Chapter 3

Elements of the Theory of Lie Groups and Algebras

3.1 Groups

A *group* is a set G in which a multiplication operation with the following properties is defined:

1. associativity: for all $a, b, c \in G$, $(ab)c = a(bc)$;

2. existence of a unit element $e \in G$, such that for all $a \in G$, $ae = ea = a$;

3. existence of an inverse element $a^{-1} \in G$ for each $a \in G$ such that $a^{-1}a = aa^{-1} = e$.

If the multiplication operation is commutative (i.e. $ab = ba$ for all $a, b \in G$), the group is said to be *Abelian*, otherwise it is *non-Abelian*.

Groups G_1 and G_2 are *isomorphic* if there exists a bijective mapping $f : G_1 \to G_2$ consistent with the multiplication operations

$$f(g_1 g_2) = f(g_1) f(g_2), \quad f(g^{-1}) = [f(g)]^{-1}.$$

In what follows, we shall write group isomorphisms as $G_1 = G_2$ and we shall often not distinguish between isomorphic groups.

A *subgroup* H of a group G is a subset H of G, which is itself a group with respect to the multiplication operation defined in G. In other words, for $h, h_1, h_2 \in G$, the product $h_1 h_2$ and the inverse element h^{-1} are defined; $h_1 h_2$ and h^{-1} are required to be elements of the set H, if $h, h_1, h_2 \in H$.

Let us give some examples.

1. The group $U(1)$ is the set of complex numbers z with modulus equal to unity, $|z| = 1$. Multiplication in $U(1)$ is the multiplication of complex numbers (since for $|z_1| = |z_2| = 1$ we have $|z_1 z_2| = 1$, multiplication is indeed an operation in $U(1)$). The unit element is $z = 1$ and the inverse element to $z \in U(1)$ is z^{-1} ($z^{-1} \in U(1)$, since $|z^{-1}| = 1$ for $|z| = 1$).

2. The group Z_n is the set of integers modulo n, i.e. integers k and $(k+n)$ are identified (in other words, the set Z_n consists of n integers $0, 1, \ldots, (n-1)$). Multiplication in Z_n is defined as addition of integers modulo n; in other words, if $0 \leq k_1 \leq n-1$, $0 \leq k_2 \leq n-1$, then

$$(k_1 + k_2) \pmod n = \begin{cases} k_1 + k_2 & \text{for } (k_1 + k_2) \leq n - 1 \\ k_1 + k_2 - n & \text{for } (k_1 + k_2) > n - 1. \end{cases}$$

   Subtraction modulo n is defined analogously. We note that addition modulo n is commutative. The unit element in Z_n is $k = 0$, the inverse to the element k is equal to

$$(-k) \pmod n = \begin{cases} 0 & \text{for } k = 0 \\ n - k & \text{for } 0 < k \leq n - 1. \end{cases}$$

Problem 1. *Show that the group Z_n is isomorphic to the group of nth roots of unity, i.e. the group consisting of all complex numbers z such that $z^n = 1$ (group multiplication is multiplication of complex numbers). Thus, Z_n is a subgroup of the group $U(1)$.*

3. The group $GL(n, C)$ is the set of complex $n \times n$ matrices with a non-zero determinant. Multiplication in $GL(n, C)$ is matrix multiplication; the unit element is the unit $n \times n$ matrix, the inverse element to $M \in GL(n, C)$ is the inverse matrix M^{-1} (which always exists because $\det M \neq 0$ by the definition of the group $GL(n, C)$).

Problem 2. *Describe the group $GL(1, C)$.*

The groups $U(1), Z_n$ and $GL(1, C)$ are Abelian groups, the groups $GL(n, C)$ with $n \geq 2$ are non-Abelian.

The groups in the following examples are subgroups of the group $GL(n, C)$. In other words, we are dealing with $n \times n$ matrices and the multiplication operation is matrix multiplication.

4. The group $GL(n, R)$ is the group of real matrices with non-zero determinant.

5. The group $U(n)$ is the group of unitary $n \times n$ matrices, i.e. such that

$$U^\dagger U = 1 \tag{3.1}$$

(we shall write the unit $n \times n$ matrix simply as 1; this is the matrix on the right-hand side of (3.1)). In order to see that $U(n)$ is indeed a group, we shall show that $U_1 U_2$ and U^{-1} are unitary if U_1, U_2, U are unitary. We have

$$(U_1 U_2)^\dagger (U_1 U_2) = U_2^\dagger U_1^\dagger U_1 U_2 = 1$$
$$(U^{-1})^\dagger (U^{-1}) = U U^\dagger = 1,$$

as required. We note that it follows from (3.1) that

$$|\det U|^2 = \det U \det U^\dagger = 1,$$

i.e. $|\det U| = 1$ for all $U \in U(n)$.

6. The group $SU(n)$ is the group of unitary matrices with unit determinant ($SU(n)$ is evidently a subgroup of $U(n)$). The fact that the group operations (matrix multiplication and inversion) are closed in $SU(n)$ (i.e. $SU(n)$ is indeed a group) follows from the equations

$$\det (U_1 U_2) = \det U_1 \det U_2 = 1$$
$$\det U^{-1} = (\det U)^{-1} = 1,$$

when $\det U_1 = \det U_2 = \det U = 1$.

7. The group $O(n)$ is the group of real orthogonal matrices, i.e. such that

$$O^T O = 1. \tag{3.2}$$

$O(n)$ is clearly a subgroup of $GL(n, R)$ and also of $U(n)$. We note that it follows from (3.2) that $\det O = \pm 1$, since

$$\det O^T O = \det O^T \det O = (\det O)^2 = 1.$$

Thus, the group $O(n)$ divides into two disjoint subsets ($\det O = +1$ and $\det O = -1$).

8. The group $SO(n)$ is the subgroup of the group $O(n)$ consisting of the matrices O with $\det O = +1$.

We note that the subset of $O(n)$ consisting of matrices with $\det O = -1$ is not a subgroup of $O(n)$. Indeed, if $\det O_1 = \det O_2 = -1$, then $\det (O_1 O_2) = +1$, i.e. this subset is not closed under matrix multiplication.

Let us continue with definitions which will be useful in the sequel. The *center* of a group G is the subset of G consisting of all elements $w \in G$, which commute with all elements of the group, i.e. such that for all $g \in G$

$$wg = gw. \tag{3.3}$$

The center of the group $W \subset G$ is a subgroup of G. Indeed, for $w_1, w_2 \in W$, we have

$$(w_1w_2)g = w_1(w_2g) = w_1gw_2 = g(w_1w_2),$$

so that $w_1w_2 \in W$. Multiplying (3.3) by w^{-1} on the left and on the right, we obtain

$$gw^{-1} = w^{-1}g,$$

so that the set W is closed under group operations.

