Homework 2 Solutions

Problem 1 :

The calculations of the N flavor φ^4 theory is very similar to what we did in the usual φ^4 theory, thus we will frequently use its result, which shall be denoted with a subscript $_0$.

(a) The propagator from flavor i to j is:

$$\tilde{\Delta}_{ij}(k) = \frac{\delta_{ij}}{k^2 + m^2 - i\epsilon}.$$

And the 4-pt vertex with labeling $ijkl$ is:

$$iV_{ijkl} = -2i\lambda(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}).$$

The correction to the propagator thus become:

$$i\Pi_{ij}(k^2) = 2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk})\delta_{kl} i\Pi_0(k^2) = 2(N+2)i\Pi_0(k^2), \quad (1)$$

where the extra factor of 2 coming from the absent of symmetric factor 2 comparing to $N = 1$ case. The Z-factors for propagators in MS scheme are:

$$Z_\varphi = 1 + O(\lambda^2), \quad Z_m = 1 + \frac{2(N+2)\lambda}{16\pi^2\varepsilon} + O(\lambda^2). \quad (2)$$

Similarly, for the vertex labeling $ijkl$ at s-channel, we have:

$$iV^s_{ijkl} = 2(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})(\delta_{mn}\delta_{kl} + \delta_{mk}\delta_{nj} + \delta_{nk}\delta_{ml})iV^s_0 \quad (3)$$

$$= 2[(N+4)\delta_{ij}\delta_{kl} + 2\delta_{ik}\delta_{lj} + 2\delta_{il}\delta_{jk}]iV^s_0. \quad (4)$$

The extra factor of 2 also comes from the absent of symmetric factor 2 in the loop comparing to $N = 1$ case. After summing the contribute from t- and u-channed, the result for Z-factor is:

$$Z_\lambda = 1 + \frac{2(N+8)\lambda}{16\pi^2\varepsilon}. \quad (5)$$

The results of β and γ_m now are:

$$\beta(\lambda) = \frac{(N+8)\lambda^2}{8\pi^2}, \quad \gamma_m(\lambda) = \frac{(N+2)\lambda}{16\pi^2}. \quad (6)$$

(b) At $d = 4 - \varepsilon$, we find the equations for the WF fixed point is:

$$-\varepsilon\lambda + \beta(\lambda) = 0, \quad \mu\frac{dm^2}{d\mu} = 0,$$

which give:

$$\lambda_* = \frac{8\pi^2\varepsilon}{(N+8)}, \quad m_* = 0. \tag{7}$$

(c) This scalar field theory may be viewed as an effective description of some statistical model. The massive field φ gives rise to a Yukawa potential $-e^{-mr}/r$, which is corrected in interacting theories by the renormalization flow $d\log m/d\log\mu = \gamma_m$, ie. $(m/m_0) = (\mu = mu_0)^{\gamma_m} = (r/r0)^{-\gamma_m}$, where we have taken the mass scale to be $\mu = 1/r$. Therefore, the corrected Yukawa potential is $-e^{(r/\xi)^{1/2\nu}}$, where $\nu = \frac{1}{2}(1-\gamma_m)$ is the critical exponent and $\xi = m_0^{-2\nu} r_0^{1-2\nu}$ is the correlation length.
To obtain the critical exponent in $d=3$ we can set $\varepsilon = 1$ (this is certainly not small, but it seems to give good estimates in certain theories). Then:

$$\nu = \frac{1}{2(1-\gamma_m(\lambda_*))} = \frac{N+8}{N+14}. \tag{8}$$

Problem 2 : a) A term that respects the $O(N)$ symmetry can be written as dot-products of the vector $\vec{\phi}$ and its derivatives. The only term that cannot be written in this form is the last one. The other terms can be written as

$$S = \int d^dx \left(\frac{1}{2}(\partial^\mu \vec{\phi}) \cdot (\partial_\mu \vec{\phi}) + t_0 \vec{\phi} \cdot \vec{\phi} + u_0 (\vec{\phi} \cdot \vec{\phi})^2 \right). \tag{9}$$

b)As usual

$$t_0 = Z_t Z_\phi^{-1} \mu^2 t \tag{10}$$

$$u_0 = Z_u Z_\phi^{-2} \mu^\epsilon u \tag{11}$$

$$v_0 = Z_t Z_\phi^{-2} \mu^\epsilon v \tag{12}$$

In the second problem we saw that the linear term in the beta functions comes from the powers of μ in the above equations. More specifically, the coefficient in front of ϵ is equal to d minus the dimension of the corresponding operator. Hence

$$c_1 = 2, \quad c_2 = c_3 = \epsilon \tag{13}$$

c) A fixed point is a solution of the set of equations

$$\beta_t = 0, \quad \beta_u = 0, \quad \beta_v = 0. \tag{14}$$

Using the expressions for the beta functions we find the following solutions

1. $t = 0, \quad u = 0, \quad v = 0,$

2. $t=0, \quad u=0, \quad v=\frac{\epsilon}{72}$,

3. $t=0, \quad u=\frac{\epsilon}{8(N+8)}, \quad v=0$,

4. $t=0, \quad u=\frac{\epsilon}{24N}, \quad v=\frac{(N-4)\epsilon}{72N}$.

