PHY 610 QFT, Spring 2017

HW8 Solutions

1. Since β is numerically equal to γ^0, it commutes with γ^0 while anticommuting with γ^i and γ_5. To derive a few nice rules, we use the fact that γ^0 and γ_5 are hermitian in Srednicki's conventions while everything else is antihermitian.

$$\begin{aligned} \overline{\gamma^0} &= \beta\gamma^{0\dagger}\beta = \beta\gamma^0\beta = \gamma^0 \\ \overline{\gamma^i} &= \beta\gamma^{i\dagger}\beta = -\beta\gamma^i\beta = \gamma^i \\ \overline{i\gamma_5} &= -\beta i\gamma_5^\dagger = -\beta i\gamma_5\beta = i\gamma_5 \end{aligned}$$

The bar operator is order-reversing ($\overline{AB} = \beta B^\dagger A^\dagger \beta = \beta B^\dagger \beta\beta A^\dagger \beta = \overline{B}\ \overline{A}$) and this will help in evaluating the others.

$$\begin{aligned} \overline{\gamma^\mu\gamma_5} &= \overline{\gamma_5}\,\overline{\gamma^\mu} = -\gamma_5\gamma^\mu = \gamma^\mu\gamma_5 \\ \overline{S^{\mu\nu}} &= \overline{\frac{i}{4}[\gamma^\mu,\gamma^\nu]} = -\frac{i}{4}[\overline{\gamma^\nu},\overline{\gamma^\mu}] = -\frac{i}{4}[\gamma^\nu,\gamma^\mu] = S^{\mu\nu} \\ \overline{i\gamma_5 S^{\mu\nu}} &= -i\overline{S^{\mu\nu}}\,\overline{\gamma^5} = iS^{\mu\nu}\gamma_5 = i\gamma_5 S^{\mu\nu} \end{aligned}$$

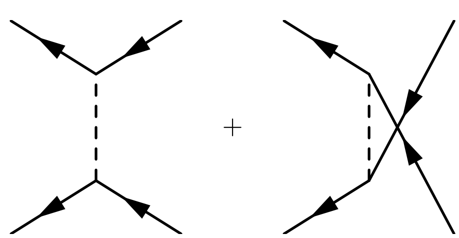
2. (a) Relations derived in chapter 40 (based on general properties of momentum and angular momentum) reveal three transformation laws for Dirac spinors:

$$\begin{aligned} P^{-1}\Psi(\vec{x},t)P &= i\beta\Psi(-\vec{x},t) \\ T^{-1}\Psi(\vec{x},t)T &= \mathcal{C}\gamma_5\Psi(\vec{x},-t) \\ C^{-1}\Psi(\vec{x},t)C &= \mathcal{C}\bar{\Psi}^{\mathrm{T}}(\vec{x},t) \end{aligned}$$

From this, it follows that the $\bar{\Psi}\Psi$ bilinear is even under C, P and T. The scalar φ must therefore be as well for the $g\varphi\bar{\Psi}\Psi$ interaction to preserve these symmetries.

(b) The same transformations tell us that $\bar{\Psi}i\gamma_5\Psi$ is even under C but odd under P and T. With a $g\varphi\bar{\Psi}i\gamma_5\Psi$ interaction, we therefore need the field φ to be odd under P (pseudoscalar) and odd under T (possibly a property without a name).

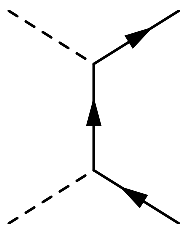
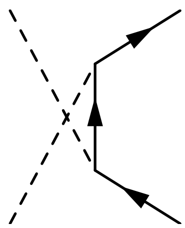
3. The tree level graphs for $e^+e^+ \to e^+e^+$ (incoming momenta and spins $p_1, s_1; p_2, s_2$, outgoing momenta and spins $p_3, s_3; p_4, s_4$) are the t and u diagrams

$i\mathcal{T} =$ 

$$= (-ig)^2(\bar{v}_{s_1}(\mathbf{p}_1)v_{s_3}(\mathbf{p}_3))(\bar{v}_{s_2}(\mathbf{p}_2)v_{s_4}(\mathbf{p}_4))\frac{-i}{-t+M^2-i\epsilon}$$
$$- (-ig)^2(\bar{v}_{s_1}(\mathbf{p}_1)v_{s_4}(\mathbf{p}_4))(\bar{v}_{s_2}(\mathbf{p}_2)v_{s_3}(\mathbf{p}_3))\frac{-i}{-u+M^2-i\epsilon}.$$

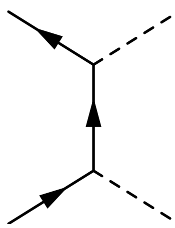
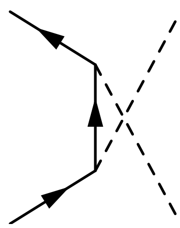
Note the relative minus sign between the two diagrams, arising from anticommuting Ψ_3 and Ψ_4.