Problem 3. *Describe the center of the group $SU(n)$ and show that it is isomorphic to Z_n.*

Problem 4. *Show that the center of the group $GL(n, C)$ consists of matrices of the form $\lambda \cdot 1$, where λ is an arbitrary, non-zero complex number and 1 is the unit $n \times n$ matrix (the non-trivial part of the problem is to show that all matrices which commute with any matrix in $GL(n, C)$ are multiples of unity).*

The *direct product* $G_1 \times G_2$ of the groups G_1 and G_2 is the set of pairs $\{g, h\}$ where $g \in G_1$ and $h \in G_2$, in which the multiplication operation and the inverse element take the form

$$\begin{aligned} \{g, h\}\{g', h'\} &= \{gg', hh'\} \\ \{g, h\}^{-1} &= \{g^{-1}, h^{-1}\}, \end{aligned}$$

the unit element is the pair $\{e_1, e_2\}$ where e_1 and e_2 are the unit elements in G_1 and G_2, respectively. Thus, $G_1 \times G_2$ is a group. We note that G_1 is a subgroup of the group $G_1 \times G_2$; more precisely, G_1 is isomorphic to the subgroup of the group $G_1 \times G_2$, consisting of the elements of the form $\{g, e_2\}$ for $g \in G_1$.

This definition is useful because, if one succeeds in identifying that some group G is a direct product of two other groups G_1 and G_2, then properties of the group G can be determined by studying the properties of the groups G_1 and G_2 individually.

A *group homomorphism* is a mapping f from a group G to a group G', consistent with the multiplication operations, i.e. for all $g, g_1, g_2 \in G$

$$f(g_1g_2) = f(g_1)f(g_2)$$

(the product g_1g_2 is given in the sense of multiplication in G, while the product $f(g_1)f(g_2)$ is given in the sense of multiplication in G'),

$$f(e) = e'$$

(e, e' are the units of G, G', respectively)

$$f(g^{-1}) = [f(g)]^{-1}$$

(the inverse elements on the left- and right-hand sides of the equation are taken in the sense of the groups G and G', respectively).

Here are some examples of homomorphisms.

1. A homomorphism from $SU(2)$ to $SU(3)$ under which the 2×2 matrix g ($g \in SU(2)$) is mapped to the 3×3 matrix of the form

$$\begin{pmatrix} & & 0 \\ & g & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.4}$$

   which clearly belongs to the group $SU(3)$.

2. The homomorphism from the group $G_1 \times G_2$ to the group G_1 under which the element $\{g, h\}$ is mapped to $g \in G_1$.

Suppose f is a homomorphism from G to G'. The set of all elements of G' which can be represented in the form $f(g)$ for some $g \in G$ is called the image of the homomorphism, $\mathrm{Im}\, f$. The set of elements $g \in G$ such that $f(g) = e'$ is called the kernel of the homomorphism, $\mathrm{Ker}\, f$. In the first example, $\mathrm{Im}\, f$ is the set of all matrices of the form (3.4), and $\mathrm{Ker}\, f$ is the unit 2×2 matrix. In the second example, $\mathrm{Im}\, f = G_1$, while $\mathrm{Ker}\, f$ is the set of elements of the form $\{e, h\}$, where h is arbitrary (i.e. $\mathrm{Ker}\, f = G_2$).

Problem 5. *Show that* $\mathrm{Im}\, f$ *is a subgroup of* G' *(*f *is a homomorphism from* G *to* G'*). Show that* $\mathrm{Ker}\, f$ *is a subgroup of* G.

Let us now introduce the concept of the (right) *coset space*, G/H of a group G by its subgroup H. Let H be a subgroup of a group G. Let us define equivalence in G: we shall say that g_1 is equivalent to g_2 ($g_1 \sim g_2$) if $g_1 = g_2 h$ for some $h \in H$. We recall that the following properties are required for an equivalence relation: 1) if $g_1 \sim g_2$, then $g_2 \sim g_1$; 2) if $g_1 \sim g_2$ and $g_2 \sim g_3$, then $g_1 \sim g_3$. In our case, these properties are easy to verify: 1) if $g_1 = g_2 h$, then $g_2 = g_1 h^{-1}$, i.e. $g_2 \sim g_1$, since $h^{-1} \in H$; 2) if $g_1 = g_2 h_{12}$, $g_2 = g_3 h_{23}$, then $g_1 = g_3(h_{23} h_{12})$, and $g_1 \sim g_3$ since $h_{23} h_{12} \in H$.

This equivalence relation allows us to divide the set G into disjoint sets (cosets): a coset consists of elements of G which are all equivalent to one another. We note that the coset containing the unit element $e \in G$ is the subgroup H itself.

The set of cosets is called the (right) coset space G/H.

Another definition of equivalence is possible: $g_1 \sim g_2$ if $g_1 = h g_2$ for some $h \in H$. This is used to construct the left coset space, which is sometimes denoted by $G \backslash H$.

Problem 6. *Take the subgroup isomorphic to $SO(2)$ in the group $SO(3)$ to be the group of matrices of the form*

$$\begin{pmatrix} & g & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g \in SO(2).$$

Show that there is a one-to-one correspondence between the coset space of $SO(3)$ by this subgroup and the two-dimensional sphere

$$SO(3)/SO(2) = S^2.$$

The coset space G/H is closely related to *homogeneous spaces.* A set A is said to be a homogeneous space with respect to the group G if the group G acts transitively on A, i.e. to each $g \in G$ there corresponds an invertible mapping of the space A to itself, such that

$$a' = F(g)a.$$

Here, the operation F is required to be consistent with the group operations, i.e.

$$\begin{aligned} F(g_1 g_2)a &= F(g_1)F(g_2)a \\ F(e)a &= a \\ F(g^{-1})a &= [F(g)]^{-1}a, \end{aligned} \tag{3.5}$$

where F^{-1} is a mapping from A to A which is the inverse of the mapping F; a is an arbitrary element of A; g, g_1, g_2 are arbitrary elements of the group G. In addition, it is required that for any pair $a, a' \in A$, there exists $g \in G$ such that

$$a' = F(g)a$$

(transitivity of the group action).

