;&\to\; AD_+B^{-1} \\ D_- \;&\to\; BD_-A^{-1} \end{aligned}$$

The $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ invariant action can be written as (in the imaginary-time notations)

$$\mathcal{A}_{\text{Dirac}} = \int d^4x\,\left[\bar\xi\,D_-\,\xi + \bar\eta\,D_-\,\eta + m\,\bar\xi\,\eta + m\,\bar\eta\,\xi\right]\,,$$

where m is the mass parameter. Note that the mass term necessarily mix components of ψ and $\bar\psi$.

[18] $D_+ = (D_+)_{\alpha\dot\alpha}$, $D_- = (D_-)^{\dot\alpha\alpha}$

Remarks

1. Note that the four terms in the Dirac action are $SL(2.\mathbb{C}) \times SL(2,\mathbb{C})$ invariant separately. If $m = 0$, the "left" and "right" components enter separately. In particular, in this case it is consistent with *proper* Lorentz transformation (but not inversion!) to set, say, $\eta = \overline{\eta} = 0$, so that the action reduces to

$$\mathcal{A}^{(\text{left})}_{\text{Weyl}} = \int d^4x \; \overline{\xi}\, D_-\, \xi \,.$$

The fermi field obeying this action is called left-handed *Weyl fermion* (of course, one can define similarly the right-handed Weyl field). Before discovery of finite neutrino mass, the neutrinos were thought to be described by a Weyl field. There is no way to add the mass term without adding also the opposite chirality fields $\eta, \overline{\eta}$. This will lead to violation of the so called "leptonic family numbers", but not necessarily the full lepton number.

2. The components $\overline{\eta}^{\alpha}$ and $\overline{\xi}_{\dot{\alpha}}$ transform according to representations equivalent to those associated with ξ_α and $\eta^{\dot{\alpha}}$, respectively. Therefore, it is consistent to identify

$$\overline{\eta}^{\alpha} = \varepsilon^{\alpha\beta}\, \xi_\beta \,, \qquad \overline{\xi}_{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\, \eta_{\dot{\beta}} \,.$$

The mass term is still possible, but the identification is inconsistent with the $U(1)$ phase rotations, so the field is "neutral". Fermi field of this properties is called *Majorana field*.

Using the gamma matrices $\gamma^a,\ a = 1,2,3,4$

$$\gamma_4 = i\gamma^0 = i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} , \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$$

the action can be written as [19]

$$\mathcal{A}_{\text{Dirac}} = \int d^4x\, \bar{\psi}(x)\; (-i\, \gamma_a \partial_a + m)\; \psi(x) \,.$$

[19]

$$D_- = \partial/\partial x_4 - i\, \boldsymbol{\sigma}\boldsymbol{\nabla} = -i\; (\partial/\partial t + \boldsymbol{\sigma}\boldsymbol{\nabla}) \,,$$
$$D_- = \partial/\partial x_4 - i\, \boldsymbol{\sigma}\boldsymbol{\nabla} = -i\; (\partial/\partial t + \boldsymbol{\sigma}\boldsymbol{\nabla}) \,,$$

This transforms to a familiar real-time form as follows. The imaginary-time paths integral has the form

$$\int D[\psi(x),\bar{\psi}(x)]...\exp\{-\mathcal{A}_{\rm Dirac}[\psi,\bar{\psi}]\}\,.$$

Upon continuation

$$x_4 = it\,, \qquad \partial/\partial x_4 = -i\partial/\partial t$$

we have

$$-\mathcal{A}_{\rm Dirac}[\psi,\bar{\psi}] \;\rightarrow\; iS_{\rm Dirac}(\psi,\bar{\psi})$$

with familiar real-time Dirac action

$$S_{\rm Dirac} = \int dt\, d^3x\ \bar{\psi}(x)\ (i\gamma_4\partial_4 + i\gamma\nabla - m)\ \psi(x) = \int dt d^3x\ \bar{\psi}(x)\ (i\,\gamma^\mu\partial_\mu - m)\ \psi(x)\,.$$

where $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ are conventional gamma-matrices which satisfy

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \qquad (\{\gamma_a,\gamma_b\} = -2\,\delta_{ab})$$

The path integral with the above action is evaluated using the formulae for the Gaussian integration over the Grassmann variables. It is easiest to work in imaginary-time (=Euclidean space-time) formalism. Thus, for the generating functional

$$W[\Xi] = Z^{-1} \qquad \int D[\psi(x),\bar{\psi}(x)]$$
$$\exp\left\{-\int d^4x\,,\bar{\psi}(x)\,(-i\gamma_a\partial_a + m)\,\psi(x) + \int d^4x\ \left(\bar{\Xi}(x)\psi(x) + \bar{\psi}(x)\Xi(x)\right)\right\}$$

with Grassmann-valued bispinor "sources" $\Xi(x), \bar{\Xi}(x)$ evaluates to

$$W[\Xi,\bar{\Xi}] = \exp\left\{\int d^4x\, d^4x'\ \bar{\Xi}(x)\, G(x-x')\,\Xi(x)\right\}$$

where the kernel $G(x)$ (4 by 4 matrix) is inverse to the operator $-i\gamma_a\partial_a + m$, i.e. the (unique) solution of the equation

$$\left(-i\gamma_a\frac{\partial}{\partial x_a} + m\right) G(x-x') = I\,\delta^{(4)}(x-x')\,.$$

which is bounded at $x-x' \to \infty$. The solution is easily obtained by Fourier transform

$$G(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \tilde{G}(k)\,,$$

with 4-dimensional wave vectors k. This leads to the equation

$$(\gamma_a k_a + m)\ \tilde{G}(k) = I\,,$$

stating that the matrix $\tilde{G}(x)$ is inverse to $\gamma_a k_a + m$. Consider the identity

$$(m+\gamma_a k_a)(m-\gamma_b k_b) = m^2 + \delta_{ab} k_a k_b = m^2 + k^2$$

which follows from

$$\{\gamma_a, \gamma_b\} = -2\,\delta_{ab}\,.$$

We find

$$\tilde{G}(k) = \frac{m - \gamma_a k_a}{m^2 + k^2}\,,$$

and finally

$$G(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \frac{m-\gamma k}{m^2+k^2}$$

This is Dirac propagator in the Euclidean space-time. It can be written as

$$G(x) = (m + i\gamma\partial) D(x)\,,$$

where

$$D(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx}\, \frac{1}{m^2+k^2}$$

is the Klein-Gordon propagator.

The correlation functions for the Dirac field are derived by taking variational derivatives of the generating functional. For the two-point function one finds

$$\langle \bar{\psi}(x)\psi(x')\rangle = \frac{\delta}{\delta\bar{\Xi}(x)}\frac{\delta}{\delta\Xi(x')} W[\Xi,\bar{\Xi}] = G(x-x')$$

Multipoint correlation functions may be obtained by taking more variations. Alternatively, one may use recurrent relations.

Exercise: The functional measure has the property $D[\psi+\epsilon,\bar{\psi}+\bar{\epsilon}] = D[\Psi,\bar{\psi}]$ with fixed grassmann bispinors $\epsilon,\bar{\epsilon}$. Using this shift of integration variables with infinitesimal $\epsilon,\bar{\epsilon}$, derive the relation

$$\langle [(-i\gamma_a\partial_a + m)\psi(x)]_\omega \; \psi(x_1)...\psi(x_N)\bar{\psi}(y_1)...\bar{\psi}(y_n)\rangle = \tag{16.2}$$
$$\sum_{k=1}^{M} (-)^{N+k-1} I_{\omega\omega_k}\delta(x-y_k)\,\langle\psi(x_1)...\psi(x_N)\bar{\psi}(y_1)...\cancel{\bar{\psi}_{\omega_k}(x_k)}...\bar{\psi}(x_M)\rangle$$

where $I_{\omega\omega'} = \delta_{\omega\omega'}$ is the identity matrix, and from (16.2)

$$\langle\,\psi_\omega(x)\;\psi(x_1)...\psi(x_N)\bar{\psi}(y_1)...\bar{\psi}(y_M)\rangle = \tag{16.3}$$
$$\sum_{k=1}^{M} (-)^{N+k-1}\, G_{\omega\omega_k}(x-y_k)\,\langle\psi(x_1)...\psi(x_N)\bar{\psi}(y_1)...\cancel{\bar{\psi}_{\omega_k}(x_k)}...\bar{\psi}(x_M)\rangle$$

The relation (16.3) is similar to Eq.(6.3) in the KG theory, it represents in a compact form the **Fermionic Wick's Theorem**, i.e. the representation of the correlation functions

$$\langle\psi(x_1)...\psi(x_N)\bar{\psi}(y_1)...\bar{\psi}(y_N)\rangle$$

in terms of a product of the Wick's pairings

$$\overbrace{\psi_\omega(x_i)\bar{\psi}_{\omega'}}^{1}(y_j) = G_{\omega\omega'}(x_i-y_j)\,.$$

with the signs appropriately defined signs (see **§4.7 of PS**).

17 Lecture 17

Local Integrals of Motion in QFT

Recall that the structure of the KG theory can be understood in terms of infinitely many local integrals of motion represented by conserved currents

$$J^\mu_f(x) = \partial^\mu\varphi(x)\, f(x) - \varphi(x)\, \partial^\mu(x) \tag{17.1}$$

where $f(x)$ is any solution of the KG equation

$$\left(\partial_\mu\partial^\mu + m^2\right) f(x) := \left(\partial_t^2 - \boldsymbol{\nabla}^2 + m^2\right) f(t,\mathbf{x}) = 0\,. \tag{17.2}$$

In classical KG theory the currents (17.1) satisfy

$$\partial_\mu J^\mu_f(x) = 0$$

in virtue of its classical equation of motion - the KG equation for the field $\varphi(x)$. These are the Noether's currents associated with the easily verified symmetry of the KG action with respect to the infinitesimal shift $\varphi(x) \to \varphi(x) + \epsilon\, f(x)$ with any $f(x)$ which obeys (17.2). See Lecture 2.

That analysis - as well as the Noether's theorem itself - was discussed in the context of classical field theory. It is possible make modifications suitable for QFT defined in terms of a functional integral.

Symmetries and conserved currents

First, lat us clarify the meaning of Noether's theorem in QFT. As in the classical version, assume that the action $\mathcal{A}[\Phi]$ (now $\Phi(x)$ stands for any collection of "fundamental fields" which play role of functional integration variables in the path integral, and may include grassmann-valued as well as c-number fields) is invariant w.r.t. infinitesimal transformations of the form

$$\Phi(x) \;\to\; \tilde\Phi(x) = \Phi(x) + \epsilon\, E(x,\Phi(x))\,, \tag{17.3}$$

where $E(x,\Phi(x))$ is a specific local function of $\Phi(x)$ and possibly the derivatives $\partial_\mu\Phi(x)$[20], which may separately depend on x. The term "invariant" implies that $\mathcal{A}[\tilde\Phi]$ expressed through $\Phi(x)$ has the same functional form as $\mathcal{A}[\Phi]$, up to terms $O(\epsilon^2)$. The transformations (17.3) represent infinitesimal form of the continuous

[20] Remark on higher derivatives

symmetry group of the theory, with the function $E(x,\Phi(x))$ defining the action of the group[21].

Regarding (17.3) as the change of the functional integration variables in the functional integral defining a correlation function

$$\langle\Phi(x_1)...\Phi(x_N)\rangle = Z^{-1}\int D[\Phi]\,\Phi(x_1)...\Phi(x_N)\,e^{-\mathcal{A}[\Phi]} \tag{17.4}$$

and remembering that change of integration variables does not change the value of the functional integral over $D[\Phi]$, one finds the equation

$$\sum_{k=1}^{N}\langle\Phi(x_1)...\cancel{\Phi(x_k)}\overset{E(x_k,\Phi(x_k))}{\nearrow}...\Phi(x_N)\rangle = 0$$

which expresses the invariance of the correlation functions under the symmetry group. I wrote this equation for the correlation function of the "fundamental fields" Φ, but clearly similar relation extends to correlation functions involving any local composite fields $O_i(x)$,

$$\sum_{k=1}^{N}\langle O_1(x_1)...\cancel{O_k(x_k)}\overset{\delta_E O_k(x_k))}{\nearrow}...O_N(x_N)\rangle = 0$$

where $\delta_E O_k$ are variations of $O_k(x)$ under the transformation (17.3), $O_k(x) \to O_k(x) + \epsilon\delta_E O_k(x)$.

But there is also more interesting relation which follows from the invariance of the action under (17.3). Consider a similar transformation

$$\Phi(x) \;\to\; \tilde{\Phi}(x) = \Phi(x) + \epsilon(x)\,E(x,\Phi(x))\,, \tag{17.5}$$

with the same $E(x,\Phi(x))$ as in (17.3), but now with the parameter ϵ replaced by an arbitrary (infinitesimal) function $\epsilon(x)$, and again regard it as the change of variables in the functional integral. Now the action is not exactly invariant, but its variation must vanish at constant $\epsilon(x) = \epsilon$. Therefore, under (17.5)

$$\delta\mathcal{A} = \int d^4x\,\partial_\mu\epsilon(x)\,J^\mu_E(x)\,, \tag{17.6}$$

where $J^\mu_E(x)$ is some local field, function of $\Phi(x)$ and its derivatives taken at x,

$$J^\mu_E(x) = J^\mu_E(x,\Phi(x),\partial\Phi(x),...)\,,$$

[21]Given $E(x,\Phi(x))$ generates a family of finite transformations $\Phi(x) \to \Phi_\epsilon(x)$ by solving $(d/d\epsilon)\Phi_\epsilon(x) = E(x,\Phi_\epsilon(x))$ which represent a one-parameter continuous group.

which is determined by the transformation E. Alternatively, integrating by parts, one can write

$$\delta\mathcal{A} = -\int d^4x\,\epsilon(x)\,\partial_\mu J^\mu_E(x)\,,$$

Again, since the transformation of the integration variables doe not change the value of the functional integral defining the correlation function (17.4), the identity

$$\int d^4x\,\epsilon(x)\,\frac{\partial}{\partial x^\mu}\langle J^\mu_E(x)\,\Phi(x_1)...\Phi(x_N)\rangle + \sum_{k=1}^{N}\epsilon(x_k)\,\langle\Phi(x_1)...\cancelto{E(x_k,\Phi(x_k))}{\Phi(x_k)}...\Phi(x_N)\rangle = 0$$

must hold for any function $\epsilon(x)$, i.e. the correlation functions must satisfy

$$\frac{\partial}{\partial x^\mu}\langle J^\mu_E(x)\,\Phi(x_1)...\Phi(x_N)\rangle + \sum_{k=1}^{N}\delta(x-x_k)\,\langle\Phi(x_1)...\cancelto{E(x_k,\Phi(x_k))}{\Phi(x_k)}...\Phi(x_N)\rangle = 0\,. \quad (17.7)$$

Relations of this type which follow from symmetries is generally known as the *Ward Identities*.

Simple and most important conclusion which follow from (17.7) is that the correlation functions involving $J^\mu_E(x)$ vanishes at all x different from the other insertion points $x_1, ..., x_N$. This property of course extends to correlation functions with any local composite fields,

$$\frac{\partial}{\partial x^\mu}\langle J^\mu_E(x)\,O_1(x_1)...O_N(x_N)\rangle = 0 \qquad \text{for} \quad x \neq x_1, ..., x_N\,, \quad (17.8)$$

although of course exact form of delta-like terms at supported at $x = x_1, ..., x_N$ depends on the transformation properties of the fields O_i under (17.3).

Integrals of Motion

According to Gauss theorem

$$\int_D d^4x\,\frac{\partial}{\partial x^\mu}J^\mu_E(x) = \int_{\partial D} d^3n_\mu(x)\,J^\mu_E(x)\,, \quad (17.9)$$

where D is any domain D in $\mathbb{R}^4$ (Euclidean or Minkowski, the metric does not matter here), ∂D is the boundary of D, and $d^3n_\mu(x)$ is a normal element of ∂D, $d^3n_\mu = e_{\mu\nu\lambda\rho}\,dx^\nu \wedge dx^\lambda \wedge dx^\rho$. Then, as the consequence of (17.8), the integral

$$\int_\Sigma d^3n_\mu(x)\,\langle J^\mu_E(x)\,O_1(x_1)...O_N(x_N)\rangle$$

over 3-hypersurface Σ does not change under topologically trivial deformations of Σ. We call

$$Q_E(\Sigma) = \int_\Sigma d^3 n_\mu(x)\, J_E^\mu(x)$$

the *Integral of Motion*, which means precisely that the expectation values

$$\langle Q_E(\Sigma)\, O_1(x_1)...O_N(x_N)\rangle$$

depend only on the topological class of Σ, but not on the details of its shape.

In the special case when Σ is chosen to be an "equal-time slice" in space-time $(t, \mathbf{x})$,

$$\Sigma = \Sigma_{t_0} : \qquad (t, \mathbf{x}) = (t_0, \mathbf{x})$$

with a given t_0, that means that

$$\langle Q_E(t)\, O_1(t_1, \mathbf{x}_1)...O_N(t_N, \mathbf{x}_N)\rangle$$

is a piecewise constant function of t, with the jumps at $t = t_1, ..., t_N$.

Σ_t

Suppose $t_1, ...t_k > t > t_{k+1}, ..., t_N$. It is natural to signify this arrangement by placing the insertion $Q_E(t)$ in between the corresponding local insertions, i.e. to denote the expectation value with this arrangement as

$$\langle O_1(t_1, \mathbf{x}_1)...O_k(t_k, \mathbf{x}_k)\, Q_E\, O_{k+1}(t_{k+1}, \mathbf{x}_{k+1})...O_N(t_N, \mathbf{x}_N)\rangle$$

With this convention, the integral of motion Q_E does not commute with the local insertions $O_i(x_i)$, even in the Euclidean QFT. (The commutator $[Q_E, O(x)]$ is determined by the variation $\delta_E O(x)$ under the transformation (17.3).)

Likewise, if there are more then one symmetry of the type (17.3), say E_1 and E_2, the expectation values

$$\langle Q_{E_1}(t) Q_{E_2}(t')\, O_1(t_1, \mathbf{x}_1)...O_N(t_N, \mathbf{x}_N)\rangle \tag{17.10}$$

is a piecewise constant function on both t and t', whose value depends on whether $t > t'$ or $t < t'$ (as well as the on how t and t' relate are arranged with respect to

to $t_1, ..., t_N$ along the t-axis. And again, it is natural to associate the order in which the insertions Q_{E_1} and Q_{E_2} are placed inside the expectation value symbol with the order of t and t' relative to $t_1, ..., t_N$, as well as relative to each other. Thus,

$$\langle O_1(t_1, \mathbf{x}_1)...O_k(t_k, \mathbf{x}_k)\, Q_{E_1} Q_{E_2}\, O_{k+1}(t_{k+1}, \mathbf{x}_{k+1})...O_N(t_N, \mathbf{x}_N)\rangle$$

corresponds to (17.10) at

$$t_1, ...t_k > t > t' > t_{k+1}, ..., t_N.$$

Then the insertions "operators" Q_{E_1} and Q_{E_2} generally do not commute with each other. It is easy to see that the commutator $[Q_{E_1}, Q_{E_2}]$ (anti-commutator, if both Q_{E_1} and Q_{E_2} are fermionic) is another integral of motion, which is determined by the singularity of the correlation function

$$\langle J^\mu_{E_1}(x) J^\nu_{E_2}(x')\, O_1(x_1)...O_N(x_N)\rangle$$

at $x \to x'$. Indeed,

$$[Q_{E_1}, Q_{E_2}] = Q_{[E_1,E_2]} := \int_\Sigma d^3 n_\mu(x)\, J^\mu_{[E_1,E_2]}(x)\,, \tag{17.11}$$

where

$$J^\mu_{[E_1,E_2]}(x) = \int_{\Sigma_x} d^3 n_\nu(x')\, J^\nu_{E_1}(x') J^\mu_{E_2}(x) \tag{17.12}$$

with the integration in the r.h.s. performed over a small 3-surface Σ_x enclosing the point x. This transformation is illustrated in Fig.17.2.

