Homework 4 Solutions

Problem 1 :
(a) The Lagrangian is:

$$S = \int d^4x \left(-\frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - \frac{\lambda}{4}(\phi_i\phi_i)^2 \right), \quad i = 1, \dots N.$$

The model do not contain any irrelevant terms, and the model is renormalizable by power-counting as written. However, a mass term is missing, and should be added.Thus the Lagrangian with Z-factors is:

$$S = \int d^4x \left(-\frac{1}{2}Z_\phi \partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{2}Z_m m^2 \phi_i^2 - \frac{\lambda}{4}Z_\lambda(\phi_i\phi_i)^2 \right), \quad i = 1, \dots N,$$

with (at leading order):

$$\begin{aligned} &Z_\phi(\Lambda) = 1 + O(\lambda^2) \quad \text{(no correction at 1-loop)}, \\ &Z_m - 1 \sim \Lambda^2, \qquad Z_\lambda(\Lambda) - 1 \sim \log \Lambda. \end{aligned}$$

(b) The Lagrangian is:

$$S = \int d^4x \left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}M^2\phi^2 + \bar{\Psi}(i\partial\!\!\!/ - m)\Psi + g\phi\bar{\Psi}\gamma_5\Psi \right).$$

The model do not contain any irrelevant terms. However, additional terms should be added to cancel all UV divergence. Assume the parity is conserved, and ϕ must be a pseudoscalar field, say $P^{-1}\phi(\mathbf{x}, t)P = -\phi(-\mathbf{x}, t)$, when the modified Yukawa interaction involves γ_5. Thus, polynomials of odd order of ϕ (i.e. ϕ and ϕ^3) are no longer need, and the Lagrangian with Z-factors is:

$$S = \int d^4x \left(-\frac{1}{2}Z_\phi \partial_\mu \phi \partial^\mu \phi - \frac{1}{2}Z_M M^2\phi^2 + \bar{\Psi}(iZ_\Psi \partial\!\!\!/ - Z_m m)\Psi + Z_g g\phi\bar{\Psi}\gamma_5\Psi - \frac{Z_\lambda \lambda}{24}\phi^4 \right),$$

with (at leading order):

$$Z_\phi(\Lambda) - 1 \sim \log \Lambda, \quad Z_M(\Lambda) - 1 \sim \Lambda^2, \quad Z_\lambda(\Lambda) - 1 \sim \log \Lambda,$$

and

$$Z_\Psi(\Lambda) - 1 \sim \log \Lambda, \quad Z_m(\Lambda) - 1 \sim m \log \Lambda, \quad Z_g(\Lambda) - 1 \sim \log \Lambda.$$

The corrections of field strengths are reduced by imposing $\frac{\partial}{\partial k^2}\Pi(k^2) = -\mu^2$, and the correction to fermion mass should proportional to m due to chiral symmetry, as $m_{\text{renor.}} \propto m$.

(c) The Lagrangian is:

$$S = \int d^4x \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \bar{\Psi}(i\partial\!\!\!/ - m)\Psi + g\phi\bar{\Psi}\Psi \right).$$

The model do not contain any irrelevant terms. However, additional terms of ϕ should be added to cancel all UV divergence, and since ϕ is a real scalar field, any ϕ^n with $n \leq 4$ should be added, and the Lagrangian with Z-factors is:

$$S = \int d^4x \Big(-\frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_M M^2\phi^2 + \bar{\Psi}(iZ_\Psi\partial\!\!\!/ - Z_m m)\Psi + Z_g g\phi\bar{\Psi}\gamma_5\Psi$$
$$+ Y\phi + \frac{Z_3 g_3}{6}\phi^3 - \frac{Z_\lambda\lambda}{24}\phi^4 \Big).$$

The UV divergence of conterterms are the same as in (b), and the additional terms have:

$$Y(\Lambda) \sim \Lambda^2, \quad Z_3(\Lambda) - 1 \sim \log\Lambda.$$

(d) The Lagrangian is:

$$S = \int d^3x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D^\mu\phi)^\dagger D_\mu\phi - m^2\phi^\dagger\phi - \frac{g}{4}(\phi^\dagger\phi)^2 \right), \quad D_\mu = \partial_\mu - ieA_\mu.$$

Here in $d = 3$, we have $[\phi] = [A_\mu] = \frac{1}{2}$, and $[g] = 1$ with $[e] = \frac{1}{2}$. All terms are relevant, and with respect to gauge-invariance, no additional conterterms should be added. The Lagrangian with Z-factors is:

$$S = \int d^3x \left(-\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} - Z_1(D^\mu\phi)^\dagger D_\mu\phi - m^2 Z_m\phi^\dagger\phi - \frac{Z_g g}{4}(\phi^\dagger\phi)^2 \right).$$