The *stationary subgroup* H for the element $a_0 \in A$ consists of all elements $h \in G$ which leave a_0 unchanged:

$$F(h)a_0 = a_0.$$

The fact that this set is a subgroup can be checked using (3.5); for example, if $h_1, h_2 \in H$, then

$$F(h_1 h_2)a_0 = F(h_1)F(h_2)a_0 = F(h_1)a_0 = a_0,$$

i.e. $h_1 h_2 \in H$.

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For a homogeneous space the stationary subgroups for all elements $a \in A$ are the same. Indeed, suppose H_0 and H_1 are stationary subgroups for the elements a_0 and a_1, respectively. Take $g \in G$ such that

$$a_1 = F(g)a_0.$$

Then an isomorphism of the subgroups H_0 and H_1 is given by the mapping

$$h' = ghg^{-1}, \tag{3.6}$$

where h is any element of H_0. First, we check that $h' \in H_1$, i.e. $F(h')a_1 = a_1$. We have

$$\begin{aligned} F(h')a_1 &= F(ghg^{-1})F(g)a_0 = F(g)F(h)F(g^{-1}g)a_0 \\ &= F(g)F(h)a_0 = F(g)a_0 = a_1, \end{aligned}$$

as required. The correspondence (3.6) is clearly one-to-one: the inverse mapping is given by the formula

$$h = g^{-1}h'g.$$

Finally, the mapping (3.6) is consistent with the group operations, for example, if $h_1, h_2 \in H$, then

$$gh_1h_2g^{-1} = gh_1g^{-1}gh_2g^{-1} = h_1'h_2',$$

where $h_{1,2}' = gh_{1,2}g^{-1}$.

Problem 7. *We define the action of the group $SO(3)$ on the two-dimensional sphere S^2 as follows. Let g be a matrix of $SO(3)$ and $\vec{a}$ a (unit) vector with components a_i, $i = 1, 2, 3$. Every such vector corresponds to a point on the unit two-dimensional sphere in three-dimensional Euclidean space. Define $F(g)\vec{a}$ to be the vector $\vec{b}$ with components $b_i = g_{ij}a_j$. Since $g^T g = 1$, we have $\vec{b}^2 = \vec{a}^2$, i.e. the action of $F(g)$ takes the sphere to the sphere. Show that $SO(3)$ acts transitively on S^2, and that the stationary subgroup of any point of the sphere S^2 is equal to $SO(2)$.*

If the group G acts transitively on the space A (i.e. A is a homogeneous space under G) then there is an isomorphism

$$A = G/H, \tag{3.7}$$

where H is the stationary subgroup of any element of the space A.

Indeed, let a_0 be some element of A, with H its stationary subgroup. Let us define the element $a_k \in A$ which corresponds to the coset $k \in G/H$, as follows

$$a_k = F(g_k)a_0, \tag{3.8}$$

where g_k is a representative of the coset k. The element a_k does not depend on the choice of representative g_k: if $g'_k = g_k h$ is another representative of the coset k, then $F(g'_k)a_0 = F(g_k)F(h)a_0 = F(g_k)a_0$. Thus, the mapping (3.8) is indeed a mapping from G/H to A. Let us check that it is one-to-one. Let a be some element of A. It is always possible to find some $g \in G$ such that $a = F(g)a_0$. It belongs to some coset $k \in G/H$. We show that if $F(g)a_0 = F(g')a_0$, then g and g' belong to the same coset (which proves the invertibility of the mapping (3.8)). From $F(g)a_0 = F(g')a_0$ we have the equation

$$F(g^{-1})F(g')a_0 = a_0,$$

which means that $g^{-1}g' \in H$, i.e. $g^{-1}g' = h$, where $h \in H$. Hence, $g' = gh$ and, consequently, g' and g belong to the same coset.

Illustrations of equation (3.7) are provided by assertions formulated in the following two problems.

Problem 8. *Show that $SO(n)/SO(n-1) = S^n$, where S^n is the n-dimensional sphere. Here, the embedding of $SO(n-1)$ in $SO(n)$ is given by*

$$\begin{pmatrix} SO(n-1) & 0 \\ 0 & 1 \end{pmatrix} \subset SO(n).$$

Problem 9. *Show that $SU(n)/SU(n-1) = S^{2n}$, where the embedding of $SU(n-1)$ in $SU(n)$ is defined analogously to in the previous problem.*

The subgroup H of the group G is said to be a *normal subgroup* of the group G if for all $h \in H$ and all $g \in G$

$$ghg^{-1} \in H.$$

If H is a normal subgroup, then $K = G/H$ is a group. Indeed, we construct the multiplication operation in K as follows. Let $k_1, k_2 \in K$, where k_1 and k_2 are cosets, and choose representatives of these, $g_1 \in k_1$, $g_2 \in k_2$. Then $k_1 k_2$ is the coset which contains the element $g_1 g_2$ of the group G. The unit $e_k \in K$ is the equivalence class which contains the unit element of the group G (observe that, from the definition of the coset space, it follows that $e_k = H$), and k^{-1} is the coset containing g^{-1}, where g is a representative of the coset k.

For these operations indeed to be operations in K, it is required that the result of their actions should not depend on the choice of representatives in the cosets. Let us verify this for the multiplication operation. Suppose $g_1, g'_1 \in k_1$, $g_2, g'_2 \in k_2$ are two sets of representatives, such that

$$g'_1 = g_1 h_1, \quad g'_2 = g_2 h_2,$$

where $h_1, h_2 \in H$. We check that $g_1' g_2' = g_1 g_2 h$ for some $h \in H$. We have

$$g_1' g_2' = g_1 h_1 g_2 h_2 = g_1 g_2 g_2^{-1} h_1 g_2 h_2.$$

But $g_2^{-1} h_1 g_2 \in H$, and so $(g_2^{-1} h_1 g_2) h_2$ also belongs to H, as required.

Problem 10. *Let $G = G_1 \times G_2$. Show that G_2 is a normal subgroup of the group G, and*

$$G_2 = G/G_1.$$

Problem 11. *Show that the subgroup $U(1)$ of the group $U(n)$, consisting of matrices which are multiples of unity, is a normal subgroup of the group $U(n)$.*

Problem 12. *Show that the center of any group is a normal subgroup of that group.*

Problem 13. *Show that*

$$U(n)/U(1) = SU(n)/Z_n,$$

where Z_n is the center of the group $SU(n)$.