In principle there is mixing between operators with the sane scaling dimensions. For this reason, the Z factor will be a matrix defined as

$$O_0^i = Z^{ij} O^j. \tag{15}$$

Let M be the matrix that diagonlize Z^{ij}. Then the interaction terms in the Langranzian can be written as

$$L_{int} = g^i O_0^i = g_i Z^{ij} O_j = (g\Lambda^{-1})_i (\Lambda Z \Lambda^{-1})^{ij} (\Lambda O)_j = g'^i Z'^i O'^i \tag{16}$$

where the primes denote the diagonalized quantities. The scaling dimensions of the diagonalized operators minus the spacetime dimensions are then equal to the derivative of the beta function at the fixed point (see Peskin & Schroeder pages 428-435).

A priori we don't know which combinations of operators appearing in the Lagrangian diagonalize the Z matrix so instead we have to diagonalize the matrix

$$H^{ij} = \frac{\partial \beta_i}{\partial g^j}, \tag{17}$$

whose eigenvalues h_i satisfy

$$\Delta_i = d + h_i \tag{18}$$

The first fixed point corresponds to the free theory. There is no mixing between the operators and their scaling dimensions are

$$\Delta_m = c_1 = d-2, \quad \Delta_u = \Delta_v = d - \epsilon. \tag{19}$$

For the other three fixed points there is mixing between the two quartic operators and the scaling dimensions are

2)

$$\Delta_m = d - 2 + \frac{\epsilon}{3}, \quad \Delta_1 = d + \epsilon, \quad \Delta_2 = d - \frac{\epsilon}{3}, \tag{20}$$

3)

$$\Delta_m = d - 2 + \frac{N+2}{N+8}\epsilon, \quad \Delta_1 = d + \epsilon, \quad \Delta_2 = d - \frac{N-4}{N+8}\epsilon. \tag{21}$$

4)

$$\Delta_m = d - 2 + \frac{2}{3}\frac{N-1}{N}\epsilon, \quad \Delta_1 = d + \epsilon, \quad \Delta_2 = d + \frac{N-4}{3N}\epsilon, \tag{22}$$

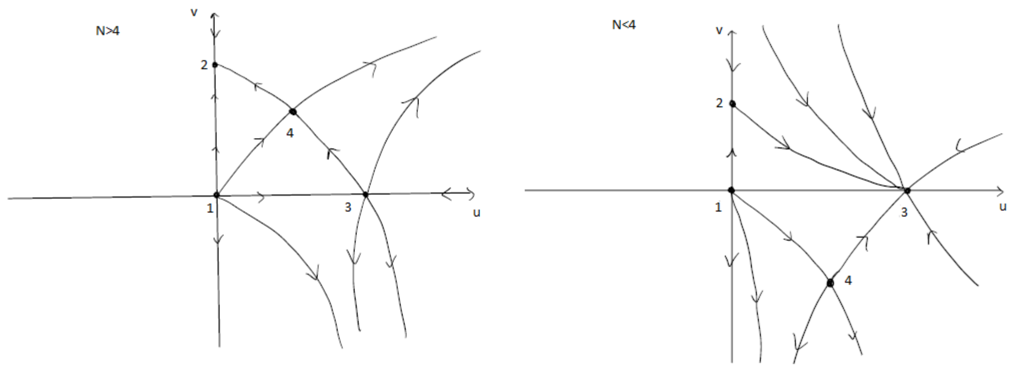

Figure 1: These are the two RG-graphs depending on the value of N.

d) A fixed point is more stable is the there are no relevant operators at the fixed point. That is because we have to tune less bare parameters in order to end up at the fixed point at some lower energies. However, in problem we see that the stability of the fixed points change for $N = N_c = 4$. For $N < 4$ the most stable fixed point (apart from the trivial one) is the third one and for $N > 4$ the most stable fixed point is the fourth one.

e) The RG flow are shown in figure (1).