Similarly, for $\varphi\varphi \to e^+e^-$ (incoming momenta p_1, p_2, outgoing momenta and spins $p_3, s_3; p_4, s_4$), we have the t and u diagrams

$i\mathcal{T} =$  $+$ 

$$= (-ig)^2 \bar{u}_{s_3}(\mathbf{p}_3) \left(\frac{-i(-\not{p}_3 + \not{p}_1 + m)}{-t + m^2 - i\epsilon} + \frac{-i(-\not{p}_3 + \not{p}_2 + m)}{-u + m^2 - i\epsilon} \right) v_{s_4}(\mathbf{p}_4).$$

4. At tree level, $e^+e^- \to \varphi\varphi$ proceeds via the t and u channels (writing u_i for $u_{s_i}(\mathbf{p}_i)$, etc. to reduce clutter):

$i\mathcal{T} =$  $+$ 

$$= (-ig)^2 \bar{v}_2 \left(\frac{-i(-(\not{p}_1 - \not{p}_3) + m)}{-t + m^2} + \frac{-i(-(\not{p}_1 - \not{p}_4) + m)}{-u + m^2} \right) u_1.$$

It helps simplify things if we use the on-shell condition $(\not{p}_1 + m)u_1 = 0$ at this point, so

$$\mathcal{T} = g^2 \bar{v}_2 \left(\frac{\not{p}_3 + 2m}{m^2 - t} + \frac{\not{p}_4 + 2m}{m^2 - u} \right) u_1.$$

Thus

$$|\mathcal{T}|^2 = g^4 \bar{u}_1 \left(\frac{\not{p}_3 + 2m}{m^2 - t} + \frac{\not{p}_4 + 2m}{m^2 - u} \right) v_2 \bar{v}_2 \left(\frac{\not{p}_3 + 2m}{m^2 - t} + \frac{\not{p}_4 + 2m}{m^2 - u} \right) u_1.$$

To simplify this, we think of $|\mathcal{T}|^2$ as a trace, and use cyclicity to move the basis spinors next to each other, in the form $u_1\bar{u}_1$, and so on, and then use the spin averaging

$$\sum_{s_1=\pm} u_{s_1}(\mathbf{p}_1)\bar{u}_{s_1}(\mathbf{p}_1) = -\not{p}_1 + m, \qquad \sum_{s_2=\pm} v_{s_2}(\mathbf{p}_2)\bar{v}_{s_2}(\mathbf{p}_2) = -\not{p}_2 - m$$

to eliminate the basis spinors. This yields

$$\begin{aligned}
\langle|\mathcal{T}|^2\rangle =& \frac{1}{4}\sum_{s_1,s_2=\pm} g^4 \operatorname{tr} u_1\bar{u}_1 \left(\frac{\not{p}_3 + 2m}{m^2 - t} + \frac{\not{p}_4 + 2m}{m^2 - u} \right) v_2 \bar{v}_2 \left(\frac{\not{p}_3 + 2m}{m^2 - t} + \frac{\not{p}_4 + 2m}{m^2 - u} \right) \\
=& \frac{1}{4} g^4 \operatorname{tr}(-\not{p}_1 + m) \left(\frac{\not{p}_3 + 2m}{m^2 - t} + \frac{\not{p}_4 + 2m}{m^2 - u} \right) (-\not{p}_2 - m) \left(\frac{\not{p}_3 + 2m}{m^2 - t} + \frac{\not{p}_4 + 2m}{m^2 - u} \right) \\
=& g^4 \left(\frac{\Phi_{tt}}{(m^2 - t)^2} + \frac{\Phi_{uu}}{(m^2 - u)^2} + \frac{\Phi_{tu} + \Phi_{ut}}{(m^2 - t)(m^2 - u)} \right),
\end{aligned}$$

