Homework 5 Solutions

Problem 1 : (Srednicki problem 32.1)
(a)Follow the lagrangian given in (32.1), we can easily calculate the U(1) current:

$$j^\mu = i\partial^\mu \varphi^\dagger \varphi - i\partial^\mu \varphi \varphi^\dagger; \tag{1}$$

so that the U(1) charge becomes:

$$Q(t) = \int j^0(\mathbf{x}, t) d^3x = -i \int \left(\Pi\varphi + \Pi^\dagger \varphi^\dagger\right)(\mathbf{x}, t) d^3x. \tag{2}$$

Based on $[\varphi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = [\varphi^\dagger(\mathbf{x}, t), \Pi^\dagger(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y})$, one easy to find $[\varphi(\mathbf{x}, t), Q(t)] = \varphi(\mathbf{x}, t)$. Exponentiate the U(1) charge, we find that:

$$e^{-i\alpha Q}\varphi e^{i\alpha Q} = \varphi + e^{-i\alpha Q}[\varphi, e^{i\alpha Q}] = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!}\varphi = e^{i\alpha}\varphi. \tag{3}$$

(b)Since $[H, Q] = 0$, $He^{-i\alpha Q}|\theta\rangle = 0$, so $e^{-i\alpha Q}|\theta\rangle$ must be a linear combination of vacua. By $e^{-i\alpha Q}\varphi e^{i\alpha Q} = e^{i\alpha}\varphi$, we have:

$$\langle\theta|e^{i\alpha Q}\varphi e^{-i\alpha Q}|\theta\rangle = \frac{v}{\sqrt{2}}e^{-i(\theta+\alpha)} = \langle\theta + \alpha|\varphi|\theta + \alpha\rangle, \tag{4}$$

then we have $e^{-i\alpha Q}|\theta\rangle = |\theta + \alpha\rangle$.

(c)Expand $e^{-i\alpha Q}|\theta\rangle$ to linear order in α, we have $(1 - i\alpha Q)|\theta\rangle = (1 + \alpha\frac{d}{d\theta})|\theta\rangle$, the second term of right hand side is not zero. Thus, $Q|\theta\rangle \neq 0$.

Problem 2 : (Srednicki problem 83.1)
(a) Since a single Dirac field involves 2 Weyl fields (one left and one right), we can denote all left hand Weyl fermions as χ_i^α, where $\alpha = 1, 2, 3$ are color indices, and $i = 1, 2, \dots n_F$ are flavor indices.
In massless case, the kinetic lagrangian is $\mathcal{L}_K = i(\chi_i^\alpha)^\dagger \sigma^\mu \partial_\mu \chi_i^\alpha$, and invariants under $U(2n_F)$ rotation mixing χ_i^α, thus the flavor symmetry is $SU(2n_F)$.
(b) The color group is $SO(3)$, and the corresponding invariant tensor is $\delta_{\alpha\beta}$. The fermionic condensate becomes $\delta_{\alpha\beta}\langle\chi_i^\alpha \chi_j^\beta\rangle = \langle\chi_i^\alpha \chi_j^\alpha\rangle = -v^3\delta_{ij}$, where we have δ_{ij} for rhs because i and j are symmetric. The transformation rule for Weyl fields was $\chi_i^\alpha \to M_i{}^j \chi_j^\alpha$, and now the condition becomes $M_i{}^{i'} M_j{}^{j'} \delta^{ij} = \delta^{i'j'}$. Thus, the unbroken symmetry group is $SO(2n_F)$.
(c) When $n_F = 2$, we have $\#$goldstones $= \#SU(4)$generators$-\#SO(4)$generators $= 15 - 6 = 9$.
(d)When the color group is $SU(2)$ rather than $SO(3)$, we have the same flavor symmetry $SU(2n_F)$. However, the condensate condition now becomes $\varepsilon_{\alpha\beta}\langle\chi_i^\alpha \chi_j^\beta\rangle = -v^3\Omega_{ij}$, where $\varepsilon_{\alpha\beta}$ is the invariant ε-tensor for $SU(2)$, thus $\Omega_{ij} =$

$-\Omega(ji)$, and the unbroken symmetry is $Sp(2n_F)$. For $SU(4) \to Sp(4)$, thus we have #goldstones = #$SU(4)$generators − #$Sp(4)$generators = 15 − 10 = 5.

