PHY 610 QFT, Spring 2017

HW9 Solutions

1. For d=3, we may try:

$$\gamma^0 = \sigma^2$$
$$\gamma^1 = i\sigma^1$$
$$\gamma^2 = i\sigma^3$$

with σ here the Pauli matrices. And then the S matrices are:

$$S^{01} = \frac{i}{2}\gamma^0\gamma^1 = \frac{i}{2}\sigma^3$$
$$S^{02} = \frac{i}{2}\gamma^0\gamma^2 = \frac{-i}{2}\sigma^1$$
$$S^{12} = \frac{i}{2}\gamma^1\gamma^2 = \frac{-1}{2}\sigma^2$$

For d=4, we can take advantage of the tensor product:

$$\gamma^0 = \sigma^2 \otimes \sigma^1$$
$$\gamma^1 = i\sigma^1 \otimes \sigma^1$$
$$\gamma^2 = i\sigma^3 \otimes \sigma^1$$
$$\gamma^3 = I \otimes i\sigma^3$$

And

$$S^{01} = \frac{i}{2}\gamma^0\gamma^1 = \frac{i}{2}\sigma^3 \otimes I$$
$$S^{02} = \frac{i}{2}\gamma^0\gamma^2 = \frac{-i}{2}\sigma^1 \otimes I$$
$$S^{03} = \frac{i}{2}\gamma^0\gamma^3 = \frac{1}{2}\sigma^2 \otimes \sigma^1$$
$$S^{12} = \frac{i}{2}\gamma^1\gamma^2 = \frac{1}{2}\sigma^2 \otimes I$$
$$S^{13} = \frac{i}{2}\gamma^1\gamma^3 = \frac{-1}{2}\sigma^1 \otimes \sigma^2$$
$$S^{23} = \frac{i}{2}\gamma^2\gamma^3 = \frac{-1}{2}\sigma^3 \otimes \sigma^2$$

In d=3 Euclidean space, we could write down a spinor representation with Pauli matrices. Yet taking $C = \sigma^2$, we have $C\sigma^i C = -(\sigma^i)^*$, which indicates that this is a pseudoreal representation representation. Therefore there is no Majorana basis.

2. (a) The first integral vanishes because it is odd in q^μ. Indeed, under the change in dummy variable $q^\mu \to -q^\mu$, the measure and $f(q^2)$ are invariant and

$$\int d^d q \; q^\mu \; f(q^2) = \int d^d q \; -q^\mu \; f(q^2).$$

The second integral is some Lorentz covariant matrix with two symmetric indices, and thus must be proportional to $g^{\mu\nu}$. Contracting the integral with $g_{\mu\nu}$ allows one to find the constant of proportionality. We get

$$\int d^d q \; q^\mu q^\nu \; f(q^2) = \frac{1}{d} g^{\mu\nu} \int d^d q \; q^2 \; f(q^2).$$

(b) This integral is some Lorentz covariant matrix with four totally symmetric indices, and must be proportional to $(1/4) g^{(\mu\nu} g^{\rho\sigma)} = g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}$. The constant of proportionality can be found by contracting with $g_{\mu\nu} g_{\rho\sigma}$. The result is

$$\int d^d q \; q^\mu q^\nu q^\rho q^\sigma \; f(q^2) = \frac{1}{d(d+2)} (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \int d^d q \; (q^2)^2 \; f(q^2).$$

3. We are to comptue the $O[\lambda]$ correction to the propagator in φ^4 theory, with action

$$\mathcal{L} = -\frac{1}{2} Z_\varphi (\partial_\mu \varphi)^2 - \frac{1}{2} Z_m m^2 \varphi^2 - \frac{1}{4!} Z_\lambda \lambda \tilde{\mu}^\epsilon \varphi^4.$$

We have included a factor of $\tilde{\mu}^\epsilon$ in the φ^4 coupling, so that in $d = 4 - \epsilon$, λ remains dimensionless. (φ has dimension $1 - \epsilon/2$ in $d = 4 - \epsilon$.) The relevant diagrams are

$$i\Pi(k^2) = \qquad\qquad + $$

$$= \frac{1}{2}(-i\lambda\tilde{\mu}^\epsilon) \int \frac{d^d l}{(2\pi)^d} \frac{-i}{l^2 + m^2 - i\epsilon} - i((Z_\varphi - 1)k^2 + (Z_m - 1)m^2).$$

Now, let us compute the integral, which is divergent for $d \geq 2$. The dimensional regularization scheme is to compute this integral in $d < 2$, and analytically continuing the resulting expression to $d = 4 - \epsilon$. One may simply use the master formula (14.27), but I will do the integral here step by step. These steps illustrate the generic case except that it does not require a Feynman reparametrization of the denominator.

First, Wick rotate the time component. Recall that the poles of the propagator sit at $\pm(\omega - i\epsilon)$ in the l^0 plane, so we may rotate the l^0 contour clockwise by $\pi/2$. The time component then becomes a euclidean component $l^0 = i\bar{l}^d$. The i arising from the measure then cancels the $-i$ in the propagator, and the $i\epsilon$ may be set to zero since there are no poles in the euclidean propagator. This leaves us with

$$I := \int \frac{d^d l}{(2\pi)^d} \frac{-i}{l^2 + m^2 - i\epsilon} = \frac{d^d \bar{l}}{(2\pi)^d} \frac{1}{\bar{l}^2 + m^2}.$$

(Note that in the mostly minus metric convention, l^2 will become $-\bar{l}^2$.)

