Lecture 1

Quantum field theory arises "inevitably" from the marriage of quantum mechanics and special relativity.

I am taking for granted a working knowledge of quantum mechanics and of special relativity. A mastery of the covariant formalism (manipulations with Lorentz indices, Minkowski metric etc.) is required. If you need to brush up on this, there are many standard textbooks. For example, most general relativity books start with a review of special relativity, see *e.g.* the first three chapters of Schutz' book "A first course in general relativity".

I presented various arguments for the *impossibility* of *single-particle* relativistic quantum mechanics. I do not necessarily recommend that you spend much time on these topics now. After you are comfortable with the main ideas QFT, it will be instructive to go back to them. My lecture was partly based on chapter 1 of Srednicki and also on chapter 1 of the book by F. Strocchi, "An introduction to non-perturbative foundations of QFT".

- Heuristic reasoning from $\Delta p \Delta q \sim \hbar$ and $E = mc^2$ leading to creation of particle/anti-particle pairs. Necessary to have a *multi-particle* framework.
- Inconsistency of *single-particle* interpretation of various covariant generalization of the Schroedinger equation.
  - Most naive attempt, the relativistic Schroedinger equation (setting $c = \hbar = 1$)
    $$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \sqrt{m^2 - \nabla^2}\psi(\mathbf{x},t)\,.$$
    The square root is very problematic: non-locality in space. There exist no solutions compact support. If they existed, their Fourier transform $\tilde{\psi}(\mathbf{p})$ would be an analytic function, but this is in contradiction with
    $$i\frac{\partial}{\partial t}\tilde{\psi}(\mathbf{p},t) = \sqrt{m^2 + \mathbf{p}^2}\tilde{\psi}(\mathbf{p},t)\,.$$
    The non-existence of localized solutions is incompatible with relativistic causality (one should be able to localize observables their support must evolve in time with velocity less than c).
    One is then led to look for local relativistic equations, such as the Klein-Gordon and Dirac equation.
  - Klein-Gordon equation, of second order in spacetime derivatives. Single component $\phi(x)$ describing a spin zero particle:
    $$(\partial^2 - m^2)\phi(x) = 0\,.$$

– Dirac equation, of first order in spacetime derivatives. Four-component $\psi(x)$ describing a spin 1/2 particle:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0\,.$$

Both the Klein-Gordon equation and the Dirac equation have negative energy solutions, making a conventional interpretation of $\phi(x)$ and $\psi(x)$ as single-particle wavefunctions very problematic. The presence of negative energy solutions is an inevitable consequence of the relativistic dispersion relation $E^2 - \mathbf{p}^2 = m^2$, which has two solutions a priori on equal footing, namely $E = \pm\sqrt{\mathbf{p}^2 + m^2}$. (And we saw, choosing *by hand* the plus sign is not an option.) In the presence of interactions (for example, by coupling to an external electromagnetic field by minimal coupling $p^\mu \to p^\mu - eA^\mu$) positive energy states will transition to negative energy states. There is no stable ground state.

Dirac mistakenly thought that his equation was conceptually superior to the Klein-Gordon equation. His initial reasoning involved the requirement of a positive definite probability density. This can be achieved if the differential operator is linear in the time derivative (as in the Dirac equation), but not if it is quadratic, as in the Klein-Gordon equation. Indeed, the conserved current associated to the Dirac equation is $j_\mu = \bar{\psi}\gamma_\mu\psi$, so $j_0 = \bar{\psi}\gamma_0\psi = \psi^\dagger\psi \geq 0$. By contrast, the conserved current associated to the Klein-Gordon equation is $j_\mu = i/2m(\phi^*\partial_\mu\phi - (\partial_\mu\phi^*)\phi)$. For solutions of energy E, which have time dependence e^{-iEt}, we find $j_0 = E/m$, so $j_0 < 0$ when $E < 0$. We now understand this issue to be a red herring. As we shall see, in QFT these conserved currents are not associated to probability but to (electric) *charge*, which can have either sign.

How inconsistencies arise in the single-particle interpretation of the Dirac equation is explained in detail in chapter 2 of Itzykson and Zuber.

The history of the Dirac equation and of Dirac's wildly creative resolution of the impasse are discussed for example in chapter 1 of Weinberg's book. (If you like history of physics, you will enjoy Schweber's beautiful book "QED and the men who made it"). Dirac quickly realized that the single-particle interpretation of his equation was untenable and developed a multi-particle interpretation based on the "Dirac sea" : negative energy states are supposed to be "already occupied". Transitions from positive energy states are forbidden by Pauli exclusion principle. A *hole* in the Dirac sea is interpreted as an antiparticle. Dirac's hole theory, when treated with care, gives correct results, but it is a very cumbersome approach and it is best thought of as a historical curiosity. We need a formalism that is able to deal with spin zero and spin one half particle on the same footing.

In summary, the single-particle interpretation of both the Dirac and Klein-Gordon equation is simply *wrong*. Dirac's reinterpretation of his equation (with the Dirac sea) is a radical departure from single-particle quantum

mechanics. It points the way to the modern viewpoint: quantum field theory.

There seem to be a priori two routes towards a consistent relativistic quantum mechanics:

1. Promote time T to an operator $\hat{T}$, on equal footing with the familiar position operator $\hat{X}^i$ (usually called $\hat{q}^i$). In covariant language, we have $\hat{X}^\mu(\tau) = (\hat{T}(\tau), \hat{X}^i(\tau))$, where τ is proper time.

2. Demote the position operator to an ordinary label, just like time is treated in ordinary non-relavistic QM. Both time and position are labels on operators, the *quantum fields* $\hat{\varphi}_a(x^\mu)$, which are operator-valued functions of spacetime.

Although at this stage this is far from obvious, when developed correctly the two routes are in fact completely equivalent – using one or the other viewpoint is a matter of taste or of convenience. By a horrible terminology, approach 1 is sometimes called the first-quantized viewpoint, and approach 2 the second-quantized viewpoint. In both cases we are quantizing exactly *once*. Approach 1 emphasizes the role of particles, approach 2 emphasizes the role of fields. Particles are quanta of fields, and fields are coherent states of quanta. Both ideas are equally fundamental, in fact they are *dual* to each other. Don't worry if you don't follow these vague philosophical pronouncements now, hopefully you will one day! We will follow approach 2, which is the conventional one, and the most convenient in most applications.