Problem 3 : (Srednicki problem 83.3)
From

$$\mathcal{L} = -\frac{1}{4} f_\pi^2 \mathrm{Tr} \partial^\mu U^\dagger \partial_\mu U, \tag{5}$$

where

$$U(x) = \exp[2i\pi^a(x)T^a/f_\pi] = \sum_{n=0}^{\infty} \frac{2i\pi^a(x)T^a}{f_\pi}, \tag{6}$$

we have

$$\partial_\mu U(x) = \sum_{n=0}^{\infty} \frac{2i\partial_\mu \pi^a(x)T^a}{f_\pi}. \tag{7}$$

By using $\mathrm{Tr}(T^aT^b) = \delta^{ab}/2$, we shall directly calculate the result:

$$\mathcal{L} = -\frac{1}{2}\partial^\mu \pi^a \partial_\mu \pi^a + \frac{1}{6f_\pi^2}(\pi^a\pi^a\partial^\mu\pi^b\partial_\mu\pi^b - \pi^a\pi^b\partial^\mu\pi^a\partial_\mu\pi^b) + \cdots . \tag{8}$$

Problem 4 : (Srednicki problem 83.6)
a) From (83.19) one can easily get

$$m_{\pi^\pm}^2 = 2v^3 f_\pi^{-2}(m_u + m_d)$$
$$m_{K^\pm}^2 = 2v^3 f_\pi^{-2}(m_u + m_s)$$
$$m_{K^0}^2 = 2v^3 f_\pi^{-2}(m_d + m_s)$$
$$m_{\pi^0,\eta}^2 = \frac{4}{3}v^3 f_\pi^{-2}\left[(m_u + m_d + m_s) \mp \sqrt{m_u^2 + m_d^2 + m_s^2 - m_u m_d - m_u m_s - m_d m_s}\right]$$

b) Expanding for small $m_{u,d}/m_s$ we get

$$\Delta m_{EM}^2 = m_{\pi^\pm}^2 - m_{\pi^0}^2 = 0.00138 GeV^2$$
$$m_u v^3 f_\pi^{-2} = \frac{1}{4}(+m_{K^\pm}^2 - m_{K^0}^2 + m_{\pi^0}^2 - \Delta m_{EM}^2) = 0.00288 GeV^2$$
$$m_d v^3 f_\pi^{-2} = \frac{1}{4}(-m_{K^\pm}^2 + m_{K^0}^2 + m_{\pi^0}^2 + \Delta m_{EM}^2) = 0.00624 GeV^2$$
$$m_s v^3 f_\pi^{-2} = \frac{1}{4}(+m_{K^\pm}^2 + m_{K^0}^2 - m_{\pi^0}^2 - \Delta m_{EM}^2) = 0.11777 GeV^2$$

c) $m_u/m_d = 0.46$ and $m_s/m_d = 19$.
d) Using equations (83.50)-(83.52) in (83.48), we find $m_\eta = 0.566 GeV$ which is 3% larger than its observed value, $0.548 GeV$.

Problem 5: (Srednicki problem 83.7)
a) Requiring the coefficient of $\partial_\mu \pi^9 \partial^\mu \pi^9$ to be $\frac{1}{2}$ yields

$$F^2 = \frac{1}{9}(2f_9^2 - 3f_\pi^2). \tag{9}$$

b) Only the mass terms for π_0, η, and π^9 are different and one can easily get

$$m_\pi^2 = 4m\frac{v^3}{f_\pi^2},\ m_\eta^2 = \frac{8}{3}m_s\left(f_\pi^{-2} + \frac{3}{4}f_9^{-2}\right)v^3,\ m_{\pi^9}^2 = \frac{9f_\pi^2}{4f_9^2 + 3f_\pi^2}m_\pi^2. \tag{10}$$