Homework 1 Solutions

Problem 1 :

Srednicki 27.1:
Using the definition of a beta function we can easily get (to linear order in α):

$$\frac{d\alpha}{dm} = \frac{\frac{d\alpha}{\ln\mu}}{\frac{dm}{\ln\mu}} = \frac{\beta(\alpha)}{m\gamma_m(\alpha)} = \frac{b_1}{c_1}\frac{\alpha}{m}, \tag{1}$$

which can be written as

$$\frac{dm}{m} = \frac{c_1}{b_1}\frac{d\alpha}{a}. \tag{2}$$

Integrating both sides we find

$$\frac{m_1}{m_2} = \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{c_1}{b_1}}. \tag{3}$$

Srednicki 28.1:
Starting with the analog of equations (28.3)-(28.6)

$$\varphi_0 = \sqrt{Z_\varphi}\varphi, \tag{4}$$

$$m_0 = \sqrt{\frac{Z_m}{Z_\varphi}}m, \tag{5}$$

$$\lambda_0 = Z_\varphi^{-2} Z_\lambda \tilde{\mu}^\epsilon \lambda, \tag{6}$$

the strategy will be to repeat the same steps as in the φ^3-theory. The 1-loop results for Z-factors of the φ^4-theory are (see Srednicki Sec. 31):

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2}\frac{1}{\epsilon} + O(\lambda^2), \tag{7}$$

$$Z_m = 1 + \frac{\lambda}{16\pi^2}\frac{1}{\epsilon} + O(\lambda^2), \tag{8}$$

$$Z_\varphi = 1 + O(\lambda^2). \tag{9}$$

As usual, all the bare quantities must be independent from the renormalization scale μ since it is an auxiliary parameter. Proceeding as in the φ^3 case we define the analog of equation (28.14)

$$G(\lambda,\epsilon) = \ln Z_{\varphi}^{-2} Z_{\lambda} \approx \frac{3\lambda}{16\pi^2}\frac{1}{\epsilon}. \tag{10}$$

From (6) we get

$$\ln \lambda_0 = G + \epsilon \ln \mu + \ln \lambda, \tag{11}$$

and requiting that λ_0 is independent of μ, or in other words

$$\frac{d \ln \lambda_0}{d \ln \mu} = 0, \tag{12}$$

we obtain (keep λ/ϵ small)

$$\frac{d\lambda}{d\ln\mu} = -\frac{\epsilon\lambda}{1+\frac{3\lambda}{16\pi^2}\frac{1}{\epsilon}} = -\epsilon\lambda\big[1 - \frac{3\lambda}{16\pi^2}\frac{1}{\epsilon} + O(\lambda^2)\big] \overset{\epsilon\to 0}{=} \frac{3\lambda^2}{16\pi^2}, \tag{13}$$

and therefore

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3). \tag{14}$$

Similarly for the mass renormalization we define

$$M = \ln\sqrt{\frac{Z_m}{Z_\varphi}} = \frac{\lambda}{32\pi^2}\frac{1}{\epsilon} + O(\lambda^2), \tag{15}$$

and requiring that $\frac{d\ln m_0}{d\ln m} = 0$ we find

$$\gamma_m = \frac{\lambda}{32\pi^2} + O(\lambda^2), \tag{16}$$

Lastly, since Z_φ doesn't have corrections at one loop order, equation (28.36) implies that

$$\gamma_\varphi = \frac{1}{2}\frac{d\ln Z_\varphi}{d\ln\mu} = O(\lambda^2). \tag{17}$$

Note: here we use Srednicki's convention that the term $-\epsilon\lambda$ is not involved in the definition of β-function. However, this term will play an important role in finding the Wilson-Fisher fixed point in $4-\epsilon$ dimensions.

Problem 2:

Generally, assume we are renormalizing operator $\mathcal{O}(x)$ involving n scalar field ϕ and several derivatives. Setting $\mathcal{O}_0(x) = Z_{\mathcal{O}}\mathcal{O}(x)$. Using Wick's contraction, the lowest order non-divergent correlator is (in Fourier space):

$$G^{(n,\mathcal{O})}(p_i;k) = \langle\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)\tilde{\mathcal{O}}(k)\rangle = Z_\phi^{-n/2} Z_{\mathcal{O}}^{-1}\langle\tilde{\phi}_0(p_1)\cdots\tilde{\phi}_0(p_n)\tilde{\mathcal{O}}_0(k)\rangle;$$

which satisfy the Callan-Symanzik equation:

$$\big[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+n\gamma_{\phi}(\lambda)+\gamma_{\mathcal{O}}(\lambda)\big]G^{(n,\mathcal{O})}(p_i;k)=0, \tag{18}$$

