# PHY 610 QFT, Spring 2017

# HW7 Solutions

1. This is a straightforward verification.

2. 2.9 a) With $\Lambda = 1 + \delta\omega$ and $U(1+\delta\omega) = I + (i/2)\delta\omega_{\mu\nu}M^{\mu\nu}$, to linear order in $\delta\omega$, the left hand side of (2.27) is

$$U(\Lambda)^{-1}\partial^\mu\varphi(x)U(\Lambda) = \partial^\mu\varphi(x) - \frac{i}{2}\delta\omega_{\nu\rho}[M^{\nu\rho}, \partial^\mu\varphi(x)]$$

while the right hand side is

$$\begin{aligned}\Lambda^\mu{}_\rho\bar{\partial}^\rho\varphi(\Lambda^{-1}x) =&\partial^\mu\varphi(x) + \delta\omega^\mu{}_\rho\partial^\rho\varphi(x) + (-\delta\omega_{\nu\rho}x^\rho)\partial^\nu\partial^\mu\varphi(x))\\ =&\partial^\mu\varphi(x) + \delta\omega_{\nu\rho}g^{\mu\nu}\partial^\rho\varphi(x) - \delta\omega_{\nu\rho}x^\rho\partial^\nu\partial^\mu\varphi(x).\end{aligned}$$

Taking the antisymmetric coefficient of $\delta\omega$ (recall that $\delta\omega$ is antisymmetric) yields the desired equality.

b) We could use the hint and follow the method of problem 2.8, but since we have not done the problem, it is probably quicker to verify the commutation relation of $S_V$ explicitly. I will show this, and then say a few words about the method of problem 2.8.

$$\begin{aligned}[S_V^{\mu\nu}, S_V^{\rho\sigma}]^\alpha{}_\gamma =&(S_V^{\mu\nu})^\alpha{}_\beta(S_V^{\rho\sigma})^\beta{}_\gamma - (\mu\nu \leftrightarrow \rho\sigma)\\ =&(-i)(g^{\mu\alpha}\delta^\nu_\beta - (\mu \leftrightarrow \nu))((-i)g^{\rho\beta}\delta^\sigma_\gamma - (\rho \leftrightarrow \sigma)) - (\mu\nu \leftrightarrow \rho\sigma)\\ =&-g^{\mu\alpha}g^{\rho\nu}\delta^\sigma_\gamma + \text{ 3 more from antisymmetry } [\mu\nu], [\rho\sigma] \; - (\mu\nu \leftrightarrow \rho\sigma)\\ =&-ig^{\rho\nu}(S_V^{\mu\sigma})^\alpha{}_\gamma + \text{ 3 more.}\end{aligned}$$

Therefore, $S_V^{\mu\nu}$ is indeed a representation of the Lorentz group.
If we were to follow the hint, we should define the total angular momentum generator $(\mathcal{J}^{\mu\nu})^\alpha{}_\beta = \mathcal{L}^{\mu\nu}\delta^\alpha_\beta + (S_V^{\mu\nu})^\alpha{}_\beta$. We showed in part (a) that $\mathcal{J}^{\mu\nu}$ are the representation matrices for fields in the vector representation. Working out problem 2.8, we find that $\mathcal{L}^{\mu\nu}$, which are the representation matrices for scalar fields, satisfies the Lorentz algebra commutation relations, by using the Jacobi identity. A similar argument will show that $\mathcal{J}^{\mu\nu}$ also satisfies the Lorentz algebra commutation relations. Finally, we notice that $[S_V^{\mu\nu}, \mathcal{L}^{\rho\sigma}] = 0$, which is to say, the spin and orbital angular momenta do not mix, so the spin part $S_V$ of the total angular momentum must also satisfy the Lorentz algebra commutation relations.

c) This is a straightforward computation. In matrix form,

$$S_V^{12} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Note that $(S_V^{12})^{2j} = \mathrm{diag}(0,1,1,0)$ and $(S_V^{12})^{2j+1} = S_V^{12}$ for integer $j$. Therefore

$$\exp(-i\theta S_V^{12}) = \sum_{j=0}^{\infty} \frac{(-i\theta)^{2j}}{(2j)!}(S_V^{12})^{2j} + \sum_{j=0}^{\infty} \frac{(-i\theta)^{2j+1}}{(2j+1)!}(S_V^{12})^{2j+1}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \cos\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - i\sin\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

d) This is a straightforward computation with the same steps as above, with a few factors of $i$ inserted.