3.2 Lie groups and algebras

For simplicity in what follows, we shall consider matrix groups whose elements are matrices (in other words, we shall consider subgroups of the group $GL(n, C)$); although the notions expounded here are of a general nature, they are most easily formulated for matrix groups.

In the space of $n \times n$ matrices the notion of neighborhood (topology) is introduced in a natural way: two matrices are said to be nearby if all their elements are nearby. We also introduce the differentiation of a family of matrices $M(t)$ with respect to a real parameter t: the elements of the matrix $(\frac{dM}{dt})_{ij}$ are the derivatives $\frac{d}{dt} M_{ij}(t)$ of the matrix elements $M_{ij}(t)$. Generally, the space of all complex $n \times n$ matrices can be viewed as a $2n^2$-dimensional (real) Euclidean space R^{2n^2}, whose coordinates are the $2n^2$ matrix elements $\mathrm{Re}\, M_{ij}$ and $\mathrm{Im}\, M_{ij}$. Smooth families of matrices are surfaces (manifolds) embedded in this Euclidean space. For example, a smooth family of matrices $M(t)$, depending on a real parameter t, is a curve in R^{2n^2}, and $\frac{dM}{dt}$ corresponds to the tangent vector to this curve.

Smooth (matrix) *groups* are groups which are smooth manifolds[1] in the space R^{2n^2} described above. These groups are called *Lie groups.*

The simplest non-trivial example of a Lie group is the group $U(1)$. It can also be understood as a matrix group by considering complex numbers as 1×1 matrices. The group $U(1)$ is a circle in the complex plane (in the two-dimensional real space of 1×1 matrices). The groups $U(n), SU(n), O(n), SO(n)$ are also Lie groups.

Two manifolds are said to be *homeomorphic* if there exists a smooth one-to-one mapping from one to the other.[2] For example, an ellipsoid is homeomorphic to a sphere, but a torus and a sphere are not homeomorphic.

Problem 14. *Show that the group $SU(2)$ is homeomorphic to the three-dimensional sphere S^3.*

For each point of a (curved) manifold of dimension k in $2n^2$-dimensional Euclidean space, one can define the tangent space to the manifold at that point: this is a real vector space of dimension k consisting of vectors tangent to the manifold at the given point.

The tangent space for a Lie group at the unit element is the *Lie algebra* of that Lie group (the unit element of the group; the unit matrix is a point of the group manifold). In other words, any curve $g(t)$ in the Lie group G is represented near unity in the form

$$g(t) = 1 + At + O(t^2), \tag{3.9}$$

where unity is the unit matrix, addition is matrix addition and A belongs to the Lie algebra of the group G. In what follows, the Lie algebra of the group G will be denoted by AG.

Equation (3.9) can be viewed as a definition of the algebra AG: its elements are all matrices A, such that (3.9) is a curve in G near unity. Let us check that the algebra AG is a real vector space. If $A \in AG$ corresponds to the curve $g(t)$, then the curve $g'(t) = g(ct)$, where c is a real number, corresponds to the element cA (because, $g'(t) = 1 + (cA)t + O(t^2)$). If $A_1, A_2 \in AG$ correspond to the curves $g_1(t), g_2(t)$ in the group, then the curve

$$g''(t) = g_1(t)g_2(t)$$

corresponds to the sum $(A_1 + A_2)$, since

$$g''(t) = (1 + A_1t + \cdots)(1 + A_2t + \cdots) = 1 + (A_1 + A_2)t + O(t^2).$$

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[1] Here and in what follows, we shall not refine the notion of smoothness. For example, we shall not encounter continuous manifolds which are not infinitely differentiable.

[2] Again, we shall not distinguish between homeomorphism (continuous but not necessarily differentiable mapping) and diffeomorphism.

Thus, the product of an element of AG by a real number and the sum of two elements of AG are also elements of the Lie algebra AG, i.e. A is a real vector space.

One more operation, commutation, is defined in a Lie algebra: the matrix $[A_1, A_2] = A_1A_2 - A_2A_1$ belongs to the algebra AG, if $A_1, A_2 \in AG$. Indeed, if

$$g_1(t) = 1 + A_1t + \cdots, \quad g_2(t) = 1 + A_2(t) + \cdots$$

then the curve

$$g(t) = g_1(\xi)g_2(\xi)g_1^{-1}(\xi)g_2^{-1}(\xi),$$

where $\xi = \sqrt{t}$, corresponds to the matrix $[A_1, A_2]$. To verify this with accuracy up to and including $t \equiv \xi^2$, we write,

$$g(t) = (1 + A_1\xi + \alpha_1\xi^2)(1 + A_2\xi + \alpha_2\xi^2)(1 - A_1\xi - \beta_1\xi^2)(1 - A_2\xi - \beta_2\xi^2), \tag{3.10}$$

where $\beta_{1,2} = \alpha_{1,2} - A_{1,2}^2$ (so that the matrix $(1 - A_1\xi - \beta_1\xi^2)$ is the inverse to the matrix $(1 + A_1\xi + \alpha_1\xi^2)$ with accuracy up to and including ξ^2). Collecting terms in (3.10), we obtain

$$g(t) = 1 + [A_1, A_2]\xi^2 + O(\xi^3),$$

so that to linear order in t,

$$g(t) = 1 + [A_1, A_2]t.$$

Thus, in a Lie algebra, in addition to multiplication by a number and addition, commutation is also defined.

Let us describe the Lie algebras of certain groups.

1. The $U(n)$ algebra (we shall sometimes denote specific groups and their algebras in the same way, provided this does not lead to confusion). Unitary matrices close to unity must have the property

   $$(1 + At + O(t^2))(1 + A^\dagger t + O(t^2)) = 1.$$

   Therefore

   $$A^\dagger = -A,$$

   i.e. the Lie algebra of the group $U(n)$ is the algebra of all anti-Hermitian matrices.

Problem 15. *Check explicitly that addition, multiplication by a number and commutation are defined in the set of anti-Hermitian matrices.*

2. The $SU(n)$ algebra. In addition to unitarity, the matrices of $SU(n)$ close to unity must satisfy the property

$$\det(1 + At + O(t^2)) = 1.$$

Since, for small t, $\det(1 + At) = 1 + (\operatorname{Tr} A)t + O(t^2)$, we have the condition

$$\operatorname{Tr} A = 0.$$

The $SU(n)$ algebra is the algebra of all anti-Hermitian matrices with zero trace.