where $\gamma_{\mathcal{O}}=\frac{d}{d\ln\mu}\ln Z_{\mathcal{O}}$, and $\gamma_{\phi}=\frac{1}{2}\frac{d}{d\ln\mu}\ln Z_{\phi}$ (Note that here we use the same definition as on p.430 of Peskin&Schroeder). From the Callan-Symanzik equation, we obtain the formula for $\gamma_{\mathcal{O}}$:

$$\gamma_{\mathcal{O}}=Z_{\mathcal{O}}\frac{\partial}{\partial\ln\mu}\big(-\delta_{\mathcal{O}}+\frac{n}{2}\delta_{\phi}\big), \tag{19}$$

where $Z_{\mathcal{O}}=1+\delta_{\mathcal{O}}$ and $Z_{\phi}=1+\delta_{\phi}$.

For pure ϕ^4 theory in $4-\epsilon$ dimensions, we have $Z_{\phi}=1+O(\lambda^2)$, and $\frac{d}{d\ln\mu}\lambda=-\epsilon\lambda+O(\lambda^2)$. Thus we can read out $\gamma_{\mathcal{O}}$ directly from $Z_{\mathcal{O}}$ (at linear order in λ):

$$Z_{\mathcal{O}}=1+\frac{a_{\mathcal{O}}\lambda}{\epsilon}+O(\lambda^2),\quad\Rightarrow\quad\gamma_{\mathcal{O}}=a_{\mathcal{O}}\lambda. \tag{20}$$

These arguments about the renormalization of composite operators can be found on Sec. 12.4 of Peskin&Schroeder.

a) The field strength renormalization for the composite operator ϕ^2 can be calculated from the divergent part of $G(p_1,p_2,x)$. Normalize the tree level contribution to be (we omitted some δ-function for momentum conservation):

$$\Big\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}^2(q)\Big\rangle_{\text{tree}}=-\frac{2}{p_1^2p_2^2}. \tag{21}$$

Then the 1-loop contribution:

$$\Big\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}^2(q)\Big\rangle_{\text{1-loop}}=-i\lambda\frac{1}{p_1^2p_2^2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{l^2}\frac{1}{(l+p_1+p_2)^2} \tag{22}$$

Following the standard procedure one finds that, in MS scheme, we have:

$$Z_{\phi^2}=1+\frac{\lambda}{16\pi^2\epsilon}+O(\lambda^2). \tag{23}$$

Hence, the anomalous dimension of ϕ^2 is

$$\gamma_{\phi^2}=\frac{\lambda}{16\pi^2} \tag{24}$$

The Callan-Symanzik equation is

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+2\gamma_{\phi^2}(\lambda)\Big)\big\langle\phi^2(x)\phi^2(0)\big\rangle=0. \tag{25}$$

Denote $\langle\phi^2(x)\phi^2(0)\rangle = f(\mu|x|)/|x|^{2(d-2)}$, the CS equation gives:

$$f(\mu|x|) = f(\mu_0|x|)\exp\Big[-2\int_{\lambda_0}^{\lambda} d\lambda'\big(\gamma_{\phi^2}(\lambda(\mu'))/\beta(\lambda')\big)\Big] \tag{26}$$

$$= f(\mu_0|x|)\exp\big[-2\int_{\mu_0}^{\mu} d\ln\mu'\gamma_{\phi^2}(\lambda(\mu'))\big]. \tag{27}$$

Change the integration argument into x', we have:

$$f(\mu|x|) = C(\lambda(|x|))\exp\big[-2\int_{1/\mu}^{|x|} d\ln x'\gamma_{\phi^2}(\lambda(x'))\big]. \tag{28}$$

In $4-\epsilon$ dimension at IR, the theory will be driven to the Wilson-Fisher fixed point for large $|x|$, where the integration is dominated by the large value of $|x'|$, and γ_{ϕ^2} can be approximate to take the value at the WF fixed point, thus $f\sim|x|^{-2\gamma_{\phi^2}(\lambda_*)}$. Therefore

$$\langle\phi^2(x)\phi^2(0)\rangle \sim \frac{1}{|x|^{2(d-2+\gamma_{\phi^2}(\lambda_*))}} = \frac{1}{|x|^{2\Delta_{\phi^2}(\lambda_*)}}, \tag{29}$$

where at the WF fixed point $\Delta_{\phi^2}(\lambda_*) = 2-\frac{2\epsilon}{3}$.