and it remains to compute the gamma traces

$$\begin{cases} 4\Phi_{tt} = \operatorname{tr}(-\not{p}_1 + m)(\not{p}_3 + 2m)(-\not{p}_2 - m)(\not{p}_3 + 2m), \\ 4\Phi_{uu} = \operatorname{tr}(-\not{p}_1 + m)(\not{p}_4 + 2m)(-\not{p}_2 - m)(\not{p}_4 + 2m), \\ 4\Phi_{tu} = \operatorname{tr}(-\not{p}_1 + m)(\not{p}_3 + 2m)(-\not{p}_2 - m)(\not{p}_4 + 2m), \\ 4\Phi_{ut} = \operatorname{tr}(-\not{p}_1 + m)(\not{p}_4 + 2m)(-\not{p}_2 - m)(\not{p}_3 + 2m). \end{cases}$$

The t and u channels are related by $p_3 \leftrightarrow p_4$, so it suffices to compute Φ_{tt} and Φ_{tu}. To compute these traces, recall that the trace of an odd number of γs vanishes, and $\operatorname{tr} I = 4$, $\operatorname{tr}\gamma^\mu\gamma^\nu = -4g^{\mu\nu}$, $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$.[1] I will illustrate the process for Φ_{tt}:

$$\begin{aligned} 4\Phi_{tt} =& \operatorname{tr}(-\not{p}_1 + m)(\not{p}_3 + 2m)(-\not{p}_2 - m)(\not{p}_3 + 2m) \\ =& -4m^4 \operatorname{tr} I + m^2 \operatorname{tr}(2\not{p}_1\not{p}_3 + 4\not{p}_1\not{p}_2 + 2\not{p}_1\not{p}_3 - 2\not{p}_3\not{p}_2 - \not{p}_3\not{p}_3 - 2\not{p}_2\not{p}_3) + \operatorname{tr}\not{p}_1\not{p}_3\not{p}_2\not{p}_3 \\ =& 4\left(-4m^4 - m^2(4p_1\cdot p_3 + 4p_1\cdot p_2 - 4p_2\cdot p_3 - p_3^2) + 2(p_1\cdot p_3)(p_2\cdot p_3) - (p_1\cdot p_2)p_3^2\right) \\ =& 4\big(-4m^4 - m^2(2(t - m^2 - M^2) + 2(2m^2 - s) - 2(u - m^2 - M^2) + M^2) \\ & \qquad + (1/2)(t - m^2 - M^2)(u - m^2 - M^2) + (1/2)M^2(2m^2 - s)\big) \\ =& 4\left(-\frac{7}{2}m^4 + 4m^2M^2 - \frac{1}{2}M^4 + \frac{1}{2}tu - \frac{9}{2}m^2t - \frac{1}{2}m^2u\right). \end{aligned}$$

In the fourth equality, we have converted to Mandelstam variables $s = -(p_1+p_2)^2 = 2m^2 - 2p_1\cdot p_2$, $t = -(p_1 - p_3)^2 = m^2 + M^2 + 2p_1\cdot p_3$, $u = m^2 + M^2 + 2p_1\cdot p_4$. ($M^2$ is the mass of the scalar.) In the fifth equality we collect terms and use that $s + t + u = 2m^2 + 2M^2$.

A similar calculation may be done for Φ_{tu}, and Φ_{uu}, Φ_{ut} may be obtained by exchanging $t \leftrightarrow u$ (in fact, $\Phi_{tu} = \Phi_{ut}$). The results are

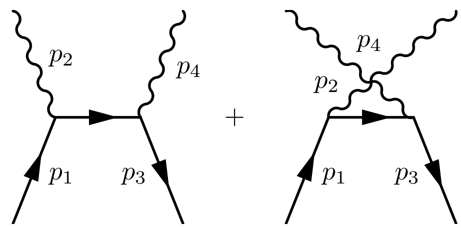
$$\begin{cases} \Phi_{tt} = \frac{1}{2}(tu - m^2(9t + u) - 7m^4 + 8m^2M^2 - M^4), \\ \Phi_{uu} = \frac{1}{2}(tu - m^2(9u + t) - 7m^4 + 8m^2M^2 - M^4), \\ \Phi_{tu} = \frac{1}{2}(-tu - 3m^2(t + u) - 9m^4 + 8m^2M^2 + M^4) = \Phi_{ut}. \end{cases}$$

Comparing with the result from $e^-\varphi \to e^-\varphi$ (48.26-29), we see that the amplitude is related by exchanging s with t, and multiplying by $-1/2$. At a diagrammatic level, this can be seen by the fact that the $e^+e^- \to \varphi\varphi$ diagrams are those of $e_-\varphi \to e^-\varphi$, but rotated by $\pi/2$. (The minus sign is due to moving a fermion from the initial to final state, and the $1/2$ is due to the fact that in $e^-\varphi \to e^-\varphi$ we are summing over the final spin states of the electron, rather than averaging (see (46.9)).)