$\Sigma_{t'}$

$\Sigma_{t'}$

Since this surface Σ_x can be deformed to be arbitrary close to the point x, the integral in (17.12) returns a local field.

Dirac Field Theory (Cont'd)

It is straightforward to verify that the Dirac action

$$\mathcal{A}_{\rm Dirac}[\psi,\bar\psi] = \int d^4x\, \bar\psi(x)\,(-i\gamma^\mu\partial_\mu + m)\,\psi(x)$$

is invariant with respect to the the change of functional variables

$$\mathcal{A}_{\rm Dirac}[\psi(x)+\epsilon\chi(x),\bar\psi(x)+\bar\epsilon\bar\chi(x)] = \mathcal{A}_{\rm Dirac}[\psi(x),\bar\psi(x)]$$

where ϵ and $\bar\epsilon$ are infinitesimal grassmann numbers, and $\chi(x)$ and $\bar\chi(x)$ are c-number solutions of the Dirac equations

$$-i\gamma^\mu\partial_\mu\chi(x) + m\chi(x) = 0\,, \tag{17.13}$$

$$i\partial_\mu\bar\chi(x)\gamma^\mu + m\bar\chi(x) = 0\,. \tag{17.14}$$

The (fermionic) conserved currents associated with these two symmetries (analogous to the current (17.1) in the KG theory) have the form

$$J^\mu_\chi(x) = \bar\psi(x)\gamma^\mu\chi(x)\,, \tag{17.15}$$

$$J^\mu_{\bar\chi}(x) = \bar\chi(x)\gamma^\mu\psi(x)\,. \tag{17.16}$$

It is important therefore to understand c-number solutions of the equations (17.13).

It is not difficult to find anti-commutators of the fermionic integrals of motion associated with the currents (17.15), (17.16). Denote

$$A_\chi = \int_\Sigma d^3n_\mu(x)\, J^\mu_\chi(x)\,, \qquad A_{\bar\chi} = \int_\Sigma d^3n_\mu(x)\, J^\mu_{\bar\chi}(x)\,,$$

Then, for example, according to the anti-commutator version of (17.11)

$$\{A_{\bar\chi_2}, A_{\chi_1}\} = \int_\Sigma d^3n_\mu(x)\, J^\mu_{\{\bar\chi_2,\chi_1\}}(x)\,,$$

where $J_{\{\bar\chi_2,\chi_1\}}$ is given by

$$J^\mu_{\{\bar\chi_2,\chi_1\}}(x) = \oint_{\Sigma_x} d^3n_\nu(x')\, J^\nu_{\bar\chi_2}(x')\, J^\mu_{\chi_1}(x) = \oint_{\Sigma_x} d^3n_\nu(x')\, \bar\chi_2(x')\gamma^\nu\psi(x')\,\bar\psi(x)\gamma^\mu\chi_1(x)$$

The singularity at $x'=x$ in the last expression appears from the contraction

$$\psi(x')\bar\psi(x) \;\to\; \overbrace{\psi(x')\bar\psi}(x) = G(x'-x)\,,$$

and we obtain

$$J^{\mu}_{\{\bar{\chi}_2,\chi_1\}}(x) = \oint_{\Sigma_x} d^3 n_\nu(x')\, \bar{\chi}_2(x')\gamma^\nu\, G(x'-x)\gamma^\mu \chi_1(x)\,.$$

The integral can be evaluated using the Gauss theorem (17.9), yielding

$$\int_{D_x} d^4x'\, \frac{\partial}{\partial x'^\nu}\, (\bar{\chi}_2(x')\gamma^\nu\, G(x'-x)\gamma^\mu \chi_1(x))$$

where D_x is a small domain including the point x. The nonzero contribution comes from the term involving

$$\frac{\partial}{\partial x'^\nu}\gamma^\nu G(x'-x) = I\,\delta^{(4)}(x'-x)$$

and we have

$$J^{\mu}_{\{\bar{\chi}_2,\chi_1\}}(x) = \bar{\chi}_2(x)\gamma^\mu\chi_1(x)\,.$$

Finally, we obtain

$$\{A_{\bar{\chi}_2}, A_{\chi_1}\} = \int_\Sigma dn_\mu(x)\, \bar{\chi}_2(x)\gamma^\mu\chi_1(x)\,. \tag{17.17}$$

It is easy to see that

$$\{A_{\chi_1}, A_{\chi_2}\} = 0\,, \qquad \{A_{\bar{\chi}_1}, A_{\bar{\chi}_2}\} = 0\,, \tag{17.18}$$

since the contractions $\overbrace{\psi\psi}$ and $\overbrace{\bar{\psi}\bar{\psi}}$ vanish.

To find a convenient form of the full algebra of the fermionic operators A_χ and $A_{\bar{\chi}}$, it is useful to introduce plane-wave solutions of the Dirac equations (17.13), (17.14).

Plane wave solutions of the Dirac equation (PS §3.3)

Let us look for solution of (17.13) in the form of a plane wave

$$\chi(x) = u(p)\, e^{-ipx}$$

where $u(p)$ is some bi-spinor amplitude. The equation (17.13) then implies

$$(\gamma^\mu p_\mu - m)u(p) = 0\,. \tag{17.19}$$

Multiplying by $(\gamma^\mu p_\mu + m)$ and using $\{\gamma^\mu, \gamma^\nu\} = 2g_{\mu\nu}$ one finds that all components of $u(p) = u_\omega(p)$ satisfy

$$(p^2 - m^2)u(p) = 0\,, \tag{17.20}$$

i.e. $u(p)$ are non-zero only on the "mass shell" $p^2 := \omega^2 - \mathbf{p}^2 = m^2$, i.e. at $\omega = \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$.

Assume that $p = (p^\mu) = (\omega_{\mathbf{p}}, \mathbf{p})$ with *positive* $\omega_{\mathbf{p}}$. Let us write

$$u(p) = \begin{pmatrix} \xi_{\mathbf{p}} \\ \eta_{\mathbf{p}} \end{pmatrix},$$

where $\xi_{\mathbf{p}}, \eta_{\mathbf{p}}$ are two-component spinor amplitudes. Then (17.19) reads

$$(p^0 - \boldsymbol{\sigma}\mathbf{p})\,\xi_{\mathbf{p}} = m\,\eta_{\mathbf{p}}\,, \tag{17.21}$$

$$(p^0 + \boldsymbol{\sigma}\mathbf{p})\,\eta_{\mathbf{p}} = m\,\xi_{\mathbf{p}}\,. \tag{17.22}$$

where $p^0 = \omega_{\mathbf{p}} > 0$. In principle, it is not difficult to solve this linear system directly, but more simple way is to start with the case $\mathbf{p} = 0, p^0 = m$, in which case we find

$$\xi_{\mathbf{p}} = \eta_{\mathbf{p}} := \xi_0\,,$$

that is

$$u(m, \mathbf{0}) = \begin{pmatrix} \xi_0 \\ \xi_0 \end{pmatrix}$$

with arbitrary ξ_0.

Let us consider the action of the spatial rotations on this solution. The generic Lorentz transformation (Minkowski space-time!) we have

$$\xi \to A\xi\,, \quad \eta \to B\eta \qquad \text{with} \quad B^{-1} = A^\dagger$$

The spatial rotations correspond to $A \in SU(2)$, so that both ξ and η transform as usual $SU(2)$ spinors. In other words, ξ_0 describes the spin 1/2 state. We will assume normalization

$$\xi_0^\dagger \xi_0 = m\,.$$

A real Lorentz boost is represented by (A, B) with hermitian A and B, $B = A^{-1}$. One can identify

$$A = e^{+\frac{\boldsymbol{\beta}\boldsymbol{\sigma}}{2}}\,, \qquad B = e^{-\frac{\boldsymbol{\beta}\boldsymbol{\sigma}}{2}}\,,$$

where $\boldsymbol{\beta}$ is the boost parameter: The momentum p_μ in the boosted frame is $(p^0, \mathbf{p})$ with

$$\begin{aligned} m\, e^{\beta} &= p^0 + |\mathbf{p}|\,, \\ m\, e^{-\beta} &= p^0 - |\mathbf{p}|\,, \end{aligned}$$

where $\beta = |\boldsymbol{\beta}|$. Then

$$\xi_{\mathbf{p}} = e^{+\frac{\boldsymbol{\beta}\boldsymbol{\sigma}}{2}}\,\xi_0 = \frac{1}{2}\,\left[e^{\frac{\beta}{2}}\,(1+\boldsymbol{n}\boldsymbol{\sigma}) + e^{-\frac{\beta}{2}}\,(1-\boldsymbol{n}\boldsymbol{\sigma})\right]\,\xi_0\,,$$

$$\eta_{\mathbf{p}} = e^{-\frac{\boldsymbol{\beta}\boldsymbol{\sigma}}{2}}\,\xi_0 = \frac{1}{2}\,\left[e^{\frac{\beta}{2}}\,(1-\boldsymbol{n}\boldsymbol{\sigma}) + e^{-\frac{\beta}{2}}\,(1+\boldsymbol{n}\boldsymbol{\sigma})\right]\,\xi_0\,,$$

where $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$. The operators

$$\Pi_\pm(\mathbf{p}) = \frac{1}{2}\,(1 \pm \boldsymbol{n}\boldsymbol{\sigma})$$

acting on ξ_0 are just the projectors onto the states with definite *helicity*

$$h(\mathbf{p}) = \frac{1}{2}\,\boldsymbol{n}\boldsymbol{\sigma})$$

which is the projection of the spin on $\mathbf{n}$. We have then the solutions

$$u(p,s) = \begin{pmatrix} \left(\sqrt{p_+}\,, \Pi_+(\mathbf{p}) + \sqrt{p_-}\,, \Pi_-(\mathbf{p})\right)\xi \\ \left(\sqrt{p_+}\,, \Pi_-(\mathbf{p}) + \sqrt{p_-}\,, \Pi_+(\mathbf{p})\right)\xi \end{pmatrix} \tag{17.23}$$

where $\xi = \frac{1}{\sqrt{m}}\,\xi_0$, and

$$p_\pm = p^0 \pm |\mathbf{p}|\,.$$

Here $\xi = \xi^{(s)}$ is an arbitrary amplitude describing the spin state. One can check that this gives the general plane-wave solution of (17.19) with $p^0 > 0$.

In many cases it is convenient to use the basis of the helicity states

$$h(\mathbf{p})\xi^{(s)} = \frac{1}{2}\,s\,\xi^{(s)}\,, \qquad s = \pm\,.$$

Then we have

$$u(p,+) = \begin{pmatrix} \sqrt{p_+}\;\xi^{(+)} \\ \sqrt{p_-}\;\xi^{(+)} \end{pmatrix}\,,$$

$$u(p,-) = \begin{pmatrix} \sqrt{p_-}\;\xi^{(-)} \\ \sqrt{p_+}\;\xi^{(-)} \end{pmatrix}\,.$$

It is then easy to verify that

$$u^\dagger(\mathbf{p},s)u(\mathbf{p},s) = 2\omega_{\mathbf{p}}\,, \qquad \omega_{\mathbf{p}} = \sqrt{m^2+\mathbf{p}^2}$$

and

$$\bar{u}(\mathbf{p},s)u(\mathbf{p},s') = 2m\,\delta_{s,s'}\,,$$

where $\bar{u} = u^\dagger\,\gamma^0$, and I assumed the normalization $\xi^\dagger\xi = 1$.

The above solution corresponds to $p^0 > 0$. The mass-shell condition (17.20) admits also negative $p^0 = -\sqrt{m^2+\mathbf{p}^2}$. The corresponding solutions can be obtained the same way, starting with the rest-frame solution

$$v(-m,\mathbf{0}) = \begin{pmatrix} \xi_0 \\ -\xi_0 \end{pmatrix}\,,$$

and then making a Lorentz boost. One finds

$$v(p,+) = \begin{pmatrix} \sqrt{p_+}\;\xi^{(+)} \\ -\sqrt{p_-}\;\xi^{(+)} \end{pmatrix}\,,$$

$$v(p,-) = \begin{pmatrix} \sqrt{p_-}\;\xi^{(-)} \\ -\sqrt{p_+}\;\xi^{(-)} \end{pmatrix}\,.$$

which satisfy

$$\bar{v}(\mathbf{p},s)v(\mathbf{p},s') = -2m\,\delta_{s,s'}\,.$$

It is straightforward to check

$$\bar{v}(\mathbf{p},s)u(\mathbf{p},s') = \bar{u}(\mathbf{p},s)v(\mathbf{p},s') = 0\,.$$

In what follows we denote

$$U_{\mathbf{p},s}(x) = u(\mathbf{p},s)\,e^{-ipx} \qquad (p^0>0)\,, \tag{17.24}$$

$$V_{\mathbf{p},s}(x) = v(\mathbf{p},s)\,e^{-ipx} \qquad (p^0<0) \tag{17.25}$$

the plane-wave solutions of (17.13).

The plane-wave solutions of the equation (17.14) can be obtained in a similar way; basic solutions have the form

$$\bar{U}_{\mathbf{p},s}(x) = \bar{u}(\mathbf{p},s)\,e^{+ipx} \qquad (p^0>0)\,, \tag{17.26}$$

$$\bar{V}_{\mathbf{p},s}(x) = \bar{v}(\mathbf{p},s)\,e^{-ipx} \qquad (p^0<0) \tag{17.27}$$

where, again, $\bar{u}(\mathbf{p},s) = u^\dagger(\mathbf{p},s)\gamma^0$, $\bar{v}(\mathbf{p},s) = v^\dagger(\mathbf{p},s)\gamma^0$ [22].

[22]The amplitudes $\bar{u}(\mathbf{p},s)$ and $\bar{v}(\mathbf{p},s)$ both satisfy $p_\mu\bar{u}(p)\gamma^\mu - m\bar{u}(p)$, which is obtained by hermitian conjugation of (17.19), with the use of ${\gamma^0}^\dagger = \gamma^0$, $\boldsymbol{\gamma}^\dagger = -\boldsymbol{\gamma}$, and $\gamma^0\boldsymbol{\gamma} = -\boldsymbol{\gamma}\gamma^0$.

Fermion creation and annihilation operators (cf PS §3.5)

Let

$$A_{\bar{U}_{\mathbf{p},s}} = \sqrt{2\omega_{\mathbf{p}}}\; a_{\mathbf{p},\mathbf{s}}\,, \qquad A_{V_{\mathbf{p},s}} = \sqrt{2\omega_{\mathbf{p}}}\; b_{\mathbf{p},\mathbf{s}}\,, \tag{17.28}$$

$$A_{U_{\mathbf{p},s}} = \sqrt{2\omega_{\mathbf{p}}}\; a^{\dagger}_{\mathbf{p},\mathbf{s}}\,, \qquad A_{\bar{V}_{\mathbf{p},s}} = \sqrt{2\omega_{\mathbf{p}}}\; b^{\dagger}_{\mathbf{p},\mathbf{s}}\,, \tag{17.29}$$

where

$$U_{\mathbf{p},s} = u(\mathbf{p},s)\, e^{-ipx}\,, \qquad V_{\mathbf{p},s}(x) = v(\mathbf{p},s)\, e^{-ipx}\,,$$

and

$$\bar{U}_{\mathbf{p},s} = \bar{u}(\mathbf{p},s)\, e^{ipx}\,, \qquad \bar{V}_{\mathbf{p},s}(x) = \bar{v}(\mathbf{p},s)\, e^{ipx}\,.$$

They obey

$$\{a_{\mathbf{p},s}, a^{\dagger}_{\mathbf{p}',s'}\} = \{a_{\mathbf{p},s}, a^{\dagger}_{\mathbf{p}',s'}\} = (2\pi)^3\, \delta_{s,s'}\, \delta^{(3)}(\mathbf{p}-\mathbf{p}')\,,$$

while the rest of the anti-commutators vanish.

$U(1)$ **charge**

The Dirac theory has an important symmetry with respect to the phase rotations of the field variables: the transformation

$$\psi(x) \;\to\; e^{i\theta}\,\psi(x)\,, \qquad \bar{\psi}(x) \;\to\; e^{-i\theta}\,\bar{\psi}(x) \tag{17.30}$$

leaves the action $\mathcal{A}_{\rm Dirac}[\psi,\bar{\psi}]$ invariant. Let us use this example to demonstrate once again how this leads to the conserved current $j^{\mu}(x)$ and associated integral of motion - the "charge".

Consider the same transformation (17.30) but this time with the phase $\theta = \theta(x)$ being an arbitrary function of x, and regard it as the functional variable transformation in the functional integral

$$Z^{-1} \int D[\psi,\bar{\psi}]\, \psi(x_1)...\psi(x_n)\, \bar{\psi}(y_1)...\bar{\psi}(y_m)\, e^{-\mathcal{A}_{\rm Dirac}[\psi,\bar{\psi}]}$$

defining the correlation function $\langle \psi(x_1)...\psi(x_n)\, \bar{\psi}(y_1)...\bar{\psi}(y_m)\rangle$. Simple calculation yields

$$\mathcal{A}_{\rm Dirac}[e^{i\theta(x)}\psi(x), e^{-i\theta(x)}\bar{\psi}(x)] = \mathcal{A}_{\rm Dirac}[\psi(x),\bar{\psi}(x)] + \int d^4x\, \partial_{\mu}\theta(x)\, j^{\mu}(x)\,, \tag{17.31}$$

where

$$j^{\mu}(x) = \bar{\psi}(x)\gamma^{\mu}\psi(x)\,.$$

Eq.(17.31) is a special case of (17.6), and repeating the arguments we have

$$\frac{\partial}{\partial x^\mu}\langle j^\mu(x)\,\psi(x_1)...\psi(x_n)\,\bar\psi(y_1)...\bar\psi(y_m)\rangle + \left[\sum_{k=1}^{n}\delta^{(4)}(x-x_k) - \sum_{l=1}^{m}\delta^{(4)}(x-y_l)\right]\langle\psi(x_1)...\psi(x_n)\,\bar\psi(y_1)...\bar\psi(y_m)\rangle = 0$$

Charge

$$Q(\Sigma) = \int_\Sigma d^3n_\mu(x)\,j^\mu(x)\,.$$

We say that the field ψ carries the $U(1)$ charge $+1$, while $\bar\psi$ has the charge -1.