b) Consider the mass term in the lagrangian as a 2-pt interaction. Any Green's function will be a power series in m^2:

$$G_m^{(n)}(x_1,\ldots,x_n) = \langle\phi(x_1)\ldots\phi(x_n)\rangle_m \tag{30}$$

$$= \sum_{l=0}^{\infty}\frac{1}{l!}(-\frac{i}{2}m^2)^l\left\langle\phi(x_1)\ldots\phi(x_n)\left(\int d^dx'\phi^2(x')\right)^l\right\rangle_0 \tag{31}$$

$$= \sum_{l=0}^{\infty}\frac{1}{l!}(-\frac{i}{2}m^2)^l\int\prod_{k=1}^{l}d^dx'_k\; G_0^{(n,l)}(x_1\ldots x_n;x'_1\ldots x'_l) \tag{32}$$

The above Green's functions obey the following Callan-Symanzik equations

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_\lambda\frac{\partial}{\partial\lambda}+2\gamma_m m^2\frac{\partial}{\partial m^2}+n\gamma_\phi\right)G_m^{(n)} = 0 \tag{33}$$

and

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_\lambda\frac{\partial}{\partial\lambda}+l\gamma_{\phi^2}+n\gamma_\phi\right)G_0^{(n,l)} = 0 \tag{34}$$

Now let's act on equation (32) with the following operator

$$
\begin{aligned}
\left(\mu\frac{\partial}{\partial\mu}+\beta_\lambda\frac{\partial}{\partial\lambda}+n\gamma_\phi\right)G_m^{(n)} &= \sum_{l=0}^{\infty}\frac{1}{l!}(\frac{-i}{2}m^2)^l\left(\mu\frac{\partial}{\partial\mu}+\beta_\lambda\frac{\partial}{\partial\lambda}+n\gamma_\phi\right)\int\prod_{k=1}^{l}d^dx'_k\ G_0^{(n,l)} \\
&= \sum_{l=0}^{\infty}\frac{1}{l!}(\frac{-i}{2}m^2)^l\left(-l\gamma_{\phi^2}\right)\int\prod_{k=1}^{l}d^dx'_k\ G_0^{(n,l)} \\
&= \left(-\gamma_{\phi^2}m^2\frac{\partial}{\partial m^2}\right)\sum_{l=0}^{\infty}\frac{1}{l!}(\frac{-i}{2}m^2)^l\ \int\prod_{k=1}^{l}d^dx'_k\ G_0^{(n,l)} \\
&= \left(-\gamma_{\phi^2}m^2\frac{\partial}{\partial m^2}\right)G_m^{(n)} \qquad (35)
\end{aligned}
$$

Comparing this equation with (33) we see that

$$\gamma_{\phi^2}=2\gamma_m, \qquad (36)$$

which match the known result of $\gamma_m=\frac{\lambda}{32\pi^2}$ to 1-loop order in ϕ^4.

c) An is part (a) one can similarly evaluate

$$\left\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4)[\tilde{\phi}^4](q)\right\rangle. \qquad (37)$$

Normalize the tree level contribution as:

$$\left\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4)[\tilde{\phi}^4](q)\right\rangle_{\text{tree}}=\frac{1}{p_1^2p_2^2p_3^2p_4^2}. \qquad (38)$$

And the 1-loop contribution is:

$$\left\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4)[\tilde{\phi}^4](q)\right\rangle_{\text{1-loop}}=\frac{1}{p_1^2p_2^2p_3^2p_4^2}\frac{i\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k+p_1+p_2)^2} \qquad (39)$$

$$+\ (\text{other 5 terms}).$$

Thus

$$Z_{\phi^4}=1+\frac{3\lambda}{8\pi^2\epsilon}+O(\lambda^2), \qquad (40)$$

and

$$\gamma_{\phi^4}=\frac{3\lambda}{8\pi^2}. \qquad (41)$$

d) Use the similar method in a), we can calculate the scaling dimension of ϕ^4 operator at the WF fixed point:

$$\Delta_{\phi^4}(\lambda_*)=2d-4+\gamma_{\phi^4}(\lambda_*). \qquad (42)$$

At the WF fixed point, $\beta(\lambda_*)=-\epsilon\lambda_*+\frac{3\lambda_*^2}{16\pi^2}=0$, thus $\lambda_*=\frac{16\pi^2\epsilon}{3}$ and $\beta'(\lambda_*)=\epsilon$. In addition, we have $\gamma_{\phi^4}(\lambda_*)=2\epsilon$. Put all these together, we find:

$$\Delta_{\phi^4}(\lambda_*)=d+\beta'(\lambda_*)=4. \qquad (43)$$

Near the WF fixed point, a scale transformation $\mu \to b\mu$ will lead to the scaling of composite operator as $\phi^4 \to b^{\Delta_{\phi^4}}\phi^4$, and the interaction term s-cales as $\lambda_0\phi^4 = \mathcal{L}_i \to b^d\mathcal{L}_i$. The bare coupling λ_0 scales as $\lambda_0 \to b^{d-\Delta_{\phi^4}}\lambda_0$ by dimensional counting. The dimensionless coupling should not scales, thus $\lambda \sim \mu^{\Delta_{\phi^4}-d}\lambda_0$, and $\beta'(\lambda) = \frac{d}{d\lambda}\frac{d}{d\ln\mu}\lambda = \Delta_{\phi^4} - d$ at the WF fixed point.