3. The $SO(n)$ algebra. This is the algebra of all real matrices satisfying the condition

$$A^T = -A$$

(in other words, the matrices of the $SO(n)$ algebra are real antisymmetric matrices).

Problem 16. *Check that the operations of a Lie algebra (addition, multiplication by a real number and commutation) are closed in (a) the set of anti-Hermitian matrices with zero trace; (b) the set of real antisymmetric matrices.*

Since every anti-Hermitian matrix can be represented in the form iA, where A is an Hermitian matrix, the $SU(n)$ algebra in physics is often defined as the algebra of Hermitian matrices with zero trace, and elements of the group $SU(n)$ near unity are written in the form

$$g = 1 + iAt + O(t^2).$$

Problem 17. *Describe the Lie algebras of the groups $GL(n, C)$ and $GL(n, R)$.*

Two Lie algebras are isomorphic if there exists a one-to-one correspondence between them which preserves addition, multiplication by a real number and commutation.

Problem 18. *Show that the Lie algebras of $SU(2)$ and $SO(3)$ are isomorphic. Show that the relation between the groups is $SU(2)/Z_2 = SO(3)$, where Z_2 is the center of the group $SU(2)$. Thus, although locally* *(close to unity) the groups $SU(2)$ and $SO(3)$ are the same, on the whole (globally), they are different.*

The dimension of the vector space, which is a Lie algebra, is called the dimension of the algebra. It is equal to the dimension of the group manifold for the corresponding group. Let us find the dimension of the $SU(n)$ algebra. Arbitrary $n \times n$ matrices are characterized by $2n^2$ parameters. In the $SU(n)$ algebra, n^2 linear conditions are imposed upon them:

$$A^\dagger = -A$$

(this is a matrix condition, i.e. $2n^2$ conditions, however, only half of them are independent, since from $A_{ij} = -A^*_{ji}$ we have the complex conjugate condition $A^*_{ij} = -A_{ji}$). In addition, another linear condition is imposed:

$$\operatorname{Tr} A = 0$$

(this is a single condition, since, from $A^\dagger = -A$ it follows that all diagonal elements are imaginary). Thus, the dimension of the $SU(n)$ algebra is equal to $(n^2 - 1)$.

Problem 19. *Show that the dimension of the $SO(n)$ algebra is equal to $n(n-1)/2$.*

In a Lie algebra, as in a vector space, one can choose a basis. The elements of this basis are k matrices T_i ($i = 1, \ldots, k$; where k is the dimension of the algebra), called the generators of the Lie algebra and of the corresponding Lie group. Since the commutator $[T_i, T_j]$ belongs to the algebra, it decomposes in terms of generators, i.e.

$$[T_i, T_j] = C_{ijk} T_k,$$

where C_{ijk} are antisymmetric in the first two indices and real. The C_{ijk} are called the *structure constants* of the algebra, or, which amounts to the same thing, the structure constants of the group. Their values, of course, depend on the choice of basis.

For example, in the space of anti-Hermitian 2×2 matrices, one can choose a basis in the form $T_i = -\frac{i}{2}\tau_i$, where the τ_i are Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The structure constants of the $SU(2)$ algebra are obtained from the equations

$$[\tau_i, \tau_j] = 2i\varepsilon_{ijk}\tau_k$$

and are equal to ε_{ijk}. However, the $SU(2)$ algebra in physics is often defined as the algebra of Hermitian 2×2 matrices; the generators (the basis in this algebra) are chosen in the form

$$T_i = \frac{1}{2}\tau_i.$$

Here, the structure constants are purely imaginary and the commutation relation for generators takes the form

$$[T_i, T_j] = i\varepsilon_{ijk}T_k.$$

The generators of the $SU(3)$ algebra (in physics, this is also defined as the algebra of Hermitian matrices with zero trace) are chosen in the form $T_a = \frac{1}{2}\lambda_a$, $a = 1, 2, \ldots, 8$, where the λ_a are the Gell–Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Problem 20. *Show that these generators of the group $SU(3)$ are linearly independent.*

Problem 21. *Calculate the structure constants of the group $SU(3)$ in the Gell–Mann basis (as mentioned earlier, the structure constants of the group and the algebra are the same).*

A *Lie subalgebra* of a Lie algebra is a real vector space in A, which is closed under the operation of commutation (i.e. it is itself a Lie algebra). For example, one subalgebra in the $SU(3)$ algebra is the set of matrices of the form

$$\begin{pmatrix} A & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where A is a 2×2 matrix in the $SU(2)$ algebra. This subalgebra is clearly isomorphic to the $SU(2)$ algebra.

Problem 22. *Let H be a Lie subgroup of the Lie group G. Considering H as a Lie group, construct its Lie algebra AH. Show that AH is a subalgebra in AG.*

Let A and B be two Lie algebras of dimensions N_A and N_B, respectively; $T_1^A, \ldots, T_{N_A}^A$ a full set of generators of the algebra A; $T_1^B, \ldots, T_{N_B}^B$ a full set of generators of the algebra B. We shall assume that the elements of the algebra A are $n_A \times n_A$ matrices, and that the elements of the algebra B are $n_B \times n_B$ matrices. We construct the set of $(N_A + N_B)$ matrices of dimension $(n_A + n_B) \times (n_A + n_B)$ such that the first N_A matrices have the form

$$\begin{pmatrix} T_i^A & O_{n_A \times n_B} \\ O_{n_B \times n_A} & O_{n_B \times n_B} \end{pmatrix}, \quad i = 1, \ldots, N_A,$$

where $O_{k \times l}$ is the zero $k \times l$ matrix. We choose the remaining N_B matrices in the form

$$\begin{pmatrix} O_{n_A \times n_A} & O_{n_A \times n_B} \\ O_{n_B \times n_A} & T_q^B \end{pmatrix}, \quad q = 1, \ldots, N_B.$$

The real vector space in which this set of $(N_A + N_B)$ matrices forms a basis is called the *direct sum* of the algebras A and B and is denoted by $(A + B)$. Clearly, the study of the direct sum of two Lie algebras reduces to the study of each algebra individually.

Problem 23. *Let $G = G_1 \times G_2$ be the direct product of the Lie groups G_1 and G_2. Show that the Lie algebra of the group G is isomorphic to the direct sum of the Lie algebras of the groups G_1 and G_2 defined above, i.e.*

$$AG = AG_1 + AG_2.$$

The Lie subalgebra C in the Lie algebra A is said to be an invariant subalgebra (or ideal), if for all $c \in C$ and $a \in A$,

$$[c, a] \in C.$$

Problem 24. *Let the subgroup H be a normal subgroup in the Lie group G. Show that the Lie algebra of the group H is an invariant subalgebra in the Lie algebra G.*

Thus, it is convenient to study local (and only local) properties of Lie groups by considering the corresponding Lie algebras. The main concepts of group theory have analogies in the theory of Lie algebras. At the same time, Lie algebras are relatively simple objects, since they are vector (linear) spaces.