Commutators

$$[Q, a^\dagger_{\mathbf{p},s}] = +a^\dagger_{\mathbf{p},s}\,, \qquad [Q, b^\dagger_{\mathbf{p},s}] = -b^\dagger_{\mathbf{p},s}\,,$$
$$[Q, a_{\mathbf{p},s}] = -a^\dagger_{\mathbf{p},s}\,, \qquad [Q, b_{\mathbf{p},s}] = +b^\dagger_{\mathbf{p},s}\,,$$

Energy and momentum

$$T_{\mu\nu} = \frac{i}{4}\,\left[\bar\psi\gamma^\mu\partial^\nu\psi + \bar\psi\gamma^\nu\partial^\mu\psi - \partial^\mu\bar\psi\gamma^\nu\psi - \partial^\nu\bar\psi\gamma^\mu\psi\right]$$

$$P^\mu(\Sigma) = \int_\Sigma d^3n_\nu(x)\,T^{\mu\nu}(x)\,.$$

$$P^\mu = (H, -\mathbf{P})$$

Commutators with the energy-momentum

$$[H, a^\dagger_{\mathbf{p},s}] = \omega_{\mathbf{p}}\,a^\dagger_{\mathbf{p},s}\,, \qquad [H, a_{\mathbf{p},s}] = -\omega_{\mathbf{p}}\,a_{\mathbf{p},s}\,,$$
$$[H, b^\dagger_{\mathbf{p},s}] = \omega_{\mathbf{p}}\,b^\dagger_{\mathbf{p},s}\,, \qquad [H, b_{\mathbf{p},s}] = -\omega_{\mathbf{p}}\,b_{\mathbf{p},s}\,,$$

and similar commutators with the spatial momentum

$$[\mathbf{P}, a^\dagger_{\mathbf{p},s}] = \mathbf{p}\,a^\dagger_{\mathbf{p},s}\,, \qquad [\mathbf{P}, a_{\mathbf{p},s}] = -\mathbf{p}\,a_{\mathbf{p},s}\,,$$
$$[H, b^\dagger_{\mathbf{p},s}] = \omega_{\mathbf{p}}\,b^\dagger_{\mathbf{p},s}\,, \qquad [H, b_{\mathbf{p},s}] = -\omega_{\mathbf{p}}\,b_{\mathbf{p},s}\,,$$

Fermionic Fock space

Vacuum:

$$a_{\mathbf{p},\mathbf{s}} \mid 0\rangle = 0\,, \qquad b_{\mathbf{p},\mathbf{s}} \mid 0\rangle = 0$$

The states

$$a^{\dagger}_{\mathbf{p}_1,s_1}...a^{\dagger}_{\mathbf{p}_n,s_n}\, b^{\dagger}_{\mathbf{p}'_1,s'_1}...b^{\dagger}_{\mathbf{p}'_n,s'_n} \mid 0\rangle$$

span the space of states of quantum Dirac theory

The operators

$$a^{\dagger}_{\mathbf{p},\mathbf{s}} \quad \text{and} \quad b^{\dagger}_{\mathbf{p},\mathbf{s}}$$

are referred to as the "creation operators" for fermion and anti-fermion, respectively, while

$$a_{\mathbf{p},\mathbf{s}} \quad \text{and} \quad b_{\mathbf{p},\mathbf{s}}$$

are called the corresponding "annihilation operators".

18 Lecture 18

Electromagnetic Field

Classical Maxwell Theory

Electromagnetic field strength is described by anti-symmetric tensor

$$F^{\mu\nu}(x) = -F^{\nu\mu}(x)$$

which satisfies the Bianci identity (the first pair of Maxwell equations)

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0\,, \tag{18.1}$$

and the equations of motion (second pair)

$$\partial_\nu F^{\mu\nu} = j^\mu \tag{18.2}$$

with the electric current in the right-hand side. The constraint is solved by introducing the 4-vector potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

There is a price to pay. Physical field $F_{\mu\nu}(x)$ does not correspond to a unique $A^\mu(x)$ -there is a whole family of fields $A^\mu(x)$ which lead to the same $F_{\mu\nu}$; members of the family differ by a gauge transfirmation

$$A_\mu(x) \;\rightarrow\; A_\mu(x) + \partial_\mu\alpha(x)$$

whth arbitrary scalar $\alpha(x)$. Then the equation of motion reads

$$\partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu + j^\nu = 0\,.$$

It can be derived using variational principle from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} + j^\mu A_\mu$$

Covariant path integral quantization

As in the case of scalar and the Dirac field it is most straightforward to define the theory in terms of the Euclidean functional integral, and then obtain the Green's functions of Minkowski theory by analytic continuation.

In the Euclidean theory we introduce

$$x_a = (x_4, \mathbf{x}) = (it, \mathbf{x})$$

and the Euclidean vector potential

$$A_a = (A_4, \mathbf{A}) = (iA^0, \mathbf{A})$$

and

$$F_{ab} = \partial_a A_b - \partial_b A_a \ .$$

The Euclidean action is then

$$\mathcal{A}_{EM} = \int d^4x \ \frac{1}{4} \ (F_{ab})^2 = \text{by parts} = \int d^4x \ \left[\frac{1}{2}(\partial_a A_b)^2 - \frac{1}{2}(\partial_a A_a)^2\right] \ ,$$

where some integrations by parts are performed.

In terms of this action one writes the functional integral

$$\int D[A] \ \big(\ \dots \big) \ e^{-\mathcal{A}_{EM}}$$

where, intuitively, $D[A] = \prod_x \prod_{a=1}^4 dA_a(x)$. Proceeding as before we find that the two-point function

$$D_{ab}(x-y) = \langle \, A_a(x) A_b(y) \rangle$$

must satisfy the equation

$$(-\delta_{ac}\partial^2 + \partial_a \partial_c) \, D_{cb}(x-y) = \delta_{ab} \ \delta^{(4)}(x-y) \ ,$$

or, making a Fourier transform

$$(k^2 \, \delta_{ac} - k_a k_c) \ \tilde{D}_{cb}(k) = \delta_{ab} \, . \tag{18.3}$$

There is a problem here. The matrix

$$k^2 \delta_{ac} - k_a k_c$$

is not invertible, and therefore $\tilde{D}$ satisfying this equation does not exist. It is not invertible because it has a zero eigenvalue associated with the eigenvector

$$\psi_a = const \ k_a \ .$$

This property is related to the gauge invariance of the EM theory the action does not change under the variable transformations

$$A_a(x) \to A_a(x) + \partial_a \alpha(x)$$

where $\alpha(x)$ is an arbitrary function. Let us fix some $A_a^{(0)}(x)$ and consider the family of field configurations

$$A_a^{(\alpha)}(x) = A_a^{(0)}(x) + \partial_a \alpha(x) \ .$$

As $\mathcal{A}_{EM}[A^{(\alpha)}] = \mathcal{A}_{EM}[A^{(0)}]$ does not depend on α, the integral

$$\int D[A^{(\alpha)}] \ e^{-\mathcal{A}_{EM}[A^{(\alpha)}]}$$

over just these configuration diverges, because the action does not provide Gaussian damping. The above difficulty with the propagator is related to this property. On the other hand, in EM theories the configurations which are different by the gauge transformation are regarded as physically identical. The gauge transformations constitute the *gauge group*. (Because this transformations have the group property – the result of two successive gauge transformations with $\alpha_1(x)$ and $\alpha_2(x)$ is equivalent to a single gauge transformation with $\alpha_1 + \alpha_2$; in EM theory the gauge group is Abelian). The set of configurations

$$A_a^{(0)}(x) + \partial_a \alpha(x)$$

constitute what is called the *orbit* of the gauge group associated with $A_a^{(0)}$. Two configuration belonging to the same orbit are physically indistinguishable. Thus we would like to write our functional integral in such a way that integration would go over not all fields $A_a(x)$, but rather over the orbits. In other words we would like to integrate over the orbits.

The way to do this it is known as *gauge fixing*. The idea is to select exactly one representative of each orbit by imposing *gauge fixing condition*, i.e. some additional constraint on A_a having the form

$$\chi(A) = 0$$

such that there is exactly one representative of each orbit $A_a^{(\alpha)}$ which satisfies this condition.

Typically, $\chi(A)$ is chosen to be local, i.e. $\chi(A)$ depends on $A_a(x)$ and its derivatives at x, so in fact $\chi(A) = 0$ represents one constraint for every point x. This is natural because of the gauge invariance is *local* symmetry.

Some example's of the gauge condition. We have already discussed the gauge condition

$$\chi(A) = \vec{\nabla} \cdot \vec{A}$$

This condition is known as *Coulomb gauge*. Note that this is not a Lorentz invariant condition. Another choice

$$\chi(A) = \partial_a A_a(x) = 0$$

is known as **Lorentz gauge**. Two configurations $A_a^{(1)}(x)$ and $A_a^{(2)}(x)$ both satisfying the gauge condition cannot be related by gauge transformation since this would imply

$$A_a^{(1)}(x) - A_a^{(2)}(x) = \partial_a \alpha(x)$$

with $\partial_a \partial_a \alpha(x) = 0$; in Euclidean space this means $\alpha(x) = 0$. There are many others interesting choices. Let me write generic gauge-fixing condition as

$$\chi(A) = 0$$

where χ can depend on A_a and it derivatives.

Once the gauge condition is chosen, one can write the integral as separate integrals over the gauge-fixed configurations, and integral over the gauge group. This is done by so called *Faddeev-Popov transformation*. Consider the identity

$$1 = \int D[\alpha(x)]\ \delta\big(\chi(A^{(\alpha)})\big)\ \det\Big(\frac{\delta\chi(A^{(\alpha)})}{\delta\alpha}\Big)$$

where $A^{(\alpha)}$ stands for the elements of the gauge orbit. This identity is continuous version of simple identity for n-dimensional integral

$$1 = \int \Big[\prod_i d\alpha_i\Big]\ \Big[\prod_j \delta\big(\chi_j(\alpha)\big)\ \det\Big(\frac{\partial\chi_j(\alpha)}{\partial\alpha_i}\Big)\ .$$

Let us insert this 1 into the integral

$$\int D[A]\ (\ldots)\ e^{-\mathcal{A}_{EM}[A]} = \int D[A]D[\alpha]\ (\ldots)\ e^{-\mathcal{A}_{EM}[A]}\ \delta\big(\chi(A^{(\alpha)})\big)\ \Delta(A^{(\alpha)})$$

where I denoted $\Delta = \det\Big(\frac{\delta A^{(\alpha)}}{\delta\alpha}\Big)$. The brackets (...) represent any **gauge invariant** insertions. Now we change variables

$$A_a \to A_a^{(\alpha)} = A_a + \partial_a \alpha$$

This is a shift, so $D[A] = D[A^{(\alpha)}]$, and also

$$\mathcal{A}_{EM}[A] = \mathcal{A}_{EM}[A^{(\alpha)}] \ .$$

Now that $A^{(\alpha)}$ became the integration variable we can rename it back as A. Thus

$$\int \mathcal{D}A \ \big(\ \dots \ \big) \ e^{-\mathcal{A}_{EM}} = \int \mathcal{D}\alpha \ \int \mathcal{D}A \ \big(\ \dots \ \big) \ e^{-\mathcal{A}_{EM}} \ \Delta[A] \ \delta\big(\chi(A)\big)$$

where $\Delta[A] = \det\big(\frac{\delta A^{(\alpha)}}{\delta \alpha}\big)\big|_{\alpha=0}$. Thus we achieved our goal: the functional integral is factorize to the integral $\int \mathcal{D}\alpha$, which is of course infinite, but should be understood as the integral over the gauge group, i.e. the volume of gauge group. The second part is the physically distinct gauge configurations, as the gauge field A there is constrained by the gauge-fixing condition $\chi(A) = 0$. Further transformations can be made once we select particular gauge-fixing condition. For example once we chose the Coulomb gauge the variational derivative $\frac{\delta A^{(\alpha)}}{\delta \alpha} = \vec{\nabla}^2$ does not depend on A (this is concerns any *linear* gauge in which G is linear in A), and we have

$$\Delta[A] = \det(\nabla^2) \ .$$

Of course, one needs a cutoff to define this determinant, and it is divergent as the cutoff distance goes to zero. But in this simple situation this is not really a problem because the determinant is an A-independent (see below). Similarly, for Lorentz gauge

$$\Delta[A] = \det({\partial_a}^2) \ .$$

Consider simple generalization of the Lorentz gauge

$$\chi(A) = \partial_a A_a(x) - \omega(x) \ .$$

The determinant $\Delta[A]$ remains the same, one still needs regularization (cutoff) to define this object, but it still does not depend on A and hence cancels out in

$$\frac{1}{Z} \ \int \mathcal{D}A \ \big(\ \dots \ \big) \ e^{-\mathcal{A}_{EM}} \ .$$

In further calculations here we can ignore the factor Δ.

$$\int \mathcal{D}A \ \big(\ \dots \ \big) e^{-\mathcal{A}_{EM}} = \int \mathcal{D}\alpha \ \int \mathcal{D}A \ \big(\ \dots \big) \ e^{-\mathcal{A}_{EM}} \ \delta\big(\partial_a A_a - \omega\big) \ .$$

This holds for any ω, so it will remain valid if we integrate both sides over ω with arbitrary functional of ω, as long as the integral

$$\int \mathcal{D}\omega(x)\ \rho[\omega(x)]$$

converges. Convenient choice is

$$\rho_\xi[\omega] = \frac{1}{Z_\xi}\ \exp\Big(-\frac{1}{2\xi}\ \int d^4x\ \omega^2(x)\Big)\ ,$$

with real parameter ξ. Denoting $Z_\xi = \int \mathcal{D}\omega(x)\ \rho_\xi[\omega(x)]$ we get

$$\Big[Z_\xi^{-1}\ \int \mathcal{D}\alpha\Big]\ \int \mathcal{D}\omega\mathcal{D}A\ e^{-\mathcal{A}_{EM}}\ \exp\Big(-\frac{1}{2\xi}\ \int d^4x\ \omega^2(x)\Big)\ \delta\big(\partial_a A_a - \omega\big)\ .$$

The integration over ω amounts to elimination of the δ-function:

$$= const\ \int\ \mathcal{D}A\ e^{-\mathcal{A}_\xi}$$

where

$$\mathcal{A}_\xi = \mathcal{A}_{EM} + \frac{1}{2\xi}\ \int d^4x\ (\partial_a A_a)^2 = \int d^4x\ \Big[\frac{1}{2}(\partial_a A_b)^2 - \frac{1}{2}\Big(1-\frac{1}{\xi}\Big)\ \ (\partial_a A_a)^2\ \Big]\ .$$

So, effectively, we have added the extra term $\frac{1}{2\xi}\ \int d^4x\ (\partial_a A_a)^2$ to the action, where ξ is an arbitrary number.

So far we considered only the integral $\int \mathcal{D}A\ e^{-\mathcal{A}_{EM}}$. The above transformations will go through for the integral

$$\int \mathcal{D}A\ e^{-\mathcal{A}_\xi}\ \big(\ \ldots\ \big)$$

as long as $\big(\ \ldots\ \big)$ is a gauge invariant insertion. Therefore, if $\ldots$ is a gauge invariant object we have

$$\big\langle\ \ldots\ \big\rangle = \frac{\int \mathcal{D}A\ e^{-\mathcal{A}_\xi}\ \big(\ \ldots\ \big)}{\int \mathcal{D}A\ e^{-\mathcal{A}_\xi}}$$

all factors cancel out like $Z_\xi^{-1}\ \det(\partial^2) \int \mathcal{D}\alpha$ cancels out. The expectations values of the gauge-invariant quantities do not depend on the choice of the gauge condition and, in particular, the dependence of the parameter ξ must disappear.

Now we can come back to the propagator. With the new action $\mathcal{A}_\xi$ it satisfies

$$\left[k^2\,\delta_{ac} - \left(1 - \frac{1}{\xi}\right) k_a k_c \right] \tilde{D}_{cb}(k) = \delta_{ab}$$

and so

$$\tilde{D}_{ab}(k) = \tilde{D}^{\perp}_{ab}(k) + \tilde{D}^{\parallel}_{ab}(k) = \frac{1}{k^2}\,\left(\,\left(\delta_{ab} - \frac{k_a k_b}{k^2}\right) + \xi\,\frac{k_a k_b}{k^2}\,\right)\,.$$

We see that $\tilde{D}$ depend on ξ which is not surprising because it is $\langle\, A_a\, A_b\,\rangle$ which is not gauge invariant. But $\tilde{D}_{ab}$ splits into **transversal** part, $\tilde{D}^{\perp}$, which is ξ independent, and $\tilde{D}^{\parallel}$, containing ξ. Obviously the correlation functions of the gauge invariant quantities, i..e the field strength F_{ab}, involves only $\tilde{D}^{\perp}$.

In calculations some particular choices of ξ can be more convenient. One is the **Landau gauge** $\xi = 0$

$$\tilde{D}_{ab}(k) = \frac{1}{k^2}\,\left(\delta_{ab} - \frac{k_a k_b}{k^2}\right)$$

Note that $\xi = 0$ corresponds to $\rho[\omega] = \delta(\omega)$, i.e. we integrate over $A_a(x)$ which satisfies the Lorentz gauge $\partial_a A_a = 0$. Indeed, the coefficients in front of the extra term $\frac{1}{2\xi}\,\int d^4x\,(\partial_a A_a)^2 \to \infty$, so

$$\exp\left(\,-\frac{1}{2\xi}\,\int d^4x\,(\partial_a A_a)^2\right) \approx \delta(\partial_a A_a)\;,\quad \xi \to 0\;.$$

Even more convenient choice is $\xi = 1$

$$\tilde{D}_{ab}(k) = \frac{\delta_{ab}}{k^2} \qquad \text{Feynman gauge}$$

Adding a source term $\int d^4x J_a(x) A_a(x)$ with arbitrary 4-vector functions $J_a(x)$ which satisfies

$$\partial_a J_a(x) = 0$$

one finds the generating functionals

$$W[J] = Z^{-1}\,\int\,D[A]\,\exp\left(\,-\mathcal{A}_{\rm EM}[A] - \int d^4x\,J_a(x)\,A_a(x)\right)$$

which is gauge invariant. Again, fixing the gauge as above, one finds

$$W[J] = \exp\left(\frac{1}{2}\,\int\,d^4x\,d^4y\,J_a(x) D_{ab}(x-y) J_b(y)\right) = \exp\left(\frac{1}{2}\,\int\,\frac{d^4k}{(2\pi)^4}\,\tilde{D}_{ab}(k)\tilde{J}_a(k)\tilde{J}_b(-k)\,\right)\,.$$

Finally, by continuation back to real-time picture (Wick rotation) k_4 it the last integral in bring the result to form previously obtained

$$Z[J]/Z[0] = \exp\left\{ -\frac{1}{2} \int d^4x\, d^4y\; D_F^{\mu\nu}(x-y)\, J_\mu(x) J_\nu(y) \right\}$$

where

$$D_F^{\mu\nu} = \int \frac{d^4k}{(2\pi)^4}\; e^{-ik(x-y)}\; \frac{(-i)}{k^2 + i\epsilon}\; \left(g^{\mu\nu} - (1-\xi)\; \frac{k^\mu k^\nu}{k^2} \right)$$

19 Lecture 19

Quantum Electrodynamics (QED)