The argument can also be found on Sec.12.4 to Sec.12.5 on Peskin, or "Space of CFTs" by Silviu Pufu in 2017 bootstrap summer school.

e) The current associated to the global $U(1)$ symmetry is

$$J_\mu = i(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*). \tag{44}$$

The one loop contribution to the three-point function

$$\langle\phi(x_1)\phi(x_2)J_\mu(x)\rangle \tag{45}$$

is given by

$$\frac{\lambda}{4}\int d^4y\,\langle\phi(x_1)\phi(x_2)J_\mu(x)\phi^4(y)\rangle\,. \tag{46}$$

It is straightforward to show that both terms in $J_\mu(x)$ give

$$\lambda\int d^4y\; D(x_1-y)D(x_2-y)D(x-y)\partial_\mu D(x-y), \tag{47}$$

and therefore the above three point function vanishes.

We can prove that $\gamma_J = 0$ to all orders in perturbation theory as follows. Consider the Ward identity for this symmetry

$$\begin{aligned}\langle\phi(x_1)\phi^*(x_2)\partial_\mu J^\mu(x)\rangle &= -i\delta^{(4)}(x-x_1)\langle\delta\phi(x_1)\phi(x_2)\rangle - i\delta^{(4)}(x-x_2)\langle\phi(x_1)\delta\phi(x_2)\rangle\\ &= \delta^{(4)}(x-x_1)\langle\phi(x_1)\phi(x_2)\rangle - \delta^{(4)}(x-x_2)\langle\phi(x_1)\phi(x_2)\rangle\\ &= \frac{1}{\pi^2}\partial_\mu\left(\frac{(x-x_1)^\mu}{(x-x_1)^4} - \frac{(x-x_2)^\mu}{(x-x_2)^4}\right)\langle\phi(x_1)\phi(x_2).\rangle \end{aligned}\tag{48}$$

We see that

$$\langle\phi(x_1)\phi^*(x_2)J^\mu(x)\rangle = \frac{1}{\pi^2}\left(\frac{(x-x_1)^\mu}{(x-x_1)^4} - \frac{(x-x_2)^\mu}{(x-x_2)^4}\right)\langle\phi(x_1)\phi(x_2)\rangle \tag{49}$$

Note that there is no integration "constant" since for $x \to \infty$ the three point function must vanish. Hence, all the divergences are coming from the propaga-tor and therefore can be taken care of by the the renormalization of ϕ.

f) From dimensional analysis any two-point function in a generic massless theory will have the form

$$G^{(2)}(p) = \frac{i}{p^2} + \frac{i}{p^2}(A\log\frac{\Lambda^2}{p^2} + \dots) + \frac{i}{p^2}(ip^2\delta_Z)\frac{i}{p^2} \tag{50}$$

$$= \frac{i}{p^2} + \frac{i}{p^2} f\left(\frac{\Lambda}{\mu}\right) - \frac{i\delta_Z}{p^2} \tag{51}$$

for some function f. So we see that all the divergent terms (terms that diverge for $\Lambda \to \infty$) are functions of $\frac{\Lambda}{\mu}$, and therefore

$$\mu\frac{d\delta_Z}{d\mu} = -\Lambda\frac{d\delta_Z}{d\Lambda}. \tag{52}$$

from which it follows that

$$\gamma_O = -\Lambda\frac{dZ_O}{d\Lambda} \tag{53}$$

Consider now the integral in (22). After using Feynman parameters, shifting the loop momentum (assuming $\Lambda \to \infty$) and Wick rotate, we have

$$\begin{aligned}
\langle\phi(p_1)\phi(p_2)\phi^2(x)\rangle &= \frac{i\lambda}{2}\frac{1}{p_1^2 p_2^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k+p_1+p_2)^2} \\
&= -\frac{\lambda}{2}\frac{1}{p_1^2 p_2^2}\int_0^1 dx\int_0^\Lambda \frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2 - x(1-x)(p_1+p_2)^2)^2} \\
&= -\frac{\lambda}{(4\pi)^2}\frac{1}{p_1^2 p_2^2}\int_0^1 dx\int_0^\Lambda dk_E\frac{1}{(k_E^2 - x(1-x)(p_1+p_2)^2)^2} \\
&\overset{\Lambda\to\infty}{=} -\frac{\lambda}{(4\pi)^2}\frac{1}{p_1^2 p_2^2}\int^\Lambda dk\frac{1}{k} \\
&= -\frac{\lambda}{(4\pi)^2}\frac{1}{p_1^2 p_2^2}\log\Lambda
\end{aligned}$$