3.3 Representations of Lie groups and Lie algebras

A *representation T of a group G* in a linear space V is a mapping under which each element $g \in G$ is mapped to an invertible linear operator $T(g)$, acting on V; this mapping must be consistent with the group operations, so that the unit element of the group G is mapped to the unit operator and the following equations are satisfied:

$$\begin{aligned} T(g_1 g_2) &= T(g_1)T(g_2) \\ T(g^{-1}) &= [T(g)]^{-1}. \end{aligned} \tag{3.11}$$

Correspondingly, *a representation T of the Lie algebra AG* in the space V is a mapping under which each element $A \in AG$ is mapped to a linear operator $T(A)$, where this mapping is consistent with the operations in the algebra AG, i.e.

$$\begin{aligned} T(A+B) &= T(A)+T(B) \\ T(\alpha A) &= \alpha T(A) \\ T([A,B]) &= [T(A),T(B)] \end{aligned} \tag{3.12}$$

for all $A, B \in AG$ and any real number α. Here, the commutator of two operators acting on V is, as usual,

$$[T(A),T(B)] = T(A)T(B) - T(B)T(A).$$

If $T(G)$ is a representation of the Lie group G in the space V, then it can be used to construct a representation $T(AG)$ of the corresponding Lie algebra AG in the space V, according to the formula

$$T(1+\varepsilon A) = 1 + \varepsilon T(A), \tag{3.13}$$

where ε is a small parameter. On the left-hand side $T(1+\varepsilon A)$ is an operator corresponding to the element $(1+\varepsilon A) \in G$ which is close to the unit element of the group; on the right-hand side $T(A)$ is the operator corresponding to the element of the algebra $A \in AG$ for the representation $T(AG)$. We remark that not every representation of an algebra is generated by a representation of the group (see problem below).

Problem 25. *Check that the mapping of the algebra AG to the set of linear operators acting on V, defined by equation (3.13) is indeed a representation of the algebra AG, i.e. the properties (3.12) are satisfied.*

If V is a real vector space (i.e. only multiplication of vectors by a real number is defined in V), then a representation of a Lie group or algebra in it is said to be a *real representation.*

If $T(g)$ is a unitary operator for all $g \in G$, then the representation of the group is said to be a *unitary representation.* For a unitary representation of a group, the representation of the corresponding Lie algebra defined by formula (3.13) consists of anti-Hermitian operators,

$$[T(A)]^\dagger = -T(A)$$

for all $A \in AG$.

Let us fix a basis e_i in V. If $T(g)$ is the operator corresponding to the element $g \in G$ for the group representation $T(G)$, then its action takes e_i to some vector of V which can again be decomposed with respect to the basis e_i, so that

$$T(g)e_i = T_{ji}(g)e_j. \tag{3.14}$$

Thus, for a fixed basis, every element $g \in G$ is mapped to a matrix $T_{ji}(g)$. For a real representation the matrices $T_{ji}(g)$ are real, for a unitary representation the $T_{ji}(g)$ are unitary matrices. The matrix $T_{ji}(g)$ has dimension $n \times n$, where n is the dimension of the space V (and has nothing in common with the dimension of the group G). Any vector $\psi \in V$ can be represented in the form of a decomposition with respect to the basis e_i,

$$\psi = \psi_i e_i,$$

where the ψ_i are the components of the vector (numbers). Then

$$T(g)\psi = \psi_i(T(g)e_i) = \psi_i T_{ji} e_j.$$

Thus, the components of the vector $T(g)\psi$ are equal to

$$(T(g)\psi)_i = T_{ij}(g)\psi_j. \tag{3.15}$$

This relation explains the somewhat unusual choice of the order of the indices in (3.14).

From equation (3.15) it follows that

$$\begin{aligned} T_{ij}(g_1 g_2) &= T_{ik}(g_1)T_{kj}(g_2) && (3.16)\\ T_{ij}(e) &= \delta_{ij} && (3.17)\\ T_{ij}(g^{-1}) &= [T(g)]^{-1}_{ij}, && (3.18) \end{aligned}$$

i.e. a product of elements of the group corresponds to a product of matrices, the unit element to the unit matrix, and the inverse element to the inverse matrix. Indeed, for all ψ, we have

$$[T(g_1 g_2)\psi]_i = T_{ij}(g_1 g_2)\psi_j.$$

On the other hand,

$$[T(g_1 g_2)\psi]_i = [T(g_1)T(g_2)\psi]_i = T_{ik}(g_1)[T(g_2)\psi]_k = T_{ik}(g_1)T_{kj}(g_2)\psi_j,$$

which, by virtue of the arbitrariness of ψ, proves the equality (3.16). Properties (3.17) and (3.18) are proved analogously. We note that equations (3.16)–(3.18) could be used as the basis for the definition of a representation.

Representations of groups (or algebras) $T(G)$ and $T'(G)$ on the same space V are said to be *equivalent* if there exists an invertible operator S, acting on V, such that

$$T'(g) = ST(g)S^{-1}$$

for all $g \in G$.

Let W be a linear subspace in V. It is said to be an *invariant subspace* of the representation $T(G)$ acting on V if for all $\psi \in W$ and $g \in G$,

$$T(g)\psi \in W,$$

i.e. the action of any operator $T(g)$ does not lead out of the subspace W. The trivial invariant subspaces are the space V itself and the space consisting of the zero vector alone. The representation $T(G)$ is said to be an *irreducible* representation of the group G on V if there are no non-trivial invariant subspaces.

We now present examples of representations of Lie groups which are important for what follows.

1. The fundamental representation

Let G be a Lie group consisting of $n \times n$ matrices (for example, $SU(n)$ or $SO(n)$), and V an n-dimensional space of columns

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}. \tag{3.19}$$

The fundamental representation $T(g)$ acts on this space V as follows:

$$(T(g)\psi)_i = g_{ij}\psi_j.$$

Another definition is possible: let V be an n-dimensional space, e_i a basis in V; then the action of the operator $T(g)$ on the vector e_i is of the form

$$T(g)e_i = g_{ji}e_j.$$

Problem 26. *Show that these definitions are equivalent.*

We note that for the groups $SU(n)$ the fundamental representation is complex, while for the groups $SO(n)$ it is real.