Next, we use the rotational invariance of the integral to write it in polar coordinates. The surface area of the unit d ball is $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$. This may be derived by noting that

$$\pi^{d/2} = \int e^{-x_1^2-\ldots-x_d^2}\, dx_1 \ldots dx_d = \Omega_d \int_0^\infty e^{-x^2} x^{d-1} dx = \Omega_d \int_0^\infty e^{-x^2} (x^2)^{d/2-1} \frac{1}{2} dx^2 = \frac{\Omega_d}{2}\Gamma(d/2).$$

(Recall that $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$.) Thus

$$\begin{aligned} I =& \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \bar{l}^{d-1} \frac{d\bar{l}}{(2\pi)^d} \frac{1}{\bar{l}^2 + m^2} \\ =& \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \frac{1}{2} \frac{d\bar{l}^2}{(2\pi)^d} (\bar{l}^2)^{d/2-1} \frac{1}{m^2} \frac{1}{1+\bar{l}^2/m^2} \\ =& \frac{\pi^{d/2}}{\Gamma(d/2)} (m^2)^{d/2-1} \int_0^\infty \frac{d(\bar{l}^2/m^2)}{(2\pi)^d} \frac{(\bar{l}^2/m^2)^{d/2-1}}{1+\bar{l}^2/m^2}. \end{aligned}$$

Now, we use the identity

$$\int_0^\infty dt \frac{t^{a-1}}{(1+t)^{a+b}} = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

so

$$\begin{aligned} I =& \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(d/2)} (m^2)^{d/2-1} \frac{\Gamma(d/2)\Gamma(1-d/2)}{\Gamma(1)} \\ =& \frac{1}{(4\pi)^{d/2}} (m^2)^{d/2-1} \Gamma(1-d/2). \end{aligned}$$

The analysis so far has been valid for $d < 2$, where the integral is convergent. However, the expression we have written above is actually valid away from the poles of Γ, the nonpositive integers, ie. for d not equal to a positive even number. We therefore analytically continue it, in particular to $d = 4 - \epsilon$, which is the region we are interested in.

We therefore have

$$\begin{aligned} i\Pi(k^2) =& -\frac{i\lambda m^2 \tilde{\mu}^\epsilon}{2} \frac{1}{(4\pi)^2} \left(\frac{m^2}{4\pi}\right)^{-\epsilon/2} \Gamma(-1+\epsilon/2) - i((Z_\varphi - 1)k^2 + (Z_m - 1)m^2) \\ =& i\frac{\lambda m^2}{16\pi^2} \left(\frac{1}{\epsilon} + \frac{1}{2} + \frac{1}{2}\log \frac{m^2}{4\pi e^\gamma \tilde{\mu}^2}\right) - i((Z_\varphi - 1)k^2 + (Z_m - 1)m^2). \end{aligned}$$

In the second equality, we have used the expansion of Γ near -1 (14.26)

$$\Gamma(-1+\epsilon/2) = -\frac{2}{\epsilon} + \gamma - 1 + O[\epsilon]$$

and used that

$$X^\epsilon = e^{\epsilon \log X} = 1 + \epsilon \log X + O[\epsilon]^2.$$

In order that $\Pi(k^2)$ is finite and μ-independent to order λ, while satisfying the renormalization conditions $\Pi(k^2 = -m^2) = 0$, $\Pi'(k^2 = -m^2) = 0$, we must choose

$$Z_\varphi - 1 = 0 + O[\lambda]^2, \qquad Z_m - 1 = \frac{\lambda}{16\pi^2}\left(\frac{1}{\epsilon} + \frac{1}{2} + \frac{1}{2}\log\frac{m^2}{\mu^2}\right),$$

with $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$.

4. (a) $[\Psi] = (d-1)/2$
   (b) $[g_n] = d - n(d-1) = n - (n-1)d$
   (c) A scalar field has dimension $[\varphi] = d/2 - 1$ from the kinetic term $(\partial_\mu \varphi)^2$, so $[g_{m,n}] = d - n(d-1) - m(d/2-1) = n + m - (n + m/2 - 1)d$.
   (d) The only renormalizable interaction in $d = 4$ is $g_{1,1}\varphi\bar{\Psi}\Psi$ (the Yukawa interaction).

5. For d=2 case, we have $[\phi] = 0$ and $V = g_1 f_1(\phi)\partial_\mu\phi\partial^\mu\phi + g_2 f_2(\phi)$, with $g_1 = [0]$, $[g_2] = 2$ and f_1, f_2 arbitrary functions of ϕ.
   For d=3, $[\phi] = \frac{1}{2}$. By setting $V_n = g_n\phi^n$ and solving $[g_n] = 3 - \frac{n}{2} \geq 0$, we know that the renormalizable interactions are in the form of $g_3\phi^3$, $g_4\phi^4$, $g_5\phi^5$ and $g_6\phi^6$.
   For d=4, $[g_n] = 4 - n \geq 0$, $V = g_4\phi^4$.
   For d=5, $[g_n] = 5 - \frac{3n}{2} \geq 0$, $V = g_3\phi^3$.
   For d=6, $[g_n] = 6 - 2n \geq 0$, $V = g_3\phi^3$.