Problem 3:

Srednicki 51.1:

Let's focus on the fermion parts of the partition function. By $\mathcal{L}_1 = g\varphi\overline{\Psi}G\Psi$ (here G for either 1 or $i\gamma_5$), we can write:

$$Z \sim \exp\big[ig\int d^4x\varphi(x)\frac{\delta}{\delta\eta(x)}G\frac{\delta}{\delta\bar{\eta}(x)}\big]\exp\big[i\int d^4xd^4y\bar{\eta}(x)S(x-y)\eta(y)\big],$$

where we have replaced $\frac{\delta}{i\delta J(x)}$ with $\varphi(x)$, and suppressed all spinor indices. Now, for the fermionic 1-loop correction to scalar propagator (check Fig. 51.1 for the corresponding Feynman diagram), we have (in position space):

$$\begin{aligned}\langle\varphi(x_1)\varphi(x_2)\rangle_{\text{1-loop}} &= i^4g^2\varphi(x_1)\varphi(x_2)\delta_1G\bar{\delta}_1\delta_2G\bar{\delta}_2\cdot\prod\int d^4x_i\;\bar{\eta}_xS_{xy}\eta_y\bar{\eta}_zS_{zw}\eta_w\\ &= -i^4g^2\varphi(x_1)\varphi(x_2)\prod\int d^4x_i\delta_1G\bar{\delta}_1\bar{\eta}_xS_{xy}\delta_2\eta_yG\bar{\delta}_2\bar{\eta}_zS_{zw}\eta_w\\ &= -g^2\varphi(x_1)\varphi(x_2)\mathrm{Tr}\big(GS_{12}GS_{21}\big),\end{aligned}$$

the minus sign coming from pulling $\delta_2G\bar{\delta}_2$ inside and acting on $\eta_y\bar{\eta}_z$. For higher order fermionic loops with more propagators, there are always a minus sign coming from the first $\delta G\bar{\delta}$, and all the rest $\delta G\bar{\delta}$ give plus signs. So we add an overall minus sign for ever fermionic loop by hand when we do such calculations directly from the Feynman diagrams.

Srednicki 52.2:

The method for calculating Z-factors are standard. First we focus on the scalar propagator, the nominator of the fermionic loop integrand changes from $\mathrm{Tr}[\tilde{S}(\not{l}+\not{k})i\gamma_5\tilde{S}(\not{l})i\gamma_5]$ into $\mathrm{Tr}[\tilde{S}(\not{l}+\not{k})\tilde{S}(\not{l})]$. Also, an extra φ^3 loop should be take into consideration. Thus the results are:

$$Z_\varphi = 1 - \frac{g^2}{4\pi^2\epsilon}, \quad Z_M = 1 + \frac{1}{16\pi^2\epsilon}\big(\lambda - 24\frac{g^2m^2}{M^2} + \frac{\kappa^2}{M^2}\big).$$

Next we turn to the fermionic propagator. The removal of $i\gamma_5$ will change the sign of mass at the nominator of loop integrand, after some calculations we find the Z-factors are:

$$Z_\Psi = 1 - \frac{g^2}{16\pi^2\epsilon}, \quad Z_m = 1 + \frac{g^2}{8\pi^2\epsilon}.$$

Similar replacements take place for the calculations of Z_λ and Z_g. However, changing $i\gamma_5$ to 1 does not change the sign of the leading divergence in both cases, and with the absent of any extra graphs, the results are:

$$Z_g = 1 + \frac{g^2}{8\pi^2\epsilon}, \quad Z_\lambda = 1 + \frac{3}{16\pi^2\epsilon}\big(\lambda - \frac{16g^4}{\lambda}\big).$$

One left Z-factor is from the loop correction to φ^3 vertices. The result is:

$$Z_\kappa = 1 + \frac{3}{16\pi^2\epsilon}\big(\lambda - \frac{16g^3m}{\kappa}\big).$$

Next, we turn to the calculations of beta functions and anomalous dimensions. Since the result for Z_Ψ, Z_φ, Z_λ and Z_g are the same, the following corresponding beta function and anomalous dimension results follow from Sec.52 and Problem 52.1, as:

$$\beta_g = \frac{5g^3}{16\pi^2}, \qquad \beta_\lambda = \frac{3\lambda^2 + 8\lambda g^2 - 48g^4}{16\pi^2},$$
$$\gamma_\varphi = \frac{g^2}{8\pi^2}, \qquad \gamma_\Psi = \frac{g^2}{32\pi^2}.$$

Finally, using $\kappa_0 = Z_\kappa Z_\varphi^{-3/2}\tilde{\mu}^{\epsilon/2}\kappa$, we find:

$$\beta_\kappa = \frac{6g^2\kappa + 3\lambda\kappa - 48g^3m}{16\pi^2},$$

and

$$\gamma_m = \frac{3g^2}{16\pi^2}, \quad \gamma_M = \frac{1}{32\pi^2}\big(4g^2 - 24\frac{g^2m^2}{M^2} + \frac{\kappa^2}{M^2} + \lambda\big).$$