The Ward identities ensures no additional coefficients are needed for $(D^\mu\phi)^\dagger D_\mu\phi$, thus at leading order:

$$Z_1(\Lambda) - 1 \text{ finite at 1-loop}, \quad Z_m(\Lambda) - 1 \sim \Lambda,$$

and

$$Z_3(\Lambda) - 1 \sim \Lambda, \quad Z_g(\Lambda) - 1 \quad \text{are convergent.}$$

All diagrams are shown as in figure (1)

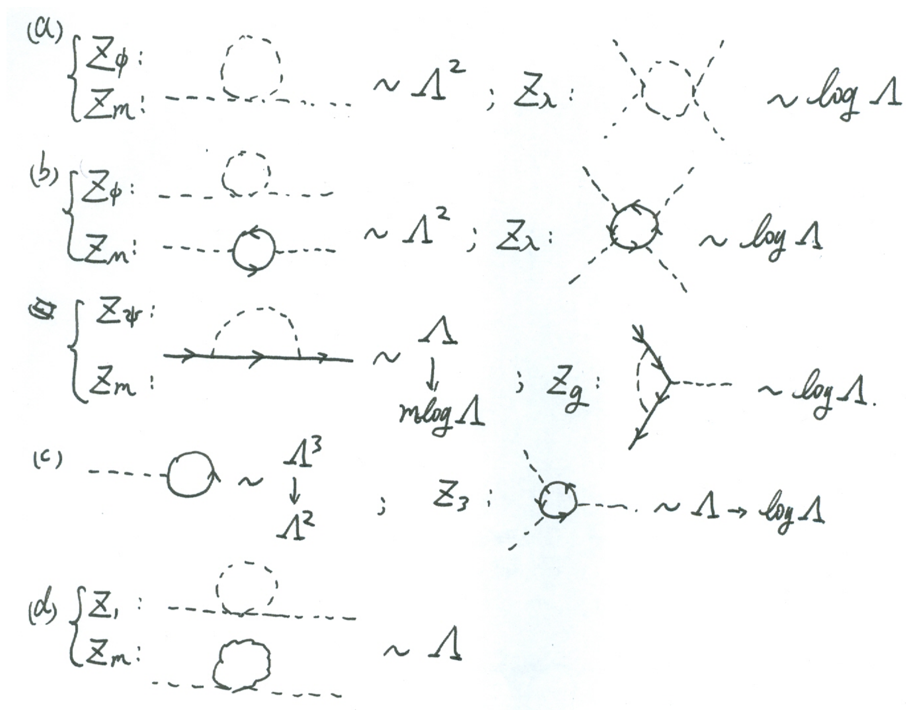

Figure 1: All loop diagrams.

Problem 2: Srednicki 78.1:
Starting with the R_ξ-gauge lagrangian in background field method:

$$\mathcal{L} = \mathcal{L}_{\rm YM}(A = \bar{A} + \mathcal{A}) - \frac{1}{2\xi}(\bar{D}^\mu \mathcal{A}_\mu)^a (\bar{D}^\nu \mathcal{A}_\nu)^a + \mathcal{L}_{\rm ghost}. \tag{1}$$

We care about the gluon and ghost vertices, which correspond to $AA\partial A$, $AAAA$, $\bar{c}\partial Ac$ and $\bar{c}AAc$ terms in the lagrangian. Note that the gluon lines A can correspond to either $\bar{A}$ or $\mathcal{A}$, and one can check section 72 for the vertices in normal cases.

First consider the 3-gluon vertex. At most one gluon might be external, and thus there should be some extra $\xi-$dependent contributions from $\mathcal{L}_{\rm gf}$. The 3-gluon vertex should be:

$$i\mathbf{V}^{\bar{a}bc}_{\bar{\mu}\nu\rho}(\bar{p},q,r) = gf^{\bar{a}bc}[(q-r)_{\bar{\mu}}g_{\nu\rho} + (r-\bar{p}+q/\xi)_\nu g_{\rho\bar{\mu}} + (\bar{p}-q-r/\xi)_\rho g_{\bar{\mu}\nu}], \tag{2}$$

where we denote the external lines with "bar".