We can describe now the interaction between EM field and Dirac field of electrons and positrons. The action is (in Euclidean space)

$$\mathcal{A}_{QED}[A,\psi,\bar{\psi}] = \mathcal{A}_{EM}[A] + \int \mathrm{d}^4x \, \left[-e\,\bar{\psi}\gamma_a\psi\, A_a \right] + \mathcal{A}_{Dirac}[\psi,\bar{\psi}] = \mathcal{A}_{EM}[A] + \int \mathrm{d}^4x \, \left[\bar{\psi}(-\mathrm{i}\,\gamma_a D_a + m_e)\psi \right], \tag{19.1}$$

where $\mathcal{A}_{EM}$ is the EM action we have considered previously, and

$$D_a = \partial_a - \mathrm{i}\,e\,A_a$$

is so called "covariant derivative", and e is the unit of charge[24]. The covariant derivative is constructed in such a way that under a gauge transformation

$$\psi(x) \to \mathrm{e}^{\mathrm{i}e\alpha(x)}\,\psi(x), \quad \bar{\psi}(x) \to \mathrm{e}^{-\mathrm{i}e\alpha(x)}\,\bar{\psi}(x), \quad A_a(x) \to A_a(x) + \partial_a\alpha(x)$$

it transforms exactly as ψ itself,

$$D_a\psi \to \mathrm{e}^{\mathrm{i}e\alpha(x)}\,D_a\psi,$$

with the transformation of A_a compensating for $\partial_a\alpha$ term appearing when ∂_a acts on $\mathrm{e}^{\mathrm{i}e\alpha(x)}$. Therefore this action is gauge invariant

$$\psi(x) \to \mathrm{e}^{\mathrm{i}e\alpha(x)}\,\psi(x), \quad \bar{\psi}(x) \to \mathrm{e}^{-\mathrm{i}e\alpha(x)}\,\bar{\psi}(x), \qquad A_a \to A_a + \partial_a\alpha$$

Therefore, the functional integral

$$\int D[A,\psi,\bar{\psi}]\,(...)\,\mathrm{e}^{-\mathcal{A}_{QED}[A,\psi,\bar{\psi}]}$$

again involves redundant integration over the gauge equivalent configurations of all fields, i.e. integration along the orbits of the gauge group. In this case the orbits is a family

$$\psi^{(\alpha)}(x) = \mathrm{e}^{\mathrm{i}e\alpha(x)}\,\psi^{(0)}(x), \quad \bar{\psi}^{(\alpha)}(x) = \mathrm{e}^{-\mathrm{i}e\alpha(x)}\,\bar{\psi}^{(0)}(x), \qquad A_a^{(\alpha)} = A_a^{(0)} + \partial_a\alpha$$

[24] It is related to the so-called fine structure constant

$$\alpha = \frac{1}{4\pi}\frac{e^2}{\hbar c} = \frac{1}{137}$$

Therefore we need the gauge fixing, which can be done exactly as before, by imposing the gauge-fixing condition $\chi[A] = 0$. We can use the same condition $\partial_a A_a - \omega = 0$, with subsequent integration over $\omega(x)$, as in the previous analysis of the free theory. As the result, the part $\mathcal{A}_{EM}$ gets replaced by $\mathcal{A}_\xi$ and integration $\mathcal{D}\alpha$ factorizes. Finally, we get the gauge -fixed action

$$\mathcal{A}_{QED} = \int \mathrm{d}^4x \left[\frac{1}{2} (\partial_a A_b)^2 - \frac{1}{2}\left(1 - \frac{1}{\xi}\right) (\partial_a A)^2 + \bar{\psi}(-\mathrm{i}\,\gamma_a \partial_a + m_e)\psi - e\, A_a\, \bar{\psi}\gamma_a\psi \right]$$

where I have deliberately separated the term containing the electron charge e as it is only term which is not quadratic in the fields.

Feynman Rules for QED

In the absence of the interaction term $-e\, A_a\, \bar{\psi}\gamma_a\psi$ we have free EM field combined with the Dirac field of electrons and positrons. In the free theory the correlation functions

$$\langle\, \psi(x_1)...\psi(x_n)\, \bar{\psi}(x'_1)...\bar{\psi}(x'_n)\, A_{a_1}(y_1)...A_{a_m}(y_m)\,\rangle$$

are expressed as the sum of all possible Wick contractions, with the elementary contractions

$$\overbrace{\psi(x)\ \bar{\psi}(x')} = G(x - x')\,,$$

$$\overbrace{A_a(y)\ A_b(y')} = D_{ab}(y - y')$$

where, as before

$$G(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{\mathrm{i}(px)}\, \frac{m - \gamma p}{m^2 + p^2}\,, \qquad \gamma p = \gamma_a p_a\,,$$

(where $(px) = p_a x_a = -p_\mu x^\mu = -p^0 t + \vec{p}\vec{x}$, $p^2 = p_a p_a = -p^\mu p_\mu$) and

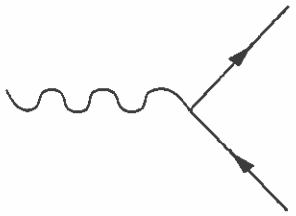
$$D_{ab}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{i(kx)}\, \frac{1}{k^2} \left(\left(\delta_{ab} - \frac{k_a k_b}{k^2}\right) + \xi\, \frac{k_a k_b}{k^2} \right)$$

are called electron and photon propagators, respectively. These propagators are represented by lines

G =

D_{ab} = a  b

The interaction term $-e\,A_a\,\bar{\psi}\gamma_a\psi$ can be treated as the perturbation. It generates the vertex

$e\,\gamma_a$ = 

More generally, if we are interested in the matrix elements between states with particles - electrons, positrons, and photons -, in particular in the scattering amplitudes, additional contractions must be added which connect initial and final particles with the field insertions. Thus, for electrons and positrons we have

$$\overbrace{\psi(x)\,a^\dagger_{\mathbf{p},s}} = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\,u(\mathbf{p},s)\,e^{ipx} \qquad \overbrace{b_{\mathbf{p},s}\psi(x)} = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\,v(\mathbf{p},s)\,e^{-ipx}\,, \quad (19.2)$$

$$\overbrace{\bar{\psi}(x)\,b^\dagger_{\mathbf{p},s}} = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\,\bar{v}(\mathbf{p},s)\,e^{ipx} \qquad \overbrace{a_{\mathbf{p},s}\bar{\psi}(x)} = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\,\bar{u}(\mathbf{p},s)\,e^{-ipx}\,, \quad (19.3)$$

and

$$\overbrace{a_{\mathbf{p},s}\,a^\dagger_{\mathbf{p}',s'}} = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{p}')\,\delta_{s,s'}\,, \qquad \overbrace{b_{\mathbf{p},s}\,b^\dagger_{\mathbf{p}',s'}} = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{p}')\,\delta_{s,s'}\,.$$

For the photons we have

$$\overbrace{c_{\mathbf{k},\alpha}\,c^\dagger_{\mathbf{k}',\alpha'}} = (2\pi)^3\,\eta_{\alpha,\alpha'}\,\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,,$$

$$\overbrace{c_{\mathbf{k},\alpha}\,A^\mu(x)} = \bar{e}^\mu_\alpha(k)\,e^{ikx}\,, \qquad \overbrace{A^\mu(x)c^\dagger_{\mathbf{k},\alpha}} = e^\mu_\alpha(k)\,e^{-ikx}\,.$$

Here e^μ_α, $\alpha = 1,2$ describe the polarization states of a photon. Generally, these vectors can be written as

$$e^\mu_\alpha := (0,\mathbf{e}_\alpha) + f\;k^\mu\,, \quad (19.4)$$

with pure transversal $\mathbf{e}_\alpha$, and the coefficient f representing the gauge ambiguity in of the vector potential $A^\mu(x)$ (the photon states are gauge invariant).

Remark 1: The gauge ambiguity represented by of the f term in (19.4) should not affect the gauge invariance of amplitudes of emission and absorbtion of physical photons. In perturbation theory, the insertion of the gauge field $A^\mu(x)$ enters through the interaction term, which (in Minkowski ST) has the form

$$-ie \int d^4x\, J^\mu(x)\, A_\mu(x)\,, \qquad J^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x)\,.$$

Contraction of $A^\mu(x)$ with an external photon $|\,(\mathbf{k}, \mathrm{e}_\alpha)(...)\rangle \sim c^\dagger_{\mathbf{k},\alpha}\,(...)\,|\,0\rangle$ generates the factor

$$-ie\; e^\mu_\alpha(k) \int d^4x\, e^{ikx} J_\mu(x)$$

so that the photon absorbtion amplitude is expressed as

$$-ie\; e^\mu_\alpha(k)\, \mathcal{M}_\mu(k, ...)\,, \qquad \mathcal{M}_\mu(k, ...) = f.t.\ \langle J_\mu(x)...\rangle$$

The gauge invariance of the absorbtion amplitude follows from the so-called *Ward-Takahashi (WT) identity*

$$k^\mu\, \mathcal{M}_\mu(k, ...)\big|_{k^\nu k_\nu = 0} = 0\,.$$

which we will derive soon.

Remark 2: The scattering may also involve electrons and positrons, which carry charge. Here the problem is more difficult. The electrons/positrons are created by the fields ψ and $\bar\psi$, which are not gauge invariant, and there is no simple (local) way to remove the gauge dependence. The local gauge-invariant operators like $\bar\psi\psi$ or $\bar\psi\gamma_a\psi$ are charge neutral and thus can not create one-electron, or one-positron states. On the other hand, the operators $\psi(x)$, $\bar\psi(x)$ which carry charge, are not gauge invariant. The correlation functions involving these fields will depend on the choice of the gauge. In particular, the electron and positron states constructed using the operators defined as suggested by (17.28), (17.29) carry gauge dependence. In principle, one can construct operators which are both gauge invariant and carry nonzero charge, but such operators are necessarily nonlocal. To have an idea, consider expression

$$\psi_{\mathcal{E}}(\mathbf{x}, t) = \psi(\mathbf{x}, t)\;\; \exp\left[-\,\mathrm{ie} \int_{x^0=t} \mathrm{d}^3\mathbf{r}\; \mathcal{E}(\mathbf{r}, \mathbf{x}) \cdot \mathbf{A}(\mathbf{r}, t) \right],$$

where $\mathcal{E}(\mathbf{r}, \mathbf{x})$ satisfies the equation

$$\nabla_r \cdot \mathcal{E}(\mathbf{r}, \mathbf{x}) = \delta^{(3)}(\mathbf{r} - \mathbf{x})\;.$$

It is easy to check that such operators are gauge invariant. The meaning is that the gauge-invariant state must contain, beside the charged particles themselves, the EM field surrounding them, like a Coulomb field of a static charge.

The scattering theory based on such gauge-invariant operators can be developed, but it is not simple. At this point we will adopt simple-minded approach: we postulate that the scattering amplitudes of electrons and positrons are given by the matrix elements between the states

$$\mid a^\dagger_{\mathbf{p}_1,s_1}\ldots a^\dagger_{\mathbf{p}_n,s_n} b^\dagger_{\mathbf{p}'_1,s'_1}\ldots b^\dagger_{\mathbf{p}'_m,s'_m}\rangle \;:=\; a^\dagger_{\mathbf{p}_1,s_1}\ldots a^\dagger_{\mathbf{p}_n,s_n} b^\dagger_{\mathbf{p}'_1,s'_1}\ldots b^\dagger_{\mathbf{p}'_m,s'_m}\mid 0\rangle\,,$$

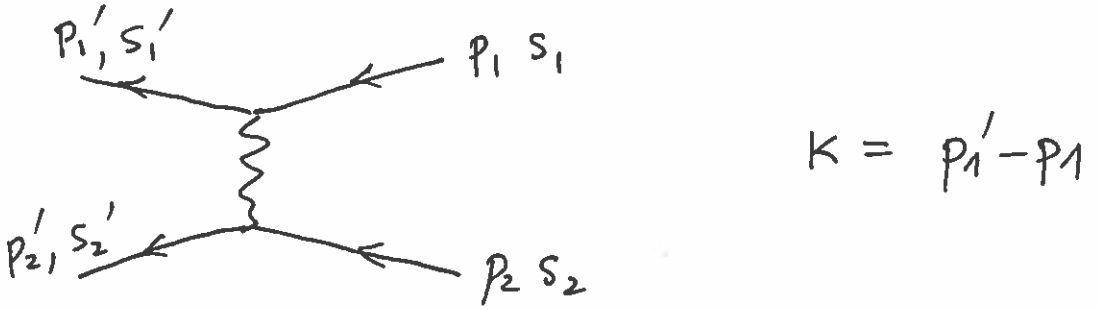
with the operators $a^\dagger$ and $b^\dagger$ defined in terms of the fields $\psi(x)$ and $\bar\psi(x)$ according to (17.28), (17.29). Then we will show that the resulting physical scattering amplitudes are gauge invariant order by order in the perturbation theory. We will also see that there is a price to pay for this simple-minded approach. Namely, we will discover that when charged particles are present in the initial and/or final states of the scattering, the diagrams beyond the leading order exhibit *infrared divergences*. They appear because we ignore the "EM dressing", surrounding the charged particles. We will see, however, how to incorporate the contributions of these "dressings" into physical quantities (using the perturbation theory) and obtain infrared-finite results.

Gauge invariance at the tree level

Let us first consider simple tree-level amplitudes to see how the dependence on the gauge cancels out. Consider, for example, the electron-electron scattering in the leading order

$$_{out}\langle\,(p'_1,s'_1),(p'_2,s'_2)\mid(p_1,s_1),(p_2,s_2)\,\rangle_{in}$$

where I used the short-hand $\mid(p_1,s_1),(p_2,s_2)\,\rangle$ for the state $a^\dagger_{\mathbf{p}_1,s_1}a^\dagger_{\mathbf{p}_2,s_2}\mid 0\rangle$ with two electrons. In the leading order we have the diagram



and similar diagram with interchanged legs, $p_1,s_1 \leftrightarrow p_2,s_2$. There is no momentum

integrations. We can compute the diagram directly in Minkowski space-time, using

$$D_{\mu\nu}(k) = \left(g_{\mu\nu} - (1-\xi)\,\frac{k_\mu k_\nu}{k^2}\right)\frac{(-\mathrm{i})}{k^2+\mathrm{i}0}\,, \qquad k^2 = k_0^2 - \vec{k}^2$$

The ξ-dependent term involves the factor $\frac{k_\mu k_\nu}{k^2}$. This factor is accompanied by the factor $\bar{u}_{s_1'}(\mathbf{p}_1')\,\gamma^\mu\,u_{s_1}(\mathbf{p}_1)$ which comes from the upper vertex in the diagram. Therefore, we have the combination

$$\bar{u}_{s_1'}(\mathbf{p}_1')\,\gamma^\mu\,u_{s_1}(\mathbf{p}_1)\,\frac{k_\mu k_\nu}{k^4}\,.$$

Since $k_\mu = p_{1\mu}' - p_{1\mu}$, this involves

$$\bar{u}_{s_1'}(p_1')\,\gamma^\mu(p_{1\mu}' - p_{1\mu})\,u_{s_1}(p_1)\,,$$

which can be written as

$$\bar{u}_{s_1'}(p_1')\,(\not{p}_1' - m - \not{p}_1 + m)\,u(p_1)\,, \qquad \not{p} = p_\mu\gamma^\mu = p_a\gamma_a$$

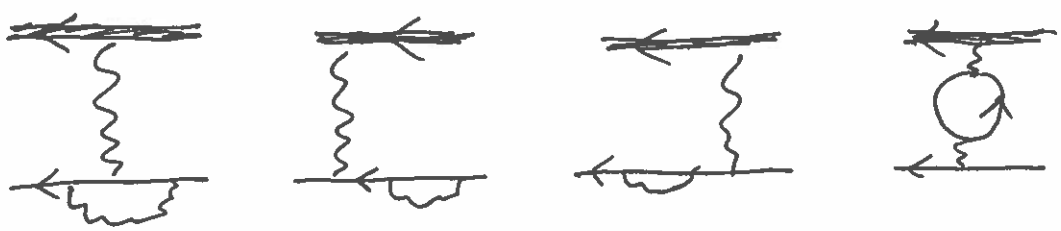
It is equal to zero because

$$(\not{p} - m)\,u(p) = 0\,, \quad \bar{u}(p)(\not{p} - m) = 0\,.$$

In fact, we will show later, using the Ward identities, that ξ-dependence disappears from all diagrams (describing on-shell amplitudes) constructed with this rules.

I am not going to elaborate this scattering process in details. If you can never done this type of calculations befpore, look up the leading order calculations of various scattering amplitudes like $e^-e^- \to e^-e^-$ or $e^-e^+ \to e^-e^+$, or Compton scattering, please take a look at Ch.5 of PS.

Instead, let us consider higher-order contributions, which involve loops. To simplify as much as possible, let me consider the scattering of an electron off a very heavy particle, which can be considered as fixed external charge, generating external EM field. In the order e^4 we have the following diagrams



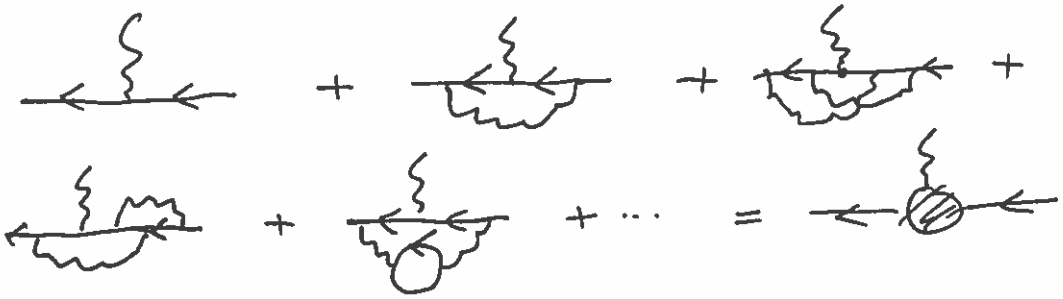
(a) (b) (c) (d)

The scattering amplitudes are defined in terms of "amputated" correlations functions, with the full propagators associated with the external legs removed. Therefore the diagrams (b) and (c), which correct the external legs, must be omitted.

The diagram (d) gives the correction to the photon propagator (the so called vacuum polarization). The photon line is not external, and the diagram has to be added. We will consider the vacuum polarization separately, and observe that its role is to renormalize the charge. At this stage we ignore it.

Let us concentrate on the diagram (a), more precisely on the part of it

with all external legs amputated. This diagram is a representative of the class of diagrams of the type



which give the corrections to the vertex

The sum of these diagrams is called the *vertex function.*

Let us first assume that neither the external photon line nor the external elections line are on-shell. Then it is easy to understand that this vertex function can be expressed through the correlation function

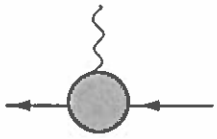
$$\langle\, J_a(x)\,\psi(y_1)\bar{\psi}(y_2)\,\rangle$$

where $J_a(x) = \bar{\psi}(x)\gamma_a\psi(x)$ is the electric current. Namely consider the Fourier transform

$$\int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\,\langle\, J_a(0)\,\psi(y_1)\bar{\psi}(y_2)\,\rangle_\xi\; \mathrm{e}^{-\mathrm{i}(p_1y_1)+\mathrm{i}(p_2y_2)} = G(p_1)\,\Gamma_a(p_1,p_2)\,G(p_2)$$

where $G(p) = \text{F.T.}\ \langle\, \psi(y_1)\bar{\psi}(y_2)\,\rangle$. It is easy to see that diagrams contributing to $\Gamma_a(p_1,p_2)$ are exactly those I listed above. I will use diagrammatic notation

$e\,\Gamma_a(p_2,p_1)$ =



Note also that this quantity is defined in terms of the operators ψ, $\bar{\psi}$, which are not gauge invariant, and so in general Γ_a may not be gauge invariant. In particular, if we calculate it in perturbation theory with $\mathcal{A}_\xi$, it will depend on ξ.