36.1 In problem 2.9, we saw that, in the defining, or vector, representation of the Lorentz group $SO(3,1)$, $D_V(\Lambda) = \exp(-i\theta S_V^{12})$ for a rotation of angle $\theta$ about the 3 axis, and $\exp(-i\eta S_V^{03})$ for a boost of rapidity $\eta$ along the 3 axis. (I have used the notation $D_V(\Lambda) = \Lambda$ to emphasize that $\Lambda$ here to distinguish between the abstract element $\Lambda \in SO(3,1)$ and its action on the vector representation.) This is the finite version of the statement (2.32), which states that the exponential map sends the vector representation $S_V^{\mu\nu}$ of the Lie algebra $\mathfrak{so}(3,1)$ to the vector representation $D_V$ of the Lie group $SO(3,1)$. Specifically, the exponential map sends $\theta S_V^{12}$ to a rotation of $\theta$ about the 3 axis and $\eta S_V^{03}$ to a boost of $\eta$ along the 3 axis.

Actually, since the representations $\Lambda \mapsto D_V(\Lambda)$ and $S^{\mu\nu} \mapsto S_V^{\mu\nu}$ are homomorphisms, the exponential map actually lifts to the abstract Lie algebra and Lie group, so in fact $\Lambda = \exp(-i\theta S^{12})$ for an abstract element $\Lambda$ representing a rotation of $\theta$ about the 3 axis. Therefore, the exponential map indeed relates any representation of the Lie algebra to one of the Lie group. In particular, for the Dirac spinor representation we have the desired relations.

3. (a) The relation $\sigma^{\mu}_{c\dot{c}}\sigma_{\mu d\dot{d}} = -2\epsilon_{cd}\epsilon_{\dot{c}\dot{d}}$ may be turned into $\bar{\sigma}^{\mu a\dot{a}}\bar{\sigma}^{\dot{b}b}_{\mu} = -2\epsilon^{\dot{a}\dot{b}}\epsilon^{ab}$ using the definition $\bar{\sigma}^{\mu a\dot{a}} = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{b\dot{b}}$. This is now what we need in order to prove the first Fierz identity.

$$\begin{aligned}(\chi_1^\dagger\bar{\sigma}^\mu\chi_2)(\chi_3^\dagger\bar{\sigma}_\mu\chi_4) &= (\chi^\dagger_{1\dot{a}}\bar{\sigma}^{\mu\dot{a}a}\chi_{2a})(\chi^\dagger_{3\dot{b}}\bar{\sigma}^{\dot{b}b}_{\mu}\chi_{4b}) \\ &= -2(\chi^\dagger_{1\dot{a}}\epsilon^{\dot{a}\dot{b}}\chi^\dagger_{3\dot{b}})(\chi_{2a}\epsilon^{ab}\chi_{4b}) \\ &= -2(\chi_1^\dagger\chi_3^\dagger)(\chi_2\chi_4)\end{aligned}$$

The second Fierz identity simply says that the same result should be obtained upon switching $\chi_2 \leftrightarrow \chi_4$. This is clear above. When we move $\chi_4$ past $\chi_2$, we get a minus sign by anticommutativity, but then we get another one by reversing the spinor index positions to agree with our conventions.

(b) The Fierz identities in Dirac form follow from the given definitions and

$$P_{\mathrm{L}} = \begin{bmatrix} \delta_a^{c} & 0 \\ 0 & 0 \end{bmatrix}, \; P_{\mathrm{R}} = \begin{bmatrix} 0 & 0 \\ 0 & \delta^{\dot{a}}_{\dot{c}} \end{bmatrix}, \; \gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu_{a\dot{b}} \\ \bar{\sigma}^{\mu\dot{a}b} & 0 \end{bmatrix}.$$

Applying these will turn $(\bar{\Psi}_1\gamma^\mu P_{\mathrm{L}}\Psi_2)(\bar{\Psi}_3\gamma^\mu P_{\mathrm{R}}\Psi_4)$ into $(\chi_1^\dagger\bar{\sigma}^\mu\chi_2)(\chi_3^\dagger\bar{\sigma}_\mu\chi_4)$ and $-2(\bar{\Psi}_1 P_{\mathrm{R}}\Psi_3^{\mathrm{C}})(\bar{\Psi}_4^{\mathrm{C}} P_{\mathrm{L}}\Psi_2)$ into $-2(\chi_1^\dagger\chi_3^\dagger)(\chi_4\chi_2)$.

