Homework 1

Review: renormalization chapters in Srednicki, such as chapters 14 and 16 (dimensional regularization and renormalization in scalar field theory), chapter 18 (power counting and renormalizability) and chapters 27 and 28 (introduction of MS scheme and of the renormalization group). I will also discuss again the paper by Polchinski, "Renormalization and effective Lagrangians". Sections 1 and 2 are required reading, the rest of the paper is very technical and you may want to read it at some later stage of your education.

Suggested reading: Cardy chapters 1, 3 and 5; Peskin chapters 12 and 13. We will mainly follow the calculational framework and notations of Srednicki, but Peskin and especially Cardy have nice physical discussions. A classic reference is the article by Wilson and Kogut, "The renormalization group and the ϵ expansion", section 12 can be read almost independently from the rest and will be briefly discussed in class.

1. (Warm-up review problems): Srednicki problems 27.1, 28.1

2. **Renormalization of composite operators.**

   Consider $\frac{\lambda}{4!}\phi^4$ theory in $d = 4$ dimensions. We want to define composite operators $[\phi^2](x)$, $[\phi^4](x)$, etc. as suitably renormalized versions of products of elementary fields at the same spacetime position x. These products are singular and we must subtract UV divergences. These are additional divergences that are *not* taken care by the usual renormalization procedure that renders the correlation functions of elementary fields finite. We will assume the usual renormalization procedure for ordinary correlation functions has been carried out to the requisite loop order, and focus on the additional renormalization needed for the composite operator.

   We will work in dimensional regularization in $d = 4 - \epsilon$ dimensions using the minimal subtraction scheme. Define the renormalized composite operators $[\phi^2]$, $[\phi^4]$, etc. by

   $$\phi_0^2(x) = Z_{\phi^2}\,[\phi^2](x)\,, \quad \phi_0^4(x) = Z_{\phi^4}\,[\phi^4](x)\,, \quad \text{etc.}. \tag{1}$$

   Here ϕ_0 is the bare field, λ the renormalized coupling and Z a divergent renormalization factor. (Recall that in MS scheme, renormalized quantities (such as λ) acquire a dependence on the renormalization scale μ, while bare quantities (such as λ_0 and ϕ_0) are μ-independent.)

   We calculate the renormalization factors $Z_{\mathcal{O}}$ by requiring that correlation functions containing the composite operator be finite. For example, we calculate Z_{ϕ^2} by considering

   $$G(p_1, p_2; x) = \langle \tilde{\phi}(p_1)\ \tilde{\phi}(p_2)\ [\phi^2](x)\rangle \tag{2}$$

and imposing that it is finite. (As customary, we have taken the elementary fields in momentum space, and kept the composite operator in position space.) In the MS scheme, we are instructed to simply subtract the poles in ϵ, so to leading order we expect

$$Z_{\mathcal{O}} = 1 + \frac{a_{\mathcal{O}}\lambda}{\epsilon} + O(\lambda^2) \tag{3}$$

where $a_{\mathcal{O}}$ is a numerical coefficient. The anomalous dimension $\gamma_{\mathcal{O}}$ is defined by

$$\gamma_{\mathcal{O}} \equiv \mu \frac{d}{d\mu} Z_{\mathcal{O}} \,. \tag{4}$$

Caveat: in the simple examples that we consider, it is ok to focus on individual composite operators. More generally, one needs to consider the phenomenon of operator mixing.

(a) To leading order in λ, calculate Z_{ϕ^2} and the anomalous dimension γ_{ϕ^2}. From the solution of the Callan-Symanzik equation in the massless theory, argue that the scaling dimension of $[\phi^2]$ at the Wilson-Fisher fixed point is given by

$$\Delta_{\phi^2} = d - 2 + \gamma_{\phi^2}(\lambda_*) \,, \tag{5}$$

where λ_* is the value of the coupling at the fixed point.

(b) Find (on general grounds) the relation between γ_{ϕ^2} and γ_m, the exponent which controls the evolution of the remormalized mass,

$$\gamma_m \equiv \frac{d}{d \log \mu} \log m(\mu) \,. \tag{6}$$

The scaling dimension controls the power in the two-point function,

$$\langle [\phi^2](x) \; [\phi^2](y)] \rangle = \frac{C}{|x-y|^{2\Delta_{\phi^2}}} \,. \tag{7}$$

(c) Repeat the exercise (a) for $[\phi^4]$, computing Z_{ϕ^4} and γ_{ϕ^4}.

(d) Calculate Δ_{ϕ^4} at the Wilson-Fisher fixed point (to leading order in ϵ). What is the relation between Δ_{ϕ^4} and $\beta'(\lambda_*)$, the derivative of the beta function evaluated at the fixed point? Explain.

(e) Consider now the theory of a complex scalar field in four dimensions with an interaction $\frac{\lambda}{4}(\phi\phi^*)^2$. Compute the leading-order anomalous dimension γ_J of the conserved Noether current associated to the $U(1)$ symmetry $\phi \to e^{i\alpha}\phi$. You should find that $\gamma_J = 0$. Argue that this is an exact result.

(f) To leading order in perturbation theory, it is equally convenient to work with an explicit momentum cutoff Λ, with Euclidean momenta satisfying $|p| \leq \Lambda$. One further defines a renormalization scale M by imposing some suitable normalization condition on $G(p_i; x)$ with external momenta $p_i \sim M$. The precise normalization condition only affects the finite part of the renormalized correlator and is thus inessential for the calculation of anomalous dimensions, which only depend on the divergence. Argue that

$$\gamma_{\mathcal{O}} = -\Lambda \frac{d}{d\Lambda} Z_{\mathcal{O}}(\Lambda, M) \,. \tag{8}$$

Repeat the one-loop calculations of γ_{ϕ^2}, γ_{ϕ^4} and γ_J using this scheme.

3. Peskin problem 12.3 (asymptotic symmetry).