For the 4-gluon vertex, at most two gluons might be external. The 4-gluon vertex with one external gluon is unchanged from the normal case, since $\mathcal{L}_{\rm gf}$

gives no contribution. If two are external, the vertex reads:

$$i\mathbf{V}^{\bar{a}\bar{b}cd}_{\bar{\mu}\bar{\nu}\rho\sigma} = -ig^2[f^{\bar{a}\bar{b}e}f^{cde}(g_{\bar{\mu}\rho}g_{\bar{\nu}\sigma} - g_{\bar{\nu}\rho}g_{\bar{\mu}\sigma}) + f^{\bar{a}ce}f^{d\bar{b}e}(g_{\bar{\mu}\sigma}g_{\bar{\nu}\rho} - g_{\bar{\mu}\bar{\nu}}g_{\rho\sigma} - g_{\bar{\mu}\rho}g_{\bar{\nu}\sigma}/\xi) + f^{\bar{a}de}f^{\bar{b}ce}(g_{\bar{\mu}\bar{\nu}}g_{\rho\sigma} - g_{\bar{\mu}\rho}g_{\bar{\nu}\sigma} + g_{\bar{\nu}\rho}g_{\bar{\mu}\sigma}/\xi)].$$

The ghost lagrangian reads:

$$\mathcal{L}_{\text{gh}} = -(\bar{D}^\mu\bar{c})^a(\mathcal{D}_\mu c)^a, \tag{3}$$

with $\bar{D}_\mu = D_\mu(\bar{A})$, $\mathcal{D}_\mu = D_\mu(\mathcal{A})$. So that the gluon-ghost-ghost vertex (with the gluon to be external) is:

$$i\mathbf{V}^{\bar{a}bc}_{\bar{\mu}}(q,r) = gf^{\bar{a}bc}(q+r)_{\bar{\mu}}. \tag{4}$$

As for the gluon-gluon-ghost-ghost vertices, there are 2 possibilities:

$$i\mathbf{V}^{\bar{a}bcd}_{\bar{\mu}\nu} = -ig^2f^{\bar{a}ce}f^{bde}g_{\bar{\mu}\nu}, \tag{5}$$

$$i\mathbf{V}^{\bar{a}\bar{b}cd}_{\bar{\mu}\bar{\nu}} = -ig^2(f^{\bar{a}ce}f^{\bar{b}de} + f^{\bar{b}ce}f^{\bar{a}de})g_{\bar{\mu}\bar{\nu}}. \tag{6}$$

Problem 3: Srednicki 73.1

The calculations of the Z factors in this case are the same as for one abelian scalar, except from some additional factors that come from group structure of the vertices.

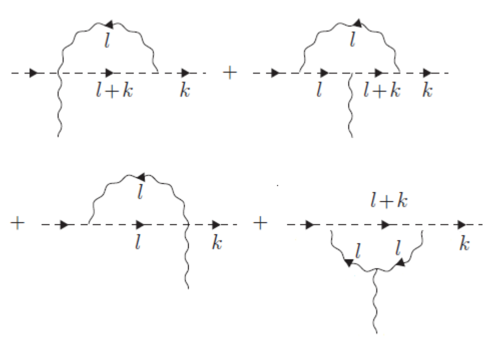

Figure 2:

For Z_1 we need to evaluate the diagrams in figure (2) and the result is

$$Z_1 = 1 + (3C(R) - T(A))\frac{g^2}{8\pi^2}\frac{1}{\epsilon} \tag{7}$$

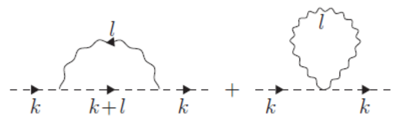

Figure 3:

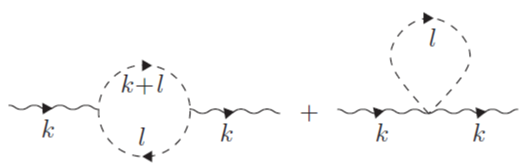

Figure 4:

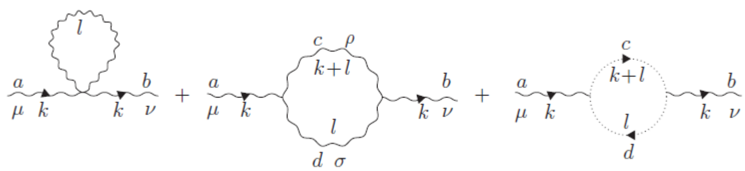

Figure 5:

For Z_2 we need to evaluate the diagrams in figure (3) and the result is

$$Z_2 = 1 + C(R)\frac{3g^2}{8\pi^2}\frac{1}{\epsilon} \tag{8}$$

For Z_3 we need to evaluate diagrams shown in figures (4) and (5)

The diagrams in figure (4) contribute

$$-\frac{1}{3}T(R)\frac{g^2}{8\pi^2}, \tag{9}$$

while the diagrams in figure (5) contribute

$$\frac{5}{3}T(A)\frac{g^2}{8\pi^2}. \tag{10}$$

Combining the contributions we get

$$Z_3 = 1 + \left(\frac{5}{3}T(A) - \frac{1}{3}T(R)\right)\frac{g^2}{8\pi^2}\frac{1}{\epsilon}. \tag{11}$$

Following the general analysis of section (52)

$$\beta(g) = -\left(\frac{11}{3}C(R) - \frac{1}{3}T(R)\right)\frac{g^3}{(4\pi)^2}. \tag{12}$$