However, we may consider the case when both p_1, p_2 are on shell, i.e.

$$p_1^2 = p_2^2 = -m_e^2\ .$$

in the Euclidean notations. Consider the quantity

$$\bar{u}(p',s')\,\Gamma_a(p',p)\,u(p,s)$$

It can be interpreted as the scattering amplitude of an electron on weak external EM field described by classical vector potential $A_a^{ext}(x)$. One can imagine a heavy particle as a source of this external field. In effect, this external field corresponds to adding a term

$$\mathcal{A}_{ext} = -\int e\,A_a^{(ext)}(x)\bar{\psi}\gamma_a\psi\;\mathrm{d}^4x$$

to the action, where $A_a^{(ext)}(x)$ is fixed function of x. If $A_a^{(ext)}$ is small, in the leading order in $A_a^{(ext)}$ we have for the scattering amplitude of the electron off this potential

$$\mathcal{M} = e\,\tilde{A}_a^{(ext)}(k)\;\bar{u}(p')\,\Gamma_a(p',p)\,u(p) + O((A_a^{(ext)})^2)$$

where $k = p' - p$, and $\tilde{A}_a^{(ext)}(k)$ is the Fourier transform $A_a^{(ext)}(x)$.

The gauge invariance suggests that if $A_a^{(ext)}$ is replaced by

$$A_a^{(ext)}(x) + \partial_a\alpha(x)\ .$$

the amplitude must remain unchanged. As

$$\tilde{A}_a^{(ext)}(k) \to \tilde{A}_a^{(ext)}(k) + \mathrm{i}\, k_a\, \alpha(k)\ ,$$

the requirement of the gauge invariance demands that

$$\bar{u}(p',s')\, k_a \Gamma_a(p',p)\, u(p,s) \ = 0 \qquad\qquad (k = p' - p)\ .$$

This follows from the Ward-Takahashi identity I mentioned before. Let me derive it now.

Ward-Takahashi identity

Consider the following transformation of variables in the paths integral

$$\psi(x) \to \mathrm{e}^{\mathrm{i}\theta(x)}\ \psi(x)\ , \qquad \bar{\psi}(x) \to \mathrm{e}^{-\mathrm{i}\theta(x)}\ \bar{\psi}(x)\ .$$

This looks exactly as gauge transformation of ψ, with $\theta(x) = e\ \alpha(x)$. If one would simultaneously gauge-transform A_a and $A_a^{(ext)}$, the action $\mathcal{A}_{QED}$ would remain unchanged. However, consider the above transformation of ψ, $\bar{\psi}$, but leave $A_a(x)$ ant $A_a^{(ext)}(x)$ unchanged. Then, the action changes by the amount

$$\mathcal{A}_{QED} \to \mathcal{A}_{QED} + \int \mathrm{d}^4x\ \partial_a\theta(x)\ \bar{\psi}\gamma_a\psi\ .$$

However, being just the transformation of the integration variables, it does not change the value of the integral. Making this transformation in the paths integral defining

$$\langle\, \psi(x)\bar{\psi}(x')\, \rangle_\xi\ ,$$

and remembering that it must leave the value of the integral unchanged, one derives the identity

$$0 = -\int \mathrm{d}^4x\ \partial_a\theta(y)\ \langle\, J_a(y)\ \psi(x)\bar{\psi}(x')\, \rangle_\xi + \mathrm{i}\,(\theta(x) - \theta(x'))\ \langle\, \psi(x)\bar{\psi}(x')\, \rangle_\xi\ .$$

As $\theta(x)$ is arbitrary, we must have

$$\langle\, \partial_a J_a(y)\ \psi(x)\bar{\psi}(x')\, \rangle_\xi = \mathrm{i}\ \big(\, \delta(y-x) - \delta(y-x')\, \big)\ \langle\, \psi(x)\bar{\psi}(x')\, \rangle_\xi\ .$$

Now make the Fourier transform of this to find (I leave this as an exercise)

$$(p'-p)_a\, G(p')\, \Gamma_a(p',p)\, G(p) = G(p') - G(p)\ .$$

Multiplying by $G^{-1}(p')$ from the right and by $G^{-1}(p)$ from the left, we obtain

$$(p'-p)_a\,\Gamma_a(p',p) = G^{-1}(p) - G^{-1}(p')\ ,$$

which itself is known as the Ward-Takahashi identity. For diagrammatic derivation see Ch.7.4 of PS.

The physical mass of the particle (the electron, in our case) m_e is determined by the location of the pole in $G(p)$ (if there is a pole):

$$G(p)\ :=\ \text{f.t.}\ \langle\psi(x)\bar\psi(x')\rangle\ =\ Z_\psi\,\frac{m_e-\gamma_a p_a}{p^2+m_e^2}+\text{regular}\ .$$

Equivalent, but more convenient definition is

$$G^{-1}(p)|_{p^2=-m_e^2}u(p)=0\ ,\qquad \bar u(p)G^{-1}(p)|_{p^2=-m_e^2}=0\ .$$

Assuming that the pole at $p^2=-m_e^2$ is present, we would arrive at the identity

$$(p'-p)_a\ \bar u(p')\Gamma_a(p',p)\,u(p)\big|_{p^2=p'^2=-m_e^2}=0\ ,$$

which expresses gauge invariance of the scattering amplitude.

The problem is that since $\psi,\bar\psi$ are not gauge invariant, the propagator $G(p)$ does not have a simple pole at the electron mass shell. Instead, it has more complicated singularity, which also depends on the gauge choice. Formally, this singularity is interpreted as in terms of creation of non-physical longitudinal "photons", which is allowed when non-gauge invariant quantities are present. This problem will be considered later.

20 Lecture 20

Vertex Function

Now let us consider the general structure of the quantity

$$\bar{u}(p')\,\Gamma_a(p',p)\,u(p)|_{p^2=p'^2=-m_e^2}$$

i.e. in the case when both p^2 and p'^2 are both on-shell. The quantity $\Gamma_a(p,p)$ is a 4×4 matrix, and as such it can be written as a combination of 16 independent 4×4 matrices. However, it is possible to reduce to just two scalar function.

Indeed, any 4x4 matrix is a linear combination of identity matrix I, and matrices generated by γ_a; there are 16 $(= 1 + 4 + 6 + 4 + 1)$ independent matrices

$$I\,,\quad \gamma_a\,,\quad \sigma_{ab}\equiv\frac{i}{2}\left[\gamma_a,\gamma_b\right],\quad e^{abcd}\gamma_b\gamma_c\gamma_d\sim\gamma_a\gamma_5\,,\quad \gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4\,.$$

Symmetrized products of the γ-matrices are all reducible in view of

$$\gamma_a\gamma_b+\gamma_b\gamma_a=-2\delta_{ab}\,.$$

The vertex function $\Gamma_a(p',p)$ also is a vector with the components labelled by a, and it depends on two on-shell vectors p'_a and p_a. Therefore the most general form of Γ_a is

$$\bar{u}(p')\Gamma_a(p',p)u(p)=\bar{u}(p')\left[\gamma_a\;A(p',p)+(p'+p)_a\;B(p',p)+(p'-p)_a\;C(p',p)\right]u(p)\,,$$

where A, B, C are scalar functions of possible scalar quantities built from p and p'. Since we assume p, p' to be on shell, all are expressed in terms of masses and $k^2=(p-p')^2$. Possible terms like $p_a(p\gamma)$ are reduce to the above structures by using

$$(m-\gamma p)\,u(p)=(m-\gamma^\mu p_\mu)\,u(p)=0\,.$$

Terms involving γ_5 never appear from the diagrams (there is no source of the parity odd object e_{abcd} in diagram calculations).

The Ward- Takahashi identity put further restrictions. The terms with A and B satisfy WT identity automatically, but

$$k_a(p'-p)_a=k_ak_a\neq 0$$

in general, so $C=0$.

It is convenient to rewrite the expression in slightly different form, using the identity (Homework problem 11)

$$-\mathrm{i}\, k_a\, \bar{u}(p')\, \sigma_{ab}\, u(p) = -2m\, \bar{u}(p')\gamma_b u(p) + (p+p')_b\, \bar{u}(p')u(p)\ ,$$

where $\sigma_{ab} = \frac{\mathrm{i}}{2}\ (\gamma_a\gamma_b - \gamma_b\gamma_a)$. With this we can write

$$\bar{u}(p')\Gamma_a(p',p)\, u(p)\big|_{p^2=p'^2=-m_e^2} = \bar{u}(p')\ \left(\gamma_a\, F_1(k^2) + \frac{\mathrm{i}}{2m_e}\ k_b\sigma_{ba}\ F_2(k^2)\right) u(p)$$

where it is assumed that $p^2 = p'^2 = -m_e^2$.

The quantities F_1 and F_2 are known as *electromagnetic form-factors* of the electron. They concentrate most important data about the electromagnetic properties of the electron. In particular, its electrostatic characteristic, like charge and magnetic moment can be extracted as follows. Consider slow varying external vector potential, $A_a^{(ext)}(x)$. Its Fourier transform $\tilde{A}_a^{(ext)}(k)$ is concentrated at k close to 0. Then the quantity

$$\mathcal{M} = e\ \bar{u}(p')\, \Gamma_a(p',p)\, u(p)\ \tilde{A}_a^{(ext)}(k)$$

(taken at Minkowski ST momenta) coincides with the elastic scattering amplitude in the external potential. Continuation amounts in setting $k = (k_4, \mathbf{k}) \to (\mathrm{i}k_0, \mathbf{k})$ and substituting

$$\Gamma_a \to (\mathrm{i}\,\Gamma^0, \boldsymbol{\Gamma})\,; \quad A_a \to (\mathrm{i}\,A_0,\ \mathbf{A})\ .$$

$$-\Gamma_a\, A_a\ \to\ \Gamma^\mu\, A_\mu = \Gamma^0\, A^0 - \boldsymbol{\Gamma}\mathbf{A}$$

so that

$$\mathcal{M} = -e\ \left(\bar{u}\,\Gamma^0\, u\right)\ \tilde{A}_0^{(ext)}(k) + e\ \left(\bar{u}\,\boldsymbol{\Gamma}\, u\right)\ \tilde{\mathbf{A}}^{(ext)}(k)$$

We will assume that $\mathbf{p},\ \mathbf{p}' \sim \mathbf{k}$ are small.

Consider first the case $\mathbf{A}^{(ext)} = 0$, and concentrate attention on the potential $\tilde{A}_0^{(ext)}(k) = \tilde{\Phi}(k)$. As $\tilde{\Phi}(k)$ concentrates around $k = 0$ we have

$$\mathcal{M} = -e\ \left(\bar{u}\Gamma^0(k=0)\, u\right)\ \tilde{\Phi}(k) = -e\ \left(\bar{u}\,\gamma^0\, u\right)\ F_1(0)\ \tilde{\Phi}(k)\ .$$

When k goes to zero, $\bar{u}_{s'}\gamma^0 u_s \to 2m\,\xi^\dagger_{s'}\xi_s$ [25], and

$$\mathcal{M} = -e\,F_1(0)\,\Phi(k)\,\cdot\,2m\,\xi^\dagger_{s'}\xi_s\,.$$

This coincides with the Born scattering amplitude in the potential

$$U(x) = e\;F_1(0)\;\Phi(x)\,.$$

Therefore measured value of the electric charge

$$e_{\text{physical}} = e\;F_1(0)\,.$$

In what follows we will develop renormalized perturbation theory, where on of the normalization conditions requires that e coincides the physical value of electron charge, as measured in electrostatic experiments. In this scheme

$$F_1(0) = 1\;.$$

Since $F_1(0) = 1$ already in zeroth approximation, all higher corrections to $F_1(0)$ will vanish (i.e. the diagram contributions will be cancelled by suitably chosen counterterms, as we will discuss in more details shortly).

Now, set instead $A_0^{(ext)} = 0$. Then

$$\mathcal{M} = e\,\bar{u}\left[\gamma^j\;F_1(k^2) + \frac{\mathrm{i}}{2m}\;k^i\,\sigma^{ij}\;F_2(k^2)\right]\,u\;\bar{A}^{(ext)\,j}(k)\;, \tag{20.1}$$

where I have assumed that $k^0 = 0$ (by choice of the Lorentz frame) . At $\mathbf{k} \to 0$ the expression $\bar{u}[\ldots]u$ here vanishes. We need the term linear in $\mathbf{k}$. For small momenta $\mathbf{p}$, $\mathbf{p}'$, from we explicit expressions for $u(p)$, $\bar{u}(p)$

$$u(p) = \sqrt{m} \times \begin{pmatrix} (\cosh\beta/2 - \mathbf{n}\sigma\,\sinh\beta/2)\;\xi_s \\ (\cosh\beta/2 + \mathbf{n}\sigma\,\sinh\beta/2)\;\xi_s \end{pmatrix} \approx \sqrt{m} \times \begin{pmatrix} (1 - \frac{\mathbf{p}\cdot\sigma}{2m})\;\xi_s \\ (1 + \frac{\mathbf{p}\cdot\sigma}{2m})\;\xi_s \end{pmatrix}$$

[25] We remind that

$$u(p) = \begin{pmatrix} \sqrt{p\sigma}\,\xi_s \\ \sqrt{p\bar{\sigma}}\,\xi_s \end{pmatrix} = \sqrt{m}\,\begin{pmatrix} (\cosh(\frac{\beta}{2}) - \vec{n}\cdot\vec{\sigma}\,\sinh(\frac{\beta}{2}))\,\xi_s \\ (\cosh(\frac{\beta}{2}) + \vec{n}\cdot\vec{\sigma}\,\sinh(\frac{\beta}{2}))\,\xi_s \end{pmatrix}$$

and

$$\gamma^\mu\;:\;\;\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\;,\quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\;,\quad \beta = \tanh(v)\;,\quad \vec{n} = \vec{p}/|\vec{p}|\;.$$

$\gamma_4 = \mathrm{i}\,\gamma_0$, $\{\gamma_a\,,\,\gamma_b\} = -2\delta_{ab}$.

one finds

$$\bar{u}(p')\gamma^i u(p) = 2m\ \xi^\dagger_{s'}\left(\frac{\mathrm{p}'\cdot\sigma}{2m}\,\sigma^i + \sigma^i\,\frac{\mathrm{p}\cdot\sigma}{2m}\right)\xi_s \tag{20.2}$$

The identity

$$\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\,\epsilon^{ijk}\,\sigma^k \tag{20.3}$$

leads to two terms. The first term - from δ^{ij} - gives the following contribution to (20.1)

$$e\,F_1(0)\left[\mathrm{p}\cdot\tilde{\mathbf{A}}^{(ext)}(k) + \mathrm{p}'\cdot\tilde{\mathbf{A}}^{(ext)}(k)\right]\xi^\dagger_s\xi_s\,. \tag{20.4}$$

This is just the non-relativistic interaction with external vector potential $\mathbf{A}$ [26]. Again we need $F_1(0) = 1$ to give e interpretation of the physical charge. The second term, from $\mathrm{i}\,\epsilon^{ijk}\,\sigma^k$ in (20.3), gives the following the contribution to (20.2),

$$-\mathrm{i}\,\xi^\dagger_{s'}\,(\epsilon^{ijk}\,k^j\sigma^k)\,\xi_s\,.$$

Term of the same structure appears from the F_2 term in (20.1) after using

$$\bar{u}\,\sigma^{ij}u = 2m\,\epsilon^{jik}\,\xi^\dagger_s\,\sigma^k\,\xi_s\,,$$

and we finally find the magnetic interaction term

$$\mathcal{M} = (20.4) - \mathrm{i}\,e\left[F_1(0) + F_2(0)\right]\epsilon^{ijk}\,k^j\,(\xi^\dagger_{s'}\,\sigma^k\,\xi_s)\,\tilde{A}^{(ext)\,i}(k)\,.$$

But $\mathrm{i}\,\epsilon^{ijk}k^j\,\tilde{A}^{(ext)\,i}(k) = \tilde{B}^k(k)$, the fourier transform of the external magnetic field. Therefore

$$\mathcal{M} = 2me\left[F_1(0) + F_2(0)\right]\frac{\xi^\dagger_{s'}\,\sigma\,\xi_s}{2m}\,\mathbf{B}(k)\,.$$

From this we can identify the magnetic moment of the electron

$$\mu = \frac{e}{2m}\left[F_1(0) + F_2(0)\right]\xi^\dagger\sigma\,\xi\,.$$

Writing this in standard form

$$\mu = \frac{e}{2m}\,g\,\mathbf{S}\,, \qquad \mathbf{S} = \frac{\sigma}{2} - \text{ spin}\,.$$

[26]

$$\hat{H} = \frac{(\hat{\mathrm{p}} - e\mathbf{A})^2}{2m} = \frac{(\hat{\mathrm{p}})^2}{2m} - \frac{e}{m}\,(\hat{\mathrm{p}}\mathbf{A} - \mathbf{A}\hat{\mathrm{p}}) + \ldots\,.$$

Thus, the electron g-factor is expressed as follows

$$\frac{g}{2} = F_1(0) + F_2(0) = 1 + F_2(0) \; .$$

In the zeroth approximation, $F_2(0) = 0$ and $g = 2$. The contribution $2F_2(0)$ is called *anomalous magnetic moment* of the electron.

Electron Propagator

As I mentioned before, in QED the statement that the electron propagator has a simple pole at $p^2 = -m_e^2$ is not exactly valid. Generally, the momentum-space correlation function is singular on-shell, but instead of a simple pole it has a branching point singularity whose nature depends on the gauge condition. One can see this evaluating higher order corrections to the propagator. There are two sources of this complication. One is that the electron propagator is not gauge invariant, and usual argument based on the spectral decomposition do not work. The state

$$\psi(x)\,|0\rangle$$

is not gauge invariant and therefore, besides the physical states, i.e. electrons and transverse photons, it contains nonphysical "photons" with longitudinal polarizations $e^\mu(k) \sim k^\mu$. These non-physical states enter the intermediate state decomposition

$$\langle \psi(0)\bar{\psi}(x)\rangle = \sum_\alpha \langle 0|\psi(0)|\alpha\rangle\langle\alpha|\bar{\psi}(0)|0\rangle \; \mathrm{e}^{-E_\alpha|x_4|} \; \mathrm{e}^{\mathrm{i}\mathbf{p}_\alpha\cdot\mathbf{x}} \; .$$

Their contributions depend on the gauge fixing condition and can not be easily separated from the one-electron state.