So we see that

$$Z_{\phi^2}(\Lambda) = -\frac{\lambda}{(4\pi)^2}\log\Lambda \tag{54}$$

which gives the expected value for γ_{ϕ^2}. One can similarly repeat the calculation for γ_{ϕ^4} and γ_J.

Problem 3 : (Peskin problem 12.3 (asymptotic symmetry))

a) The relevant Feynman diagrams for β_λ are shown in figure (1). Apart from the symmetry factors, the loop integral is the same as in the usual ϕ^4 theory. The calculation is done in detail in the book by *Peskin & Shroeder* (see p. 326-327). Using the notation of this book the above diagrams are equal to

$$-\left(\lambda^2 + \frac{1}{9}\rho^2\right)[iV(s) + iV(t) + iV(u)] - i\delta_\lambda, \tag{55}$$

where the factor in front of ρ^2 comes from the symmetry factor of the diagrams with two ρ-vertices. Following the usual steps, the beta function can be easily evaluated

$$\beta_\lambda = \frac{9\lambda^2 + \rho^2}{3(4\pi)^2} \tag{56}$$

For the other beta function, the relevant diagrams are shown in figure (2). They are equal to

$$\left(\frac{1}{3}\lambda\rho\right) 2iV(s) - \frac{1}{9}\rho^2\,[iV(t) + iV(u)] - i\delta_\rho. \tag{57}$$

As before the coefficients $\frac{1}{3}$ and $\frac{1}{9}$ are coming from the symmetry factors. In this case the beta function is

$$\beta_\rho = \frac{6\lambda\rho + 4\rho^2}{3(4\pi)^2} \tag{58}$$

b)Using the above beta functions it is easy to calculate the beta function for the ration ρ/λ as

$$\mu\frac{d(\rho/\lambda)}{d\mu} = \mu\frac{d\rho}{d\mu}\frac{1}{\lambda} - \mu\frac{d\lambda}{d\mu}\frac{\rho}{\lambda^2} = \frac{\rho}{3(4\pi)^2}\left(-\left(\frac{\rho}{\lambda}\right)^2 + 4\frac{\rho}{\lambda} - 3\right) \tag{59}$$

We see that there are two fixed point for $\rho/\lambda = 0, 1, 3$. If we start with $\rho/\lambda < 3$ then the ration will flow toward the other fixed point which satisfies $\rho = \lambda$.

c) The beta functions in $d = 4 - \epsilon$ dimensions are

$$\beta_\lambda = -\epsilon\lambda + \frac{9\lambda^2 + \rho^2}{3(4\pi)^2} \tag{60}$$

$$\beta_\rho = -\epsilon\rho + \frac{6\lambda\rho + 4\rho^2}{3(4\pi)^2} \tag{61}$$

However, the terms contacting ϵ cancel out in the beta function for the ratio and the result is the same as in part (b). The reason is that the ratio is still a dimensionless parameter. As before, the fixed points happen for $\rho/\lambda = 0, 1, 3$ and the RG diagram is shown in figure (3).

Figure 1: Relevant diagrams for β_λ. The single line correspond to ϕ_1 and the double line to ϕ_2.

Figure 2: Relevant diagrams for β_ρ. The single line correspond to ϕ_1 and the double line to ϕ_2.

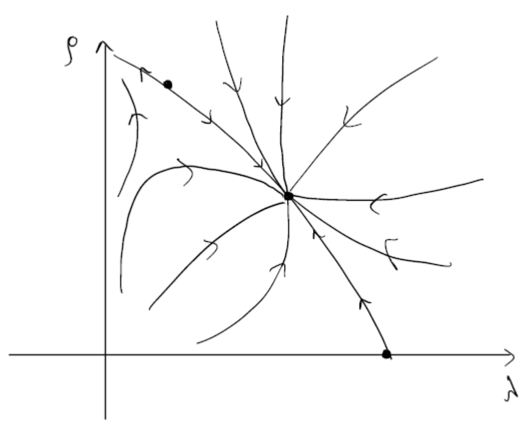

Figure 3: RG graph.