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Problem 27. *Show that the fundamental representations of the groups $SU(n)$ and $SO(n)$ are irreducible.*

2. Representation conjugate to the fundamental representation

This is a representation of a group of $n \times n$ matrices on an n-dimensional space of columns (3.19), defined by the equation

$$(T(g)\psi)_i = g^*_{ij}\psi_j.$$

Equivalent definition: the conjugate of the fundamental representation is the representation on the space of rows $\phi = (\phi_1, \dots, \phi_n)$ such that

$$(T(g)\phi)_i = \phi_j g^\dagger_{ji}.$$

Problem 28. *Show that the fundamental representation of the group $SU(2)$ is equivalent to its conjugate.*

The fundamental representation of a Lie algebra AG and the conjugate of the fundamental representation of the algebra are defined analogously.

Problem 29. *As previously mentioned, the $SU(2)$ and $SO(3)$ algebras are isomorphic. Let T be the fundamental representation of the $SU(2)$ algebra. This corresponds to some representation of the $SO(3)$ algebra, to be denoted by $\bar{T}$. Show that no representation of the group $SO(3)$ generates the representation $\bar{T}$ of the $SO(3)$ algebra according to formula (3.13).*

3. The adjoint representation $\mathrm{Ad}\,(G)$ of the Lie group G

Let AG be the Lie algebra of the group G; we shall suppose that both the group G and the algebra AG consist of $n \times n$ matrices. The algebra AG is a real vector space, which is also the space of the adjoint representation. We define the action of the linear operator $\mathrm{Ad}\,(g)$, corresponding to the element $g \in G$, on a matrix $A \in AG$ as follows:

$$\mathrm{Ad}\,(g)A = gAg^{-1}.$$

For this to be a representation, the essential requirement is that gAg^{-1} should be an element of the algebra AG for all $A \in AG$ and $g \in G$. To see this, we construct a curve in the group G of the form

$$h(t) = gg_A(t)g^{-1},$$

where $g_A(t) = 1 + tA + \cdots$ is the curve defining the element $A \in AG$. We have $h(0) = 1$ and

$$h(t) = 1 + tA_h + \cdots,$$

where A_h is some element of the algebra AG. On the other hand,

$$h(t) = 1 + tgAg^{-1} + \cdots,$$

so that $gAg^{-1} = A_h \in AG$, as required.

The properties (3.11) are easily checked; for example,

$$\begin{aligned} \mathrm{Ad}\,(g_1 g_2)A &= (g_1 g_2)A(g_1 g_2)^{-1} \\ &= g_1 g_2 A g_2^{-1} g_1^{-1} = g_1(g_2 A g_2^{-1})g_1^{-1} \\ &= \mathrm{Ad}\,(g_1)\,\mathrm{Ad}\,(g_2)A \end{aligned}$$

(as always, $\mathrm{Ad}\,(g_1)\,\mathrm{Ad}\,(g_2)$ is understood as the consecutive action of first the operator $\mathrm{Ad}\,(g_2)$ and then the operator $\mathrm{Ad}\,(g_1)$).

From formula (3.13) it follows that the *adjoint representation of a Lie algebra* is such that the element $B \in AG$ is mapped to the operator $\mathrm{ad}\,(B)$ acting on elements A of AG (the space of the representation) as follows:

$$\mathrm{ad}\,(B)A = [B, A]. \tag{3.20}$$

Indeed, if $g = 1 + \varepsilon B$, then

$$\mathrm{Ad}\,(g)A = (1 + \varepsilon B)A(1 - \varepsilon B) = A + \varepsilon[B, A],$$

which, together with equation (3.13), which in this case has the form

$$\mathrm{Ad}\,(g)A = A + \varepsilon\,\mathrm{ad}\,(B)A,$$

leads to (3.20).

The matrices of the adjoint representation of a Lie algebra coincide with the structure constants. Indeed, by the definition of a matrix of a representation

$$\mathrm{ad}\,(t_i)t_j = T^{(i)}_{kj} t_k,$$

where t_j are generators (basis elements) in AG, and $T^{(i)}_{kj}$ is the matrix of the linear operator corresponding to the generator t_i. On the other hand,

$$\mathrm{ad}\,(t_i)t_j = [t_i, t_j] = C_{ijk} t_k,$$

where C_{ijk} are the structure constants of the algebra AG. Consequently,

$$T^{(i)}_{kj} = C_{ijk}. \tag{3.21}$$

We again stress that the adjoint representation is always real. This can be seen from (3.21), since the structure constants are real.

3.4 Compact Lie groups and algebras

Lie groups are smooth manifolds (matrix Lie groups are submanifolds in the space of all matrices of a specific dimension, see Section 3.2). *Compact Lie groups* are those whose manifolds are compact.

Problem 30. *Show that the groups $SU(n)$ and $SO(n)$ are compact, while $GL(n, C)$ and $GL(n, R)$ are not compact.*

Compact Lie algebras are Lie algebras corresponding to compact Lie groups.

The following theorem holds. A Lie algebra is compact if and only if it has a (positive-definite) scalar product, which is invariant under the action of the adjoint representation of the group.

In other words, in any compact Lie algebra AG, and only in a compact Lie algebra, there exists a bilinear form (A, B) such that for all $g \in G$ and all $A, B \in AG$

$$(\mathrm{Ad}\,(g)A, \mathrm{Ad}\,(g)B) = (A, B),$$

where for all $A \in AG$

$$(A, A) \geq 0,$$

with equality only for the zero element of the algebra, $A = 0$.

For matrix groups, the scalar product in the corresponding algebra is the trace

$$(A, B) = -\mathrm{Tr}\,(AB).$$

Its invariance under the adjoint representation is evident from the possibility of permuting matrices cyclically inside the trace symbol:

$$(gAg^{-1}, gBg^{-1}) = -\mathrm{Tr}\,(gAg^{-1}gBg^{-1}) = -\mathrm{Tr}\,(AB).$$

The non-trivial part of this theorem for matrix algebras is the positive definiteness of $-\mathrm{Tr}\,(A^2)$, for compact matrix Lie algebras and only for compact matrix Lie algebras.