It is interesting to see how this happens using solvable limit, the case when $e \to 0$ but $e^2\xi$ is not necessarily small. Considering our action $\mathcal{A}_{QED,\xi}$ let us make rescaling of A_a

$$e\,A_a = a_a$$

Then

$$\mathcal{A}_{QED,\xi} = \int \mathrm{d}^4x \left[\frac{1}{4e^2}\,(f_{ab})^2 + \frac{1}{2e^2\xi}\,(\partial_b a_b)^2 + \bar{\psi}(-\mathrm{i}\gamma_b\partial_b - \gamma_b a_b + m)\,\psi \; ; \quad f_{ab} = \partial_a a_b - \partial_b a_a \; .$$

This transformation is instructive in general because it shows that the parameter e^2 controls the fluctuations of f_{ab}. Let us denote

$$\lambda = e^2\,\xi$$

and take limit $e^2 \to 0$. In this limit fluctuations of f_{ab} are completely suppressed, so integration effectively takes into account only a_a of the form of pure gauge:

$$a_a = \partial_a \chi \ .$$

Neglecting f_{ab} we rewrite the action as

$$\int \mathrm{d}^4x \left[\frac{1}{2\lambda} \, (\partial^2 \chi)^2 + \bar{\psi} \big(- \mathrm{i} \, \gamma_a \partial_a - \gamma_a \partial_a \chi + m \big) \, \psi \right]$$

Now changing variables

$$\psi = \mathrm{e}^{\mathrm{i}\chi} \, \psi_0 \ , \qquad \bar{\psi}_0 = \mathrm{e}^{-\mathrm{i}\chi} \, \bar{\psi}_0$$

we get

$$\int \mathrm{d}^4x \left[\frac{1}{2\lambda} \, (\partial^2 \chi)^2 + \bar{\psi}_0 \big(- \mathrm{i} \, \gamma_a \partial_a + m \big) \, \psi_0 \right] .$$

In these variable the action splits into non-interacting parts. Therefore

$$\langle \psi(x) \bar{\psi}(x') \rangle |_{\xi = \frac{\lambda}{e^2}, \ e \to 0} = \langle \psi_0(x) \bar{\psi}_0(x') \rangle \ \langle \, \mathrm{e}^{\mathrm{i}\chi(x)} \ \mathrm{e}^{-\mathrm{i}\chi(x')} \rangle$$

The first term is the usual free propagator

$$\langle \psi_0(x) \bar{\psi}_0(x') \rangle = G_0(x - x')$$

The second term involves integration over χ. This scalar theory contains second-order derivatives, so this theory is certainly non-unitary. (We should not worry about this because contributions of χ are not going to show up in gauge invariant quantities). The χ-propagator is

$$\langle \chi(x) \chi(0) \rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \, \frac{\lambda}{k^4} \, \mathrm{e}^{\mathrm{i}kx} \ .$$

This integral contains bad infrared divergence. Assuming L is the size of the universe we get

$$\langle \chi(x) \chi(0) \rangle = \frac{\lambda}{16\pi^2} \, \log \left(\frac{L^2}{x^2} \right) = \mathcal{D}(x) \ . \tag{20.5}$$

Exercise: Let χ be a free boson field with the propagator,

$$\langle \chi(x) \, \chi(x') \rangle = \mathcal{D}(x - x') \ .$$

Show that

$$\langle e^{ix(x)} e^{-ix(0)} \rangle = \exp\{\mathcal{D}(x-x') - \mathcal{D}(0)\} .$$

Introducing short distance cutoff $\epsilon = \frac{1}{\Lambda}$ we obtain from (20.5)

$$\langle e^{ix(x)} e^{-ix(0)} \rangle = \left[\Lambda^2 x^2\right]^{-\frac{\lambda}{16\pi^2}}$$

The divergent factor $(\Lambda^2)^{-\frac{\lambda}{16\pi^2}}$ can be eliminated by renormalization. What produces the effect we are after is the power-like dependence on (x^2). We obtain

$$\langle\psi(x)\bar{\psi}(0)\rangle|_{\xi=\frac{\lambda}{e^2},\ e\to 0} = G_0(x)\ (x^2)^{\frac{\lambda}{16\pi^2}} .$$

It is not easy to Fourier transform this exactly, but it is possible to show that for $\lambda \neq 0$ the Fourier transform does not have a pole at $p^2 = -m_e^2$, instead it has branch-cut singularity

$$\propto \frac{(m+\gamma p)}{(m^2+p^2)^{\Delta}} , \qquad \Delta = 1 - \frac{\lambda}{16\pi^2} .$$

Another source of the problem is in the massless nature of the physical photons. Even if we take into account only physical states in the sum

$$\langle\psi(0)\bar{\psi}(x)\rangle = \sum_{\alpha}\langle 0|\psi(0)|\alpha\rangle\langle\alpha|\bar{\psi}(0)|0\rangle\ e^{-E_\alpha|x_4|}\ e^{\pm i\mathbf{p}_\alpha\cdot\mathbf{x}} ,$$

there is no gap between the one electron state

$$|(\mathbf{p},s)\rangle , \qquad E = \sqrt{\mathbf{p}^2+m_e^2}$$

and the states with one or more physical transverse photons added,

$$|(\mathbf{p},s),(\mathbf{k},\mathbf{e}_\alpha)\rangle , \qquad E = \sqrt{\mathbf{p}^2+m_e^2} + |\mathbf{k}| .$$

So

$$P_\mu P^\mu = E^2 - \mathbf{P}^2 = m_e^2 + 2\ \omega_p\ |\mathbf{k}| .$$

There is no lower bound on the $|\mathbf{k}|$, so $|\mathbf{k}|$ can be arbitrary small ("soft photons"), and as the result the eigenvalue m_e^2 is not an isolated point of the spectrum of $P_\mu P^\mu$ but rather a part (endpoint) of the continuous spectrum

$$P_\mu P^\mu \geq m_e^2 .$$

This is why instead of a pole at $p^2 = -m_e^2$ we expect to have a brunch cut

$$-p^2 \geq -m_e^2$$

in $G(p)$, even irrespective to the presence of the gauge non-invariant states.

Electron Propagator: Renormalization

To familiarize ourselves with the situation, and refresh our memories of the basics of renormalized perturbation theory, let us study the leading correction to the electron self energy. Denoting $G(p)$ the momentum-space two-point function $\langle\psi\bar{\psi}\rangle$, we have as usual

$$G^{-1}(p) = m + \gamma p + \Sigma(p)$$

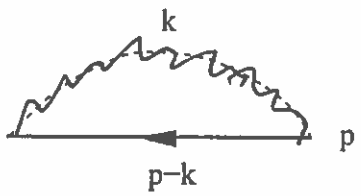
Here

$$\Sigma = -\Big[\text{ sum of one} - \text{particle irreducible diagrams}\Big]$$

This quantity is gauge dependent. For simplicity, we will do the calculations in the Feynman gauge $\xi = 1$, i.e.

$$D_{ab} = \frac{\delta_{ab}}{k^2}$$

(when ξ-dependence is needed, we have already seen that it is easy to collect contributions from the ξ-terms). The diagram which contributes to $\Sigma(p)$ is



Its contribution is

$$-e^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\gamma_a(-\gamma(p-k)+m)\gamma_a}{((p-k)^2+m^2)\,k^2}$$

We need to evaluate $\gamma_a\gamma_a = -4$,

$$\gamma_a\gamma_b\gamma_a = -\gamma_b\gamma_a\gamma_a - 2\delta_{ab}\,\gamma_a = (4-2)\,\gamma_b = 2\gamma_b$$

Therefore[27]

$$+e^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \int_0^1 \mathrm{d}x\, \frac{2\gamma(p+k)+4m}{(k^2+2pk\,x+p^2x+m^2x)^2} = (k+pkx=q) =$$
$$\frac{e^2}{(2\pi)^4} \int \mathrm{d}^4q \int_0^1 \mathrm{d}x \frac{2\gamma(q+p(1-x))+4m}{(q^2+p^2x(1-x)+m^2x)^2}$$

[27] Feynman parametrization: $\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}x}{(xA+(1-x)B)^2}$

The rest of calculation is done as usual. Integral with γq vanishes as the result of angle integration, and the integration over q reduces to[28]

$$\int \frac{d^4q}{(2\pi)^4}\,\frac{1}{(q^2+A)^2} = \int_0^\infty \frac{q^2dq^2}{16\pi^2}(q^2+A)^2 = \frac{1}{16\pi^2}\left[\log\frac{\Lambda^2}{A} - 1\right]$$

and we obtain

$$+\frac{e^2}{8\pi^2}\int_0^1 \mathrm{d}x\,\left[\gamma p\,(1-x)+2m\right]\left[\log\left(\frac{\Lambda^2}{m^2}\right) - 1 - \log\left(x(1-x)\,\frac{p^2}{m^2}+x\right)\right]$$

where I have introduced the ultraviolet cutoff Λ. We see the divergent term

$$\frac{e^2}{8\pi^2}\left(\frac{1}{2}\,\gamma p + 2m\right)\left(\log\left(\frac{\Lambda^2}{m^2}\right) - 1\right).$$

We already know how to handle this. The divergence appears if we do unrenormalized perturbation theory, where $m = m_0$ stands for the "bare mass", i.e. the mass parameter in the action, and we calculate $G = \langle\,\psi\,\bar\psi\,\rangle$ with $\psi = \psi_0$ is unrenormalized field. Instead we can transform to renormalized perturbation theory. Introduce the renormalized field

$$\psi_R = Z_\psi^{-\frac{1}{2}}\,\psi_0\,, \qquad \bar\psi_R = Z_\psi^{-\frac{1}{2}}\,\bar\psi_0\,.$$

In QED the constant Z_ψ is conventionally called Z_2

$$Z_\psi = Z_2\,.$$

We then write the Dirac part of the action as

$$\mathcal{A}_{Dirac} = \int d^4x\,\bar\psi_0\,(-i\gamma\partial + m_0)\,\psi_0 \;=\; \int \mathrm{d}^4x\,\left[\bar\psi_R(-\mathrm{i}\gamma\partial + m_e)\psi_R + \delta_2\,\bar\psi_R(-\mathrm{i}\gamma\partial)\psi_R + \delta_m\,\bar\psi_R\psi_R\right]$$

where of course

$$\delta_2 = Z_2 - 1\,, \qquad \delta_m = m_0 - Z_2 m_e\,.$$

This gives the counterterms, and, correspondingly, the diagrams must be supplemented by the counterterm vertices

$$- < - \otimes - < - \;=\; -(\delta_m + \delta_2\,\gamma p)\,.$$

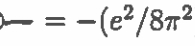
The counterterms δ_m, δ_2 depend on the coupling constants e^2, so that

[28] $\Omega_d = 2\pi^{d/2}/\Gamma(d/2) \;\to\; 2\pi^2$

$$\delta_2 = \frac{e^2}{8\pi^2}\,\delta_2^{(2)} + \dots\,, \qquad \delta_m = \frac{e^2}{8\pi^2}\,\delta_m^{(2)} + \dots\,.$$

Then in order e^2

$\Sigma(p)$ =  + 

with  $= -(e^2/8\pi^2)(\delta_2^{(2)}\,\gamma p + \delta_m^{(2)}))$.

The values of $\delta_2^{(2)}$ and $\delta_m^{(2)}$ must be chosen to satisfy the normalization conditions. One can demand that singularity in $G_R(p)$ is located at $p^2 = -m_e^2$, with additional condition fixing the normalization of the fields ψ, $\bar{\psi}$.

As we already know, $G_R(p)$ not a pole but branching point at $p^2 = -m_e^2$. Calculating the integrals over x, one obtains ($m = m_e$!)[29]

$$\Sigma(p) = \frac{e^2}{8\pi^2}\left\{ \gamma p \left[\underbrace{\frac{1}{2}\log\left(\frac{\Lambda^2}{m^2}\right) + 1 + \delta_2^{(2)}}_{a} + \frac{1}{2}\left(\frac{m^4}{p^4} - 1\right)\,\log\left(1 + \frac{p^2}{m^2}\right) - \frac{1}{2}\left(1 + \frac{m^2}{p^2}\right)\right]\right.$$

$$\left. + \underbrace{\left[\,2m\log\left(\frac{\Lambda^2}{m^2}\right) + m + \delta_m^{(2)}\right]}_{b} - 2m\left(1 + \frac{m^2}{p^2}\right)\,\log\left(1 + \frac{p^2}{m^2}\right)\right\}.$$

It is easy to adjust the constants a, b aso that $G(p)$ has singularity at $p^2 = -m_e^2$, i.e.

$$G^{-1}(p)|_{\gamma p + m = 0} = 0\,.$$

One has to chose

$$b = m\,a\,.$$

However, this singularity is not a simple pole, as the terms with

$$\log\left(1 + \frac{p^2}{m^2}\right)$$

indicate. Rather, it is a weak brunching point. The reason for that is very physical. As we discussed, the electron one-particle state $|p, s\rangle$ is a part of continuum states of the form $e(p) + \gamma(k_1) + \dots + \gamma(k_N)$ with N "soft photons", $k_i \to 0$. Therefore we expect that the branch point is not an artifact of wrong perturbation theory, and we can not make it into a pole by any renormalization. The nature of the

[29] Integrate$[-(1-x)\,\log\left(x(1-x)\,a^2+x\right), \{x, 0, 1\}, \text{Assumptions} \to a > 0] = 1+\frac{1}{2}\left(\frac{1}{a^2}-1\right)\,\log(1+a^2)-\frac{1}{2a^2}$. Integrate$[-\log\left(x(1-x)\,a^2+x\right), \{x, 0, 1\}, \text{Assumptions} \to a > 0] = 2-\left(\frac{1}{a^2}+1\right)\,\log(1+a^2)$.

singularity is not known in advance (in fact, it is gauge dependent) and any standard normalization condition which refers to residue does not make sense.

To circumvent this difficulty (rather postpone its analysis!) it is conventional to temporarily add a mass μ to photon. If we do that, the singularity of $G(p)$ at $p^2 = -m_e^2$ would become a simple pole. So let us temporarily introduce such mass, replacing

$$D_{ab}(k) = \frac{\delta_{ab}}{k^2} \to \frac{\delta_{ab}}{k^2 + \mu^2}$$

We just want to make sure that all physical (gauge-invariant) quantities, like S-matrix elements have well-defined limit $\mu^2 \to 0$. With this we obtain

$$\Sigma(p) = \frac{e^2}{16\pi^2} \left\{ \gamma p \left(\frac{1}{2} \log\left(\frac{\Lambda^2}{m^2}\right) - \frac{1}{2} + \delta_2^{(2)} \right) + \left(2m \log\left(\frac{\Lambda^2}{m^2}\right) - 2m + \delta_m^{(2)} \right) \right.$$
$$\left. - \int_0^1 \mathrm{d}x \, \left[\gamma p (1-x) - 2m \right] \left[\log\left(x(1-x)\,\frac{p^2}{m^2} + x + (1-x)\,\frac{\mu^2}{m^2} \right] \right\}$$

The propagator now has a simple pole, because at no value of x the argument of the logarithm turns to zero. Now the renormalization conditions take the same form as we had in the scalar theory[30]

$$\Sigma\big|_{\gamma p = -m} = -m\, \Sigma_1(-m^2) + \Sigma_2(-m^2) = 0 \qquad \left((\gamma p)^2 = -p^2 = m^2 \right)$$
$$\frac{\partial \Sigma}{\partial(\gamma p)}\bigg|_{\gamma p = -m} = \Sigma_1(-m^2) - 2m^2\, \Sigma_1'(-m^2) + 2m\, \Sigma_2'(-m^2) = 0$$

where $\Sigma_{1,2}(p^2)$ are defined as

$$\Sigma(p) = \gamma p\, \Sigma_1(p^2) + \Sigma_2(p^2)\ .$$

From these equations one determine $\delta_2^{(2)}$, $\delta_m^{(2)}$. Substituting these back to $\Sigma(p)$ above cancels $\log(\Lambda^2)$ terms. The dependence on μ^2 however remains. We tolerate it here only because $G(p)$ itself is not something which is directly measurable.

For future references I quote the result for the counterterm $\delta_2^{(2)}$

$$\delta_2^{(2)} = \frac{e^2}{8\pi^2} \int_0^1 \mathrm{d}x \left[(x-1) \log\left(\frac{\Lambda^2(1-x)}{m^2}\right) + (1-x) \log\left(x^2 + \frac{\mu^2}{m^2}\,(1-x) \right) \right.$$
$$\left. + \frac{2x(1-x^2)\,m^2}{x^2 m^2 + (1-x)\,\mu^2} \right]$$

[30] Since γp is the only matrix quantity appearing in this calculation, we can treat it as a number, $\Sigma(p) = \Sigma(\gamma p)$. Also, since $(\gamma p)^2 = -p^2$ we can write $\Sigma(\gamma p) = (\gamma p)\, \Sigma_1(p^2) + \Sigma_2(p^2)$, and $\Sigma_{1,2}(p^2) = \Sigma_{1,2}(-(\gamma p)^2)$

21 Lecture 21

Vertex Function: Renormalization

Let us consider again the vertex function

$$\Gamma_a(p', p) \ .$$

As we remember, this is defined in terms of the correlation function

$$\langle \, J_a(y) \ \psi(x)\bar{\psi}(x') \, \rangle$$

We want to develop renormalized theory, so we rather consider the correlation functions which involve renormalized fields $\psi = \psi_R$, $\bar{\psi} = \bar{\psi}_R$ instead of the"bare" fields ψ_0, $\bar{\psi}_0$ which appear in the bare form action. The vertex function is defined in terms of the correlation function involving the current $J_a(x)$. The current is a composite field

$$J_a(x) = \bar{\psi}(x)\gamma_a\psi(x) \ .$$

In QFT, defining composite fields often requires specific renormalizations. We have seen that in φ^4 theory. When defining a composite field, e.g. φ^2, it was not sufficient to just replace $\varphi_0 \to \varphi_R$ to absorb all UV divergences which appear in the correlation functions. We needed to introduce additional renormalization constant Z_{φ^2}, and define $(\varphi^2)_R = Z_{\varphi^2} \ \varphi^2$.

In a similar manner, renormalization of the current $J_a(x)$ requires special renormalization constant, which is conventionally called Z_1, namely

$$(J_a)_R(x) = Z_1 \ \bar{\psi}_R\gamma_a\psi_R \ , \quad \psi_R = Z_2^{-\frac{1}{2}} \ \psi$$

The factor Z_1 will absorbs specific divergencies which appear in $\langle (J_a)_R(x) \ \ldots \rangle$. As in φ^4 theory, we can write

$$Z_1 = 1 + \delta_1(e^2)$$

where

$$\delta_1 = \frac{e^2}{8\pi^2} \, \delta_1^{(2)} + O(e^4) \ .$$

Correspondingly, the renormalized perturbation theory for Γ_a involves the counterterm vertex

$(J_a)_R$: γ_a  + $\gamma_a \, \delta_1$ 

The counterterm δ_1, i.e. the renormalization constant Z_1, must be determined from certain renormalization condition, which says exactly how the field $(J_a)_R$ is normalized.

To understand the situation let us recall our derivation of the Ward-Takakhashi identity. We made the phase transformation of field variables in the paths integral

$$\psi_0 \to \mathrm{e}^{\mathrm{i}\theta(x)}\ \psi_0\ , \qquad \bar{\psi}_0 \to \mathrm{e}^{-\mathrm{i}\theta(x)}\ \bar{\psi}_0$$

and defined the current by the variation of the full action under such transformation,

$$\mathcal{A} \to \mathcal{A} + \int \mathrm{d}^4x\ J_a(x)\ \partial_a\theta(x)\,.$$

Note that this definition includes the canonical normalization of $J_a(x)$, under which $\psi(x)$ ($\bar{\psi}(x)$) creates (annihilates) a unit of charge, i.e. the charge operator

$$Q = \int d^3\mathbf{x}\, J^0(\mathbf{x})\,,$$

satisfies[31]

$$[Q, \psi(x)] = \psi(x)\,, \qquad [Q, \bar{\psi}(x)] = -\bar{\psi}(x)\,.$$

Note that these relations are unsensitive to normalization of ψ, i.e. they apply equally well to $\psi = \psi_R$ and ψ_0. It is natural to define the renormalized current according to this normalization, i.e.

$$(J_a)_R(x) = J_a(x)\,.,$$

From the form of the action, we have

$$J_a(x) = (J_a)_R(x) = \bar{\psi}_0(x)\gamma_a\psi_0(x) = Z_2\,\bar{\psi}_R(x)\gamma_a\psi_R(x)$$

We see that this normalization of the current corresponds to the choice

$$Z_1 = Z_2\,.$$

[31] In terms of correlation functions, this normalization corresponds to coefficient 1 in the r.h.s. of the Ward identity

$$\partial_a\langle\, J_a(x)\,\psi(y_1)\bar{\psi}(y_2)\rangle = i\,\left[\delta^{(4)}(x-y_1) - \delta^{(4)}(x-y_2)\right]\,\langle\psi(y_1)\bar{\psi}(y_2)\rangle$$

.