5. The calculation is virtually identical to $e^+e^- \to \varphi\varphi$ in Yukawa theory from the previous problem, and indeed a very similar example of $e^+e^- \to \gamma\gamma$ is done in the text. I will therefore be brief.

   The relevant tree level diagrams for $e^-\gamma \to e^-\gamma$ are (once again, with u_1 short for $u_{s_1}(\mathbf{p}_1)$, and ϵ_1^μ short for $\epsilon_{\lambda_1}^\mu(\mathbf{p}_1)$ and so on)

[1] The index structures of these gamma traces are fixed by the cyclicity property of the trace. Alternatively, these identities are derived in section 47.

$i\mathcal{T} =$ 

$$= \epsilon_2^{*\mu}\epsilon_4^{\nu}\bar{u}_3\left((-ie\gamma_\nu)\frac{-i(-\not{p}_1 - \not{p}_2 + m)}{-s+m^2}(-ie\gamma_\mu) + (-ie\gamma_\mu)\frac{-i(-\not{p}_1 + \not{p}_4 + m)}{-u+m^2}(-ie\gamma_\nu)\right)u_1,$$

so, using $(\gamma^\mu\gamma^\nu\ldots\gamma^\rho)^\dagger = \beta\gamma^\rho\ldots\gamma^\nu\gamma^\mu\beta$,

$$|\mathcal{T}|^2 = e^4\epsilon_2^{*\mu}\epsilon_2^{\rho}\epsilon_4^{\nu}\epsilon_4^{*\sigma}\bar{u}_1 A_{\sigma\rho}u_3\bar{u}_3 A_{\mu\nu}u_1,$$

where

$$A_{\mu\nu} = \frac{\gamma_\nu(-\not{p}_1 - \not{p}_4 + m)\gamma_\mu}{-s+m^2} + \frac{\gamma_\mu(-\not{p}_1 + \not{k}_2 + m)\gamma_\nu}{-u+m^2}.$$

Averaging over initial, and summing over final states, with $\sum_s u_s\bar{u}_s = -\not{p} + m$, $\sum_\lambda \epsilon_\lambda^{*\mu}\epsilon_\lambda^{\nu} = g^{\mu\nu}$, we arrive at

$$\langle|\mathcal{T}|^2\rangle = e^4\left(\frac{\langle\Phi_{ss}\rangle}{(s-m^2)^2} + \frac{\langle\Phi_{su}\rangle + \langle\Phi_{us}\rangle}{(s-m^2)(u-m^2)} + \frac{\langle\Phi_{uu}\rangle}{(u-m^2)^2}\right),$$

where

$$\begin{cases} \langle\Phi_{ss}\rangle = \frac{1}{4}\operatorname{tr}\gamma_\nu(-\not{p}_1 - \not{p}_2 + m)\gamma_\mu(-\not{p}_1 + m)\gamma^\mu(-\not{p}_1 - \not{p}_2 + m)\gamma^\nu(-\not{p}_3 + m), \\ \langle\Phi_{su}\rangle = \frac{1}{4}\operatorname{tr}\gamma_\nu(-\not{p}_1 - \not{p}_2 + m)\gamma_\mu(-\not{p}_1 + m)\gamma^\nu(-\not{p}_1 + \not{p}_4 + m)\gamma^\mu(-\not{p}_3 + m), \end{cases}$$

and $\langle\Phi_{uu}\rangle$ and $\langle\Phi_{us}\rangle$ are obtained from $\langle\Phi_{ss}\rangle$ and $\langle\Phi_{su}\rangle$ respectively by $s \leftrightarrow u$, ie. $p_2 \leftrightarrow -p_4$.