Problem 31. *Show that $-\mathrm{Tr}\,(A^2)$ is positive for all non-zero A in the $SU(2)$ algebra. Show that $-\mathrm{Tr}\,(A^2)$ may be both positive and negative for the $GL(2, C)$ algebra.*

The existence in a Lie algebra of a positive-definite scalar product, which is invariant under the adjoint representation is of great importance for gauge theories, therefore precisely compact Lie groups and algebras are used in their construction.

In what follows, we shall consider only compact Lie groups and algebras and will not stipulate this explicitly each time.

In an algebra, generators can be chosen so as to form an orthonormal basis. The normalization is usually taken as follows:

$$\mathrm{Tr}\,(t_i t_j) = -\frac{1}{2}\delta_{ij}. \tag{3.22}$$

In this basis the structure constants are antisymmetric with respect to all three indices. Indeed, by definition,

$$[t_i, t_j] = C_{ijk} t_k,$$

and from equation (3.22), it follows that

$$C_{ijk} = -2\mathrm{Tr}\,[t_i, t_j] t_k = -2[\mathrm{Tr}\,(t_i t_j t_k) - \mathrm{Tr}\,(t_j t_i t_k)].$$

We compare this expression with the same quantity but with the indices k, j transposed:

$$C_{ikj} = -2[\mathrm{Tr}\,(t_i t_k t_j) - \mathrm{Tr}\,(t_k t_i t_j)].$$

With cyclic permutation of the matrices within the trace symbol, we have

$$C_{ikj} = -2[\mathrm{Tr}\,(t_j t_i t_k) - \mathrm{Tr}\,(t_i t_j t_k)],$$

which coincides with $-C_{ijk}$. Thus,

$$C_{ikj} = -C_{ijk}$$

and C_{ijk} is fully antisymmetric, by virtue of the antisymmetry in the first two indices.

Problem 32. *Suppose A is an invariant subalgebra of the compact algebra B. Let $A_\perp$ be the orthogonal complement of A in B (we recall that A is a vector space with a scalar product). Show that $A_\perp$ is also an invariant subalgebra and*

$$B = A + A_\perp$$

in the sense of a direct sum of Lie algebras.

All *Abelian* compact Lie algebras are direct sums of $U(1)$ algebras.

A compact Lie algebra is said to be *semi-simple* if it does not contain an Abelian invariant subalgebra. A compact Lie algebra is said to be *simple* if it does not contain any invariant subalgebras whatsoever.

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The following statement holds. Any compact Lie algebra A is uniquely representable in the form of a direct sum of a certain number of $U(1)$ subalgebras and simple subalgebras.

$$A = U(1) + U(1) + \cdots + U(1) + A_1 + \cdots + A_n, \qquad (3.23)$$

where the A_n are simple algebras. Thus, the study of compact Lie algebras reduces to the study of simple Lie algebras. Equation (3.23) implies that *locally* every compact Lie group is represented in a unique way in the form of a direct product

$$G = U(1) \times U(1) \times \cdots \times U(1) \times G_1 \times \cdots \times G_n,$$

where the G_n are simple groups (simple Lie groups are those which correspond to simple algebras). The global (i.e. valid for groups as a whole) version of this statement is somewhat more complicated; we shall not use it and we shall not formulate it here.[3]

In the case of a simple compact Lie algebra, there exists just one invariant positive-definite scalar product (up to multiplication by a number). If the algebra is semi-simple, the full set of invariants is described as follows. Suppose, for example,

$$A = A_1 + A_2.$$

Then any vector $B \in A$ has the form

$$B = B_1 + B_2 \quad B_1 \in A_1, B_2 \in A_2. \qquad (3.24)$$

Let $(,)_1$ be an invariant scalar product in A_1 and $(,)_2$ an invariant scalar product in A_2. Then all invariant scalar products of vectors of the form (3.24) have the form

$$(B, B') = \alpha_1 (B_1, B_1')_1 + \alpha_2 (B_2, B_2')_2,$$

where α_1 and α_2 are arbitrary positive numbers. In other words, positive-definite quadratic invariants (relative to the adjoint representation) in a sum of simple algebras are linear combinations of quadratic invariants in each of the simple algebras with arbitrary positive coefficients.

The complete list of simple Lie algebras is known. In addition to the algebras with which we have become acquainted $SU(n)$, $n = 2, 3, \ldots$, and $SO(n)$, $n = 5, 7, 8 \ldots$, ($SO(3)$ and $SO(4)$ reduce to $SU(2)$ and $SO(6)$ to $SU(4)$), there is an infinite set of matrix algebras $Sp(n, C)$, $n = 3, 4, \ldots$, and a finite number (five) of so-called exceptional algebras G_2, F_4, E_6, E_7, E_8.

[3] That the analogous assertion to (3.23) for groups as a whole is not completely trivial can be seen from the fact that different Lie groups can correspond to the same Lie algebra. An example is provided by the groups $SU(2)$ and $SO(3)$.

Problem 33. *Show that the $SO(4)$ algebra is isomorphic to the $(SU(2)+SU(2))$ algebra.*

In the construction of models in particle physics, the groups $SU(n)$ are most often used; the symmetries $SO(n)$ are occasionally considered, while the groups E_6 and E_8 are used in the construction of unified theories of the strong, weak and electromagnetic interactions.

The following statement holds for representations. *Any representation of a compact Lie group is equivalent to a unitary representation*, and representations of the Lie algebra are equivalent to anti-Hermitian representations. This property is also important for the theory of gauge fields; in what follows, we shall always assume that group representations are unitary.

As previously mentioned, when the group $SU(n)$ is considered in physics, it is customary to use *Hermitian* (rather than anti-Hermitian) generators (if A is an anti-Hermitian matrix, then $A = iB$, where B is Hermitian). Then, every element of the algebra is represented in the form

$$A = iA^a t_a,$$

where t_i are Hermitian matrices, and the A^a are real coefficients. Elements close to the unit element of the Lie group are written in the form

$$g = 1 + i\varepsilon^a t_a,$$

where ε^a are small real parameters. The relations between the generators explicitly contain the imaginary unit, i,

$$[t_a, t_b] = iC_{abc}t_c,$$

where C_{abc} are fully antisymmetric real structure constants of the algebra. For complex representations of $SU(n)$ and other algebras, Hermitian generators $T(t_a) \equiv T_a$ such that

$$[T_a, T_b] = iC_{abc}T_c$$

are also used.

We shall usually employ this convention in the following study.

Problem 34. *Show that $SU(n)$, $n = 2, 3, \ldots$, and $SO(n)$, $n = 5, 6, \ldots$, are simple groups.*