Alternatively (in fact, equivalently), this relation can be derived directly from the WT identity for the vertex function,

$$(p'-p)_a\,\Gamma_a(p',p) = G^{-1}(p) - G^{-1}(p')\ .$$

Taking the limit

$$k_a \equiv (p'-p)_a \to 0\,.$$

one finds

$$\Gamma_a(p,p) = +\frac{\partial}{\partial p_a}\,G^{-1}(p)\ .$$

Note that these relations are not sensitive to normalization of $\psi, \bar{\psi}$. In particular, we can take

$$G(p) = G_R(p) = f.t.\,\langle\,\psi_R(x)\bar{\psi}_R(x')\,\rangle\,, \qquad G_R(p')\Gamma_a^{(R)}(p',p)G_R(p) = f.t.\langle\,J_a(y)\psi_R(x)\bar{\psi}_R(x')\,\rangle$$

Write $G_R^{-1}(p)$ as

$$G_R^{-1}(p) = m + \gamma p + \gamma p\ \Sigma_1(p^2) + \Sigma_2(p^2)\ .$$

This gives

$$\Gamma_a^{(R)}(p,p) = \gamma_a + \gamma_a\ \Sigma_1(p^2) + 2p_a\,(\gamma p)\ \Sigma_1'(p^2) + 2p_a\ \Sigma_2'(p^2)$$

Sandwiching this between $\bar{u}(p)$ and $u(p)$, and taking into account[32]

$$\bar{u}(p)\gamma_a u(p) = +\frac{p_a}{m}\ \bar{u}(p)u(p)\,,$$

and that $G_R(p)$ is normalized in a standard way, with the simple pole with canonical residue (before sending μ^2 to zero), i.e.

$$\Sigma_1(-m^2) - 2m^2\ \Sigma_1'(-m^2) + 2m\ \Sigma_2'(-m^2) = 0$$

we find

$$\bar{u}(p')\Gamma_a^{(R)}(p,p)u(p)|_{p^2=-m^2} = \bar{u}(p')\gamma_a u(p)\,.$$

As

$$\Gamma_a^{(R)}(p',p) = \gamma_a\ F_1(k^2) + \frac{\mathrm{i}}{2m}\ k_a\sigma_{ba}\ F_2(k^2)$$

it follows

$$F_1(0) = 1\ .$$

[32]Recall the Gordon identity: $\mathrm{i}\,k_a\,\bar{u}(p')\sigma_{ab}\,u(p) = -2m\ \bar{u}(p')\gamma_b\,u(p) + (p'+p)_b\,\bar{u}(p')\,u(p)$.

Equivalently, if we want to normalize the composite field

$$J_a = Z_1 \, \bar{\psi}_R \gamma_a \psi_R$$

so that each electron contributes $+1$ to

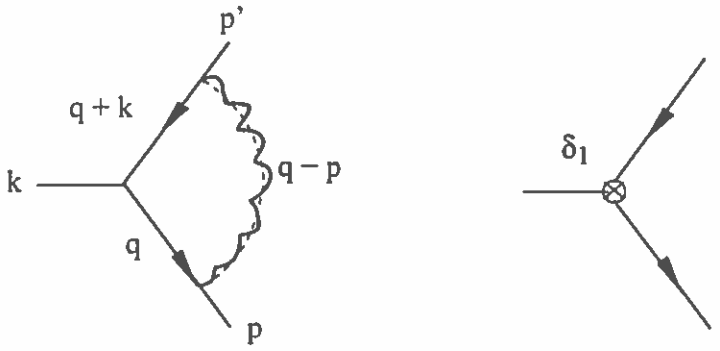
$$Q = \int \mathrm{d}^3\vec{x} \; J_0(x)$$

we should chose

$$Z_1 = Z_2 \quad \leftrightarrow \quad F_1(0) = 1 \; .$$

The above relation $Z_1 = Z_2$ itself is sometimes referred to as the WT identity.

Now it is time to do some calculations. Consider the e^2 contribution to $\Gamma_a^{(R)}$ which I'll denote simply Γ_a. There are two diagrams to this order



The second diagram represent the counterterm

$$\delta_1 = Z_1 - 1$$

in the order e^2. For the first diagram we have

$$e^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, \frac{\bar{u}(p')\gamma_b(-\gamma q' + m)\gamma_a(-\gamma q + m)\gamma_b u(p)}{(q^2 + m^2)((q')^2 + m^2)((q-p)^2 + \mu^2)}$$

Again, $\gamma_b\gamma_a\gamma_b = 2\gamma_a$, $\gamma_a\gamma_a = -4$. With some γ-matrix algebra, which involves identities in Problem 11, one finds

$$4\,e^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, \frac{\bar{u}(p') \left((\gamma q)\, \gamma_a \, (\gamma q') - 2m \, (q + q')_a + m^2 \gamma_a \right) u(p)}{(q^2 + m^2)((q')^2 + m^2)((q-p)^2 + \mu^2)}$$

To handle the integral we need little generalization of the Feynman parameterization we used before

$$\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}x}{(xA+(1-x)B)^2} = \int_0^1 \mathrm{d}x\mathrm{d}y\, \frac{\delta(x+y-1)}{(xA+yB)^2} \,.$$

This is a particular case of the general identity

$$\frac{1}{A_1 A_2 \ldots A_n} = (n-1)! \int_0^1 \mathrm{d}x_1 \ldots \mathrm{d}x_n\, \frac{\delta(x_1+\ldots+x_n-1)}{(x_1A_1+\ldots+x_nA_n)^n}$$

which can be proven by induction with the use

$$\frac{1}{A^n B} = \int \frac{\mathrm{d}x\; x^{n-1}}{(xA+(1-x)B)^{n+1}} \,.$$

We need

$$\frac{1}{ABC} = 2\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\, \frac{\delta(x+y+z-1)}{(xA+yB+zC)^3} \,.$$

Also, to keep the calculation somewhat symmetrical with respect p and p' we will use

$$Q = q - p \,, \quad Q = q' - p' \,.$$

Using these we obtain[33]

$$4e^2 \int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\; \delta(x+y+z-1) \int \frac{\mathrm{d}^4Q}{(2\pi)^4} \times \\ \frac{\bar{u}(p')\left((\gamma(Q+p))\,\gamma_a\,(\gamma\,(Q+p')) - 2m(2Q+p+p')_a + m^2\gamma_a\right)u(p)}{(Q^2+2Q(px+p'y)+z\mu^2)^3} \,,$$

where I made use of $p^2 = (p')^2 = -m^2$. The object in the denominator can be written as[34]

$$\left(P^2 + k^2xy + m^2(1-z)^2 + z\mu^2\right)^3$$

where

$$P = Q + px + p'y \,, \quad (k = p' - p) \,.$$

[33] $x\left((Q+p)^2+m^2\right) + y\left((Q+p')^2+m^2\right) + z\,(Q^2+\mu^2) = Q^2 + 2Q(px+p'y) + z\;\mu^2$ as $p^2 = (p')^2 = -m^2$.

[34] $P^2 = Q^2 + 2Q(px+p'y) + (px+p'y)^2 = Q^2 + 2Q(px+p'y) - m^2\underbrace{(x^2+y^2+2xy)}_{(1-z)^2} + \underbrace{(2(pp')+2m^2)}_{-(p-p')^2}\,xy$.

We now assume that P as the integration variable. The numerator becomes

$$\bar{u}(p')\Big[\big((\gamma P)+(1-x)(\gamma p)+my\big)\gamma_a\big((\gamma P)+(1-y)(\gamma p')+mx\big)$$
$$-2m\big(2P+(1-2x)p+(1-2y)p'\big)_a+m^2\gamma_a\Big]\,u(p)$$

where again $(\gamma p)u(p)=-mu(p)$, $\bar{u}(p')(\gamma p')=-m\,\bar{u}(p')$ are used. As the rest of the integrand depends on P^2 only, the term linear in P disappear, and numerator simplifies as

$$\bar{u}(p')\Big[(\gamma P)\gamma_a(\gamma P)+(1-x)(1-y)(\gamma p)\gamma_a(\gamma p')+m\,x(1-x)\,(\gamma p)\gamma_a+m\,y(1-y)\,\gamma_a(\gamma p')$$
$$-2m\big((1-2x)p_a+(1-2y)p'_a\big)+m^2(1+xy)\,\gamma_a\Big]\,u(p)\ .$$

While the denominator is

$$(P^2+\Delta)^3\ ,\quad \Delta=k^2xy+m^2(1-z)^2+2\mu^2\ ,$$

the numerator can be further simplified by reshuffling the γ-matrices.

Exercise 2. Show that this expression can be transformed to[35]

$$\bar{u}(p')\Big[(\gamma P)\gamma_a(\gamma P)+\big(m^2\,(1-2z-z^2)-(1-x)(1-y)k^2\big)\,\gamma_a-m\,z(1-z)\,(p+p')_a$$
$$-m\,(x-y)\,(2-z)\,(p'-p)_a\Big]\,u(p)$$

(Use $\gamma p\ u=-mu$, $\bar{u}(p')(\gamma p')=-m\,\bar{u}(p')$, and $x+y+z=1$.)

Evaluation of the P integral is now straightforward. Angular part of the P-integration replaces

$$P_aP_b\ \to\ \frac{\delta_{ab}}{4}\,P^2\ ,$$

---

[35] $\bar{u}(p')\,(p\gamma)\gamma_a(p'\gamma)\,u(p)=\big(-m^2+2(pp')\big)\,u(p')\gamma_a u(p)+2m(p'+p)_a\,\bar{u}(p')u(p)$. Thus the terms $\propto\gamma_a$:

$$(1-x)(1-y)(2pp'-m^2)+m^2x(1-x)+m^2y(1-y)+m^2(1+xy)=-(1-x)(1-y)\,k^2+m^2(1-2z-z^2)\ .$$

Also the terms

$$\propto(p'+p)_a\ :\qquad 2m(1-x)(1-y)+m\,(y(1-y)+x(1-x))-2mz$$

and the terms $\propto(p'-p)_a$: $m\,(x(1-x)-y(1-y)-2(x-y))$

and we obtain for the diagram (I) contribution to the form-factors[36]

$$F_1^{(I)} = \frac{e^2}{8\pi^2}\int_0^1 \mathrm{d}X\left[\log\left(\frac{z\Lambda^2}{\Delta}\right)+\frac{(1-4z+z^2)m^2-(1-x)(1-y)k^2}{\Delta}\right]$$
$$F_2^{(I)} = \frac{e^2}{8\pi^2}\int_0^1 \mathrm{d}X\,\frac{2m^2z(1-z)}{\Delta}\,,$$

where $\mathrm{d}X = \mathrm{d}x\mathrm{d}y\mathrm{d}z\;\delta(x+y+z-1)$, and $\Delta = k^2xy+m^2(1-z)^2+2\mu^2$. Note that the term containing $(p'-p)_a$ will vanish as result of $\mathrm{d}X$ integration $(x\leftrightarrow y)$, in agreement with WTI. Before doing this further, let us take a look at the second diagram, (II)

$$\delta_1 = \frac{e^2}{8\pi^2}\,\delta_1^{(2)}+\ldots$$

To the order e^2 its contribution is just

$$\frac{e^2}{8\pi^2}\,\delta_1^{(2)}\,\gamma_a$$

leading to

$$F_1^{(II)} = \frac{e^2}{8\pi^2}\,\delta_1^{(2)}\,,\qquad F_2^{(II)} = 0\,.$$

The coefficient $\delta_1^{(2)}$ has to be determined from the normalization condition. As we discussed before, if we want to interpret the current

$$j_a(x) = Z_1\,\bar\psi_R\gamma_a\psi_R = (1+\delta_1)\,\bar\psi_R\gamma_a\psi_R$$

as the physical current (such that each electron contribute $+1$ to $Q=\int \mathrm{d}^3\vec{x}\;j_0(\vec{x})$) the constant Z_1 has to be chosen such that

$$F_1(0) = 1$$

which, in particular, implies $Z_1 = Z_2$. As

$$F_1 = 1+(F_1^{(I+II)})+O(e^4)$$

we must have

$$(F_1^{(I+II)})(0) = 0$$

[36] $\int\frac{\mathrm{d}^4P\;(\gamma P)\gamma_a(\gamma P)}{(P^2+\Delta)^3} = \frac{\gamma_b\gamma_a\gamma_b}{4}\int\frac{\mathrm{d}^4P\;P^2}{(P^2+\Delta)^3} = \frac{\gamma_a}{2}\int\frac{\mathrm{d}^4P\;P^2}{(P^2+\Delta)^3} = \frac{\gamma_a}{2}\,2\pi^2\Big[\int\frac{\mathrm{d}PP^3}{(P^2+\Delta)^2}-\Delta\int\frac{\mathrm{d}PP^3}{(P^2+\Delta)^3}\Big]$

This determines the constant $\delta_1^{(2)}$:

$$\delta_1^{(2)} = -\int_0^1 \mathrm{d}X \left[\log\left(\frac{z\Lambda^2}{\Delta_0}\right) + \frac{(1-4z+z^2)m^2}{\Delta_0} \right]$$

where $\Delta_0 = m^2(1-z)^2 + z\mu^2$. The integrand here depends on x, y only through δ-function, and we can integrate out

$$\int_0^1 \mathrm{d}x \delta(x+y+z-1) = \begin{cases} 1 & \text{if } y+z<1 \\ 0 & \text{if } y+z>1 \end{cases}$$

so

$$\int_0^1 \mathrm{d}x\mathrm{d}y\ \delta(x+y+z-1) = \int_0^{1-z} \mathrm{d}y = 1-z\ ,$$

and

$$\begin{aligned} \delta_1^{(2)} &= \int_0^1 \mathrm{d}z\ (z-1) \left[\log\left(\frac{z\Lambda^2}{\Delta_0}\right) + \frac{(1-4z+z^2)m^2}{\Delta_0} \right] \\ &= -\frac{1}{2}\log\left(\frac{\Lambda^2}{m^2}\right) + \int_0^1 \mathrm{d}x \left[x\ \log\left(\frac{m^2x^2+\mu^2(1-x)}{m^2(1-x)}\right) - \frac{x(x^2+2x-1)m^2}{m^2x^2+\mu^2(1-x)} \right], \end{aligned}$$

where I have changed $z = 1-x$. Using the identity

$$\int_0^1 \mathrm{d}x\,(1-2x)\ \log\left(\frac{m^2x^2+\mu^2(1-x)}{m^2(1-x)}\right) = \int_0^1 \mathrm{d}x\,\frac{x^2(x-2)}{m^2x^2+\mu^2(1-x)}$$

which can be proven by integrating by parts, one can rewrite this as

$$\delta_1^{(2)} = -\frac{1}{2}\log\left(\frac{\Lambda^2}{m^2}\right) + \int_0^1 \mathrm{d}x \left[(1-x)\ \log\left(\frac{m^2x^2+\mu^2(1-x)}{m^2(1-x)}\right) + \frac{2x(1-x^2)m^2}{m^2x^2+\mu^2(1-x)} \right] = \delta_2^{(2)}\ ,$$

which is exactly $\delta_2^{(2)}$ we calculated the last time from renormalization condition for $G(p)$. This explicitely agrees with W-T identity

$$Z_1 = Z_2\ .$$

With the account of this counterterm the Λ-dependence of $F_1^{(2)}(k^2)$ disappears, but it still depends on the "photon mass" μ^2. We will try to isolate and interprete this dependence little later.

Now let us take a look at the second form-factor, $F_2^{(2)}$

$$F_2^{(2)} = \frac{e^2}{8\pi^2} \int_0^1 dxdydz\ \delta(x+y+z-1)\ \frac{2m^2 z(1-z)}{\Delta} .$$

This quantity does not contain ultraviolet divergence. Moreover, the integral over the Feynman parameter remain finite if we set

$$\mu^2 = 0 .$$

We thus have

$$F_2^{(2)}(k^2) = \frac{e^2}{8\pi^2} \int_0^1 dxdydz\ \delta(x+y+z-1)\ \frac{2m^2 z(1-z)}{k^2 xy + m^2(1-z)^2} .$$

As we discussed before, the quantity $F_2(0)$ is related to the magnetic moment of the electron. We obtain

$$F_2^{(2)}(0) = \frac{e^2}{8\pi^2} \int_0^1 dz\ \frac{2m^2 z(1-z)^2}{m^2(1-z)^2} = \frac{e^2}{8\pi^2}\ 2 \int_0^1 dz\ z = \frac{e^2}{8\pi^2} .$$

Remember, the factor g in

$$\mu_e = g\ \frac{e}{2m}\ \mathbf{S} , \qquad \mathbf{S} = \frac{\sigma}{2} - \quad \text{spin}$$

was related to F_1, F_2:

$$\frac{g}{2} = F_1(0) + F_2(0) = 1 + F_2(0) .$$

Therefore

$$\frac{g-2}{2} = \frac{e^2}{8\pi^2} + O(e^4) .$$

The quantity

$$\frac{e^2}{4\pi} = \alpha$$

is known as the "fine-structure constant", its numerical value is

$$\alpha = \frac{1}{137.035989\ldots} ; \qquad \frac{g-2}{2} = \frac{\alpha}{2\pi} + O(\alpha^2) \quad \text{(Schwinger, 1948)}$$

Numerically

$$\frac{\alpha}{2\pi} = 0.0011614182...$$

This result is in reasonable agreement with the experimental value

$$\left(\frac{g-2}{2}\right)_{exp} = 0.001159652188(4)\ldots$$

The calculations of the form-factors in renormalized perturbation theory can be continued to include higher corrections in α. For example, the next correction involves the diagrams like shown in this Figure

The calculation was dome by Peterman and Sommerfeld in $50th$

$$\frac{g-2}{2} = \frac{\alpha}{2\pi} + a_4 \left(\frac{\alpha}{\pi}\right)^2 + O(\alpha^4)$$

where

$$a_4 = \frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ \log 2 + \frac{3}{4}\,\zeta(3) = -0.328479\ldots \quad \left(\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}\right).$$

The coefficient a_6 in

$$\frac{g-2}{2} = \frac{\alpha}{2\pi} + a_4 \left(\frac{\alpha}{\pi}\right)^2 + a_6 \left(\frac{\alpha}{\pi}\right)^6 + a_8 \left(\frac{\alpha}{\pi}\right)^8 + O(\alpha^{10})$$

still can not be found analytically. Contemporary numerical calculation yields

$$a_6 = 1.176\ldots$$

combining these results one finds

$$\frac{g-2}{2} = 0.00115965216\ldots$$

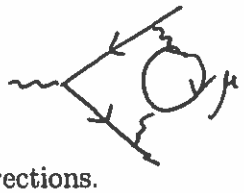
Agreement with the above experimental value is quite remarkable.