To evaluate these traces, the following identities are useful:

$$\begin{cases} \gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = -g_{\mu\nu}g^{\mu\nu} = -4, \\ \gamma^\mu\gamma_\nu\gamma_\mu = \gamma^\mu(-2g^{\mu\nu} - \gamma_\mu\gamma_\nu) = 2\gamma_\nu, \\ \operatorname{tr}\gamma_\mu\gamma_\nu = -g_{\mu\nu}\operatorname{tr}I = -4g_{\mu\nu}, \operatorname{tr}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}). \end{cases}$$

I will quote the result after performing the gamma matrix algebra:

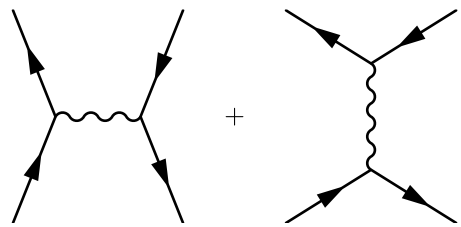
$$\begin{aligned} \langle\Phi_{ss}\rangle =& 8p_1\cdot(p_1+p_2)\,p_3\cdot(p_1+p_2) - 4(p_1+p_2)^2 p_1\cdot p_3 - 16m^2(p_1+p_2)^2 \\ &+ 16m^2(p_1+p_3)\cdot(p_1+p_2) - 4m^2 p_1\cdot p_3 + 16m^4 \\ =& 2(-su + m^2(3s+u) + m^4) \end{aligned}$$

$$\begin{aligned} \langle\Phi_{su}\rangle =& -8(p_1+p_2)\cdot(p_1-p_4)\,p_1\cdot p_3 - 4m^2((p_1-p_4)\cdot(p_1+p_2) + (p_1+p_3)\cdot(p_1-p_4) \\ &+ (p_1+p_3)\cdot(p_1+p_2) + p_1\cdot p_3) - 8m^4 \\ =& 2m^2(-t+4m^2) = \langle\Phi_{us}\rangle \end{aligned}$$

$$\langle\Phi_{uu}\rangle = 2(-su + m^2(3u+s) + m^4).$$

This is related to the $e^+e^- \to \gamma\gamma$ cross section (59.22-25) by crossing symmetry $s \leftrightarrow t$, as remarked in the problem.

6. The tree level diagrams for $e^+e^- \to e^+e^-$ are

$i\mathcal{T} =$ 

$$= \bar{v}_2(-ie\gamma^\mu)u_1 \frac{-ig_{\mu\nu}}{-s} \bar{u}_3(-ie\gamma^\nu)v_4 - \bar{u}_3(-ie\gamma^\mu)u_1 \frac{-ig_{\mu\nu}}{-t} \bar{v}_2(-ie\gamma^\nu)v_4.$$

Note the relative minus sign owing to anticommuting u_1 past $\bar{u}_3$. A similar calculation to that of the above yields

$$|\mathcal{T}|^2 = e^4 \left(\frac{\Phi_{ss}}{s^2} + \frac{\Phi_{st} + \Phi_{ts}}{st} + \frac{\Phi_{tt}}{t^2} \right),$$

where, upon averaging over initial and summing over final spins,

$$\begin{cases} \langle \Phi_{ss} \rangle = \frac{1}{4} \operatorname{tr}((-\not{p}_1 + m)\gamma^\nu(-\not{p}_2 - m)\gamma^\mu) \operatorname{tr}((-\not{p}_3 + m)\gamma_\mu(-\not{p}_4 - m)\gamma_\nu), \\ \langle \Phi_{st} \rangle = \frac{1}{4} \operatorname{tr}((-\not{p}_1 + m)\gamma^\nu(-\not{p}_3 + m)\gamma_\mu(-\not{p}_4 - m)\gamma_\nu(-\not{p}_2 - m)\gamma^\mu) \end{cases}$$

and $\langle \Phi_{tt} \rangle$, $\langle \Phi_{ts} \rangle$ are obtained via exchanging $p_2 \leftrightarrow -p_3$. Performing the gamma matrix algebra yields

$$\begin{cases} \langle \Phi_{ss} \rangle = 2(s^2 + 2st + 2t^2 - 8m^2 t + 8m^4), \\ \langle \Phi_{tt} \rangle = 2(t^2 + 2st + 2s^2 - 8m^2 s + 8m^4), \\ \langle \Phi_{st} \rangle = -2(u^2 - 8m^2 u + 12m^4) = \langle \Phi_{ts} \rangle. \end{cases}$$