Calculation of the anomalous magnetic moment of electron represents one of the examples of remarkable success of QED. Other examples is the radiational correction to the energy levels of Hydrogen atom ("Lamb shift").

In principle, the theoretical calculation can be improved, by including

$$a_8 \approx 1.4$$

(which requires calculating of $\sim$ 1000 diagrams!) At this level of accuracy one also has to include corrections for the QED theory itself, like contribution of μ-meson in α^2 term



as well as hadronic corrections.

This success is definitely related to the fact that the fine-structure constant α is relatively small. Higher order corrections are small. More importantly, it is possible to show that the expansion is not convergent- the series is asymptotic. However, since alpha is small, few leading terms give very accurate approximations.

22 Lecture 22

Vertex Function: Infrared Divergence

Let us now go back to the form-factor $F_1(k^2)$. With the account of $\delta_1^{(2)}$ (which amounts to replacing

$$F_1(k^2) \to F_1(k^2) - F_1(0)$$

we have

$$F_1^{(2)} = \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\ \delta(x+y+z-1)\left[\log\left(\frac{(1-z)^2m^2+\not{\mu}^2 z}{(1-z)^2m^2+\not{\mu}^2 z+xyk^2}\right)\right.$$
$$\left. + \frac{(1-4z+z^2)m^2-(1-x)(1-y)k^2}{xyk^2+(1-z)^2m^2+z\mu^2} - \frac{(1-4z+z^2)m^2}{(1-z)^2m^2+z\mu^2}\right]$$

The first term, with the logarithm, is regular at $\mu^2 \to 0$, so we can set there $\mu^2 = 0$. Let us concentrate on the second and third terms. Here we can not just set $\mu^2 = 0$. It is easy to see that if we do that the integrals over the Feynman parameters diverge as

$$z \to 1\ , \quad x, y \to 0\ .$$

The singularity at $\mu^2 \to 0$ comes from this domain. If we want to extract this singularity, we can set

$$x = y = 0,\ z = 1$$

in numerators, and $z = 1$ in $z\mu^2$ term in denominators. Denoting the singular part $F_1^{(2)}$ as $F_1^{(sing)}$ we can write

$$F_1^{(sing)}(k^2|\mu^2) = \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\delta(x+y+z-1)\left[\frac{2m^2}{(1-z)^2m^2+\mu^2} - \frac{2m^2+k^2}{xyk^2+(1-z)^2m^2+\mu^2}\right]$$

Again, integrating over x just eliminates the δ-function

$$\int_0^1 \mathrm{d}x\,\delta(x+y+z-1) = \theta(1-y-z)\ .$$

We also change the variables as $1-z = w$, $y = w\xi$ [37]:

$$F_1^{(sing)}(k^2|\mu^2) = \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}\xi\int_0^1 \mathrm{d}w\ w\left[\frac{2m^2}{w^2m^2+\mu^2} - \frac{2m^2+k^2}{w^2(\xi(1-\xi)k^2+m^2)+\mu^2}\right]$$

[37] Under this change

$$dydz = \left|\frac{\partial(y,z)}{\partial(w,\xi)}\right| dwd\xi = |w|dwd\xi$$

and integration domain becomes $w \in [0,1]$ and $\xi \in [0,1]$ (the last is enforced by the theta-function $\theta(1-y-z) = \theta(w-w\xi)$

Integral over w is then straightforwardly evaluated

$$F_1^{(sing)}(k^2|\mu^2) = \frac{\alpha}{4\pi}\int_0^1 \mathrm{d}\xi\left[2\,\log\left(\frac{m^2}{\mu^2}\right) - \frac{2m^2+k^2}{m^2+k^2\xi(1-\xi)}\,\log\left(\frac{m^2+k^2\xi(1-\xi)}{\mu^2}\right)\right]$$

We can rewrite this as the sum of two terms, one being the same integral with $\mu^2 \to k^2$ and the singular term

$$F_1^{(sing)}(k^2|\mu^2) = F_1^{(sing)}(k^2|k^2) - \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}\xi\left[\frac{m^2+\frac{k^2}{2}}{m^2+k^2\xi(1-\xi)} - 1\right]\log\left(\frac{k^2}{\mu^2}\right).$$

The term $(\mu^2 \to k^2)$ is of course μ-independent. The integral over ξ there does not cause any difficulties. We find therefore for $F_1^{(sing)}(k^2) := F_1^{(sing)}(k^2|\mu^2)$,

$$F_1^{(sing)}(k^2) = -\frac{\alpha}{2\pi}\,f_{IR}(k^2)\,\log\left(\frac{k^2}{\mu^2}\right) + \text{finite at } \mu^2 \to 0\,,$$

where

$$f_{IR}(k^2) = \int_0^1 \mathrm{d}\xi\left[\frac{m^2+\frac{k^2}{2}}{m^2+k^2\xi(1-\xi)} - 1\right]$$

How this μ^2-dependent term affects the differential cross-section for the scattering of the electron off the external potential $A_a^{(ext)}(k)$? $F_1(k^2)$ is just a factor that multiplies γ_a in the S-matrix element. Therefore the μ^2-dependent contribution to the differential cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_0 \left|1 - F_1^{(sing)}(k^2) + \text{nonsingular terms}\right|^2 + O(\alpha^2) =$$
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi}\,f_{IR}(k^2)\,\log\left(\frac{k^2}{\mu^2}\right) + \text{nonsing. terms}\right],$$

where the first factor is just the Born approximation

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_0 = \frac{\alpha}{\pi}\left|\bar{u}_f\gamma_a u_i\,A_a^{(ext)}(k)\right|^2.$$

So, the fictitious photon mass term seems to appear in the differential cross-section $e(p) \to e(p')$ in the external potential. How to interpret it?

Remember, our reason for introducing the photon mass μ^2 was the difficulty in distinguishing between the electron state and the states where soft photons are also present:

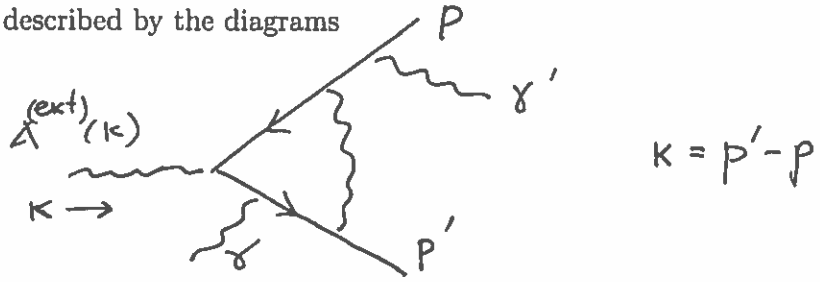
$$|\,e(\mathbf{p})\,\rangle\,, \quad |\,e(\mathbf{p})\,\gamma(\mathbf{k_1})\,\ldots\gamma(\mathbf{k_N})\rangle$$

This difficulty is of physical nature: In experiment, a detector can not detect a photon with arbitrarily low energy. Therefore it is not possible to isolate the elastic electron scattering off the potential from process in which some soft photons (with sufficiently low energy) are also created.

Therefore, with our scattering off $A^{(ext)}$ we need to consider also the processes

$$e \to e + \gamma\,, \quad e + 2\gamma, \ldots$$

These processes are described by the diagrams



Each external photon contributes e to the amplitude. So, the dominating process of this kind is when just one soft photon is emitted (e is small). Let us consider this amplitude. Two diagrams contribute to $e(p) \to e(p') + \gamma(q)$ in the leading order.

The amplitude is

$$\mathcal{M}(e(p) \to e(p')+\gamma(q)) = \bar{u}(p') \left[\mathcal{A}_0 \frac{m - \gamma(p-q)}{m^2 + (p-q)^2} \gamma_a \, e_a^*(q) + \gamma_a \, e_a^*(q) \frac{m - \gamma(p'+q)}{m^2 + (p'+q)^2} \mathcal{A}_0 \right] u(p)$$

where

$$\mathcal{A}_0 = e\gamma_b \, A_b^{(ext)}(p' - p - q)\,,$$

and $e_a(q)$ is the polarization vector of the emitted photon. We are going to consider the case of soft photon,

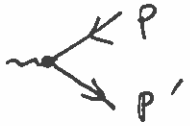
$$|q| \ll |p - p'|, \; q \to 0\,.$$

In this case we can replace

$$A_b^{(ext)}(p' - p - q) \approx A_b^{(ext)}(p' - p)\,,$$

so that $\mathcal{A}_0$ essentially coincides with the Born amplitude of the elastic scattering

$$\mathcal{A}_0 \to e\,\gamma_a\, A_a^{(ext)}(p'-p) \;=$$



For the same reason we can ignore q in the numerators of the electron propagator. Then the numerators can be simplified as

$$(m-\gamma p)\gamma_a u(p) e_a^*(q) = +2p_a e_a^*(q) u(p)$$

where I used $(m+\gamma p)u(p) = 0$. The numerator in the second term transforms similarly

$$\bar{u}(p')(m-\gamma p')e_a^*(q) = +2p_a e_a^*(q)\bar{u}(p')\ .$$

Taking into account $p^2 = (p')^2 = -m^2$, $q^2 \to 0$ we can simplify the denominator

$$m^2 + (p-q)^2 = -2pq\ ; \qquad m^2 + (p'+q)^2 = +2p'q$$

Therefore

$$\mathcal{M}(e \to e+\gamma) = e\,\bar{u}(p')\,\mathcal{A}_0\,u(p)\left[\frac{p'_a e^*_{a\lambda}(q)}{(p'q)} - \frac{p_a e^*_{a\lambda}(q)}{(pq)}\right].$$

where I explicitly indicated the polarization state λ of the photon. This must be continued to the Minkowski space by simple replacement

$$(p'e_\lambda) \to -p'e_\lambda\,,\ \ (p'q) \to -p'q\,,\ \ \text{etc} \qquad pq = p_\mu q^\mu$$

The form of the expression does not change, only Euclidean scalar producte get replaced by Minkowski ones, i.e. (pq) now stands for $p^\mu q_\mu$, etc. The differential probability of this emission of the soft photon is therefore

$$\mathrm{d}\sigma\,(e(p) \to e(p') + \gamma(q)) = \mathrm{d}\sigma\,(e(p) \to e(p'))\int \mathrm{d}\mu(q)\sum_{\lambda=1,2} e^2\left|\frac{(p'\varepsilon_\lambda)}{(p'q)} - \frac{(p\varepsilon_\lambda)}{(pq)}\right|^2$$

where, as usual

$$\mathrm{d}\mu(q) = \frac{1}{2\omega_q}\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\ .$$

Here $\omega_q = |\mathbf{q}|$ is the photon energy. We need to perform the polarization sum. The polarization sums over the photon polarizations can be simplified using the following trick. As we have seen, the W-T identity has the following consequence. If we consider arbitrary scattering amplitude involving an external photon line

$= \quad e^{\mu}_{\lambda}(q)\, \mathcal{M}_{\mu}(q, \ldots)$

and replace $e^{\mu}_{\lambda} \to q^{\mu}$, the result is zero

$$q^{\mu}\, \mathcal{M}_{\mu}(q \ldots .) = 0 \, .$$

In our case this can be checked explicitly

$$\sum \left(\frac{(p' e_{\lambda})}{(p' q)} - \frac{(p e_{\lambda})}{(pq)} \right) \to \frac{(p' q)}{(p' q)} - \frac{(pq)}{(pq)} = 0$$

Suppose we want to evaluate

$$\sum_{\lambda} \left| e^{\mu}_{\lambda}(q)\, \mathcal{M}(q, \ldots) \right|^2 .$$

Let us chose the frame where

$$q^{\mu} = (q, 0, 0, q)$$

where one can choose

$$e^{\mu}_{1} = (0, 1, 0, 0) \, , \qquad e^{\mu}_{2} = (0, 0, 1, 0)$$

are two transverse polarizations

$$\sum_{\lambda} \left| e^{\mu}_{\lambda}(q)\, \mathcal{M}_{\mu}(q, \ldots) \right|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2$$

The W-T identity in this frame reads

$$q\, \mathcal{M}_0 - q\, \mathcal{M}_3 = 0 \quad \Longrightarrow \quad \mathcal{M}_0 = \mathcal{M}_3 \, .$$

Therefore

$$\sum_{\lambda} \left| e^{\mu}_{\lambda}(q)\, \mathcal{M}_{\mu}(q, \ldots) \right|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + |\mathcal{M}_3|^2 - |\mathcal{M}_0|^2$$

i.e.

$$= -g^{\mu\nu} \mathcal{M}_{\mu} \mathcal{M}^{*}_{\nu} \, .$$

In other words, the sum

$$\sum_\lambda e^\mu_\lambda e^{*\nu}_\lambda \to -g^{\mu\nu} \ .$$

can be replaced by $-g^{\mu\nu}$. We formally include "longitudinal" and "time-like" photons in the sum, which do not change the answer because these two contributions cancel each other.

Using this trick we obtain

$$\sum_{\lambda=1,2} \left| \frac{(p'e_\lambda)}{(p'q)} - \frac{(pe_\lambda)}{(pq)} \right|^2 = -\left(\frac{(p')_\mu}{(p'q)} - \frac{p_\mu}{(pq)}\right)\left(\frac{(p')^\mu}{(p'q)} - \frac{p^\mu}{(pq)}\right) = \frac{2(pp')}{(p'q)(pq)} - \frac{m^2}{(p'q)^2} - \frac{m^2}{(pq)^2} \ .$$

With all these, we obtain for the cross section $e \to e + \gamma(q)$

$$\mathrm{d}\sigma(p' \to p + \gamma(q)) = \mathrm{d}\sigma(p' \to p) \times e^2 \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3} \frac{1}{2\omega_q} \left[\frac{2(pp')}{(p'q)(pq)} - \frac{m^2}{(p'q)^2} - \frac{m^2}{(pq)^2} \right] .$$

We write

$$\mathrm{d}^3\mathbf{q} = |\mathbf{q}|^2 \mathrm{d}|\mathbf{q}| \ \mathrm{d}\Omega_q \, .$$

Let us evaluate the angular integral. The last two terms are calculated straightforwardly

$$\int \frac{\mathrm{d}\Omega_q}{(pq)^2} = \int_0^\pi \frac{2\pi \sin(\theta)\, \mathrm{d}\theta}{(p^0q^0 - |\mathbf{p}||\mathbf{q}|\cos(\theta))^2} = 2\pi \int_{-1}^1 \frac{\mathrm{d}\cos\theta}{|\mathbf{q}|^2(p^0 - |\mathbf{p}|\cos\theta)^2} = \frac{4\pi}{|\mathbf{q}|^2((p^0)^2 - \mathbf{p}^2)} = \frac{4\pi}{m^2|\mathbf{q}|^2}$$

The first term can be done using the Feynman parameterization and[38]. One has

$$\begin{aligned} \int \frac{\mathrm{d}\Omega_q}{(p'q)(pq)} &= \int_0^1 \mathrm{d}\xi \int \frac{\mathrm{d}\Omega_q}{((pq)\xi + (p'q)(1-\xi))^2} = \int_0^1 \mathrm{d}\xi \int \frac{\mathrm{d}\Omega_q}{(q(p\xi + p'(1-\xi))^2} \\ &= \int_0^1 \mathrm{d}\xi \frac{4\pi}{|\mathbf{q}|^2(\xi p + (1-\xi)p')^2} = \frac{4\pi}{|\mathbf{q}|^2} \int_0^1 \frac{\mathrm{d}\xi}{m^2 + \xi(1-\xi)k^2} \end{aligned}$$

where for Euclidean k

$$k^2 = (p' - p)_a (p' - p)_a \ .$$

Combining all three terms we get

$$\begin{aligned} \mathrm{d}\sigma(e(p') \to e(p) + \gamma(q)) &= \mathrm{d}\sigma(p' \to p) \times \frac{e^2}{2\pi^2} \int_0^1 \mathrm{d}\xi \left[\frac{m^2 + \frac{k^2}{2}}{m^2 + k^2\xi(1-\xi)} - 1 \right] \frac{|\mathbf{q}|^2 \mathrm{d}|\mathbf{q}|}{|\mathbf{q}|^2\omega_q} \\ &= \mathrm{d}\sigma(p' \to p) \left(\frac{\alpha}{\pi}\right) f_{IR}(k^2) \frac{2\mathrm{d}|\mathbf{q}|}{\omega_q} \ . \end{aligned}$$

[38] $(\xi p + (1-\xi)p')^2 = m^2\xi^2 + (1-\xi)^2 m^2 + 2\xi(1-\xi)(pp') = m^2(\xi^2 + (1-\xi)^2 + 2\xi(1-\xi)) - k^2\xi(1-\xi)$.

Suppose that the laboratory detectors can detect a photon only if its energy is not smaller then E_l. Then the process

$$e(p) \to e(p') + \gamma(q) \quad \text{with} \quad |\mathbf{q}| < E_l$$

is indistinguishable from the elastic process. We have to add the elastic cross-section $\mathrm{d}\sigma(e(p') \to e(p))$ and the above cross-sections with the emissions of a soft photons, with the energy below E_l. Note that at low $|\mathbf{q}|$ the soft photon energy dependence of these cross-sections involves the factor $d|\mathbf{q}|/\omega_q$, so the energy integral is

$$\int_{0<|\vec{q}|<E_l} \mathrm{d}\sigma(e(p') \to e(p) + \gamma(q)) \sim 2 \int_0^{E_l} \frac{\mathrm{d}|\mathbf{q}|}{\omega_q} = \log\left(\frac{E_l^2}{\mu^2}\right) .$$

For a photon $\omega_q = |\mathbf{q}|$ and so the integral is infrared divergent. Adding again the photon mass μ^2

$$\omega_q = \sqrt{|\mathbf{q}|^2 + \mu^2}$$

and assuming $\mu^2 \ll E_l$, we get $\log\left(\frac{E_l^2}{\mu^2}\right)$. Adding this to what we obtained for the elastic cross section

$$\mathrm{d}\sigma(p' \to p + \text{soft photons}) = \left(\mathrm{d}\sigma(p' \to p)\right)_0 \left[1 - \frac{\alpha}{\pi} f_{IR}(k^2) \log\left(\frac{k^2}{\mu^2}\right) \right. \tag{22.1}$$

$$\left. + \frac{\alpha}{\pi} f_{IR}(k^2) \log\left(\frac{E_l^2}{\mu^2}\right) + \dots \right] = \left(\mathrm{d}\sigma(p' \to p)\right)_0 \left[1 + \frac{\alpha}{\pi} f_{IR}(k^2) \log\left(\frac{E_l^2}{k^2}\right) + \dots \right]$$

We see that in the measured cross-section the auxiliary photon mass cancels exactly, and instead it involves the quantity E_l, which depends on the exact setup of the laboratory experiment (what photons can not be detected).

The higher-loop calculations of $\Gamma_a(p', p)$ also lead to infrared divergences (singularity at $\mu^2 \to 0$) which are canceled by the amplitudes with emissions of more then soft photons. I am not going to demonstrate this. Look up Ch. 6.5. of the textbook P.S.