(c) As an example, we may apply the above rules to prove the middle identity. The left hand side is $\bar{\Psi}_1 P_L \Psi_2 = \xi_1 \chi_2$ while the right hand side is $\bar{\Psi}_2^C P_L \Psi_1^C = \chi_2 \xi_1$. These are equal for the same reason as before, where we need anti-commutativity and a reversal of the index positions.

4. In this problem, we see that the energy momentum tensor in a theory with Lorentz invariance can be made symmetric. Recall in homework 2, we noted that the canonical energy momentum tensor needs not be symmetric, but you may know that the energy momentum tensor is also the tensor which couples to the (symmetric) metric in general relativity. The solution to this is that the tensor which appears in general relativity is actually the improved, or Belinfante energy momentum tensor, which we will derive in this problem.

(a) Recall that the (canonical) energy momentum tensor is the Noether current corresponding to spacetime translations $\delta\varphi_A(x) = -a^\mu \partial_\mu \varphi_A(x)$. This may be derived either by the method of section 22, noting that the lagrangian varies into a total derivative $\partial_\mu K^\mu$ with $K^\mu = -a^\mu \mathcal{L}$, or by varying the action with a spacetime dependent variation $\delta\varphi_A(x) = a^\mu(x)\partial_\mu\varphi_A(x)$ and finding the coefficient of $\partial_\mu a^\nu$. Either way, we obtain the desired energy momentum tensor. (The $A$ index doesn't affect anything.)

(b) Consider an infinitesimal lorentz transformation $\delta\varphi_A(x) = (i/2)\delta\omega_{\mu\nu}(S^{\mu\nu})_A{}^B\varphi_B(x) - \delta\omega^\mu{}_\nu x^\nu \partial_\mu \varphi_A(x) = \delta\omega_{\mu\nu}((i/2)(S^{\mu\nu})_A{}^B - \delta_A^B x^\nu \partial^\mu)\varphi_B(x)$ (this is the infinitesimal version of (36.66)). The lagrangian varies into a total derivative, $\delta\mathcal{L} = -\delta\omega_{\mu\nu} x^\nu \partial^\mu \mathcal{L} = -\delta\omega_{\mu\nu}\partial^\mu(x^\nu \mathcal{L})$, where we have used that $\partial^\mu x^\nu = g^{\mu\nu}$ vanishes when contracted with the antisymmetric $\delta\omega_{\mu\nu}$. Therefore, the Noether current is

$$\frac{1}{2}\delta\omega_{\nu\rho}\mathcal{M}^{\mu\nu\rho} = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}\delta\omega_{\mu\nu}\left(\frac{i}{2}(S^{\mu\nu})_A{}^B - x^\rho\partial^\nu\delta_A^B\right)\varphi_B + \delta\omega^\mu{}_\nu x^\nu \mathcal{L}$$

$$\begin{aligned}\Rightarrow\ \mathcal{M}^{\mu\nu\rho} =& -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)}\left(i(S^{\mu\nu})_A{}^B - x^{[\rho}\partial^{\nu]}\delta_A^B\right)\varphi_B + g^{\mu[\rho}x^{\nu]}\mathcal{L} \\ =& T^{\mu[\rho}x^{\nu]} + B^{\mu\nu\rho},\end{aligned}$$

where $B^{\mu\nu\rho} = -i(\partial\mathcal{L}/\partial(\partial_\mu\varphi_A))(S^{\mu\nu})_A{}^B\varphi_B$ is the Belinfante correction term.

(c) Using $\partial_\mu\mathcal{M}^{\mu\nu\rho} = 0$ and the above expression for $\mathcal{M}^{\mu\nu\rho}$, we get

$$\begin{aligned}0 =& \partial_\mu(T^{\mu[\rho}x^{\nu]} + B^{\mu\nu\rho}) \\ =& \partial_\mu T^{\mu[\rho}x^{\nu]} + T^{\mu[\rho}\delta_\mu^{\nu]} + \partial_\mu B^{\mu\nu\rho} \\ =& 0 + T^{[\nu\rho]} + \partial_\mu B^{\mu\nu\rho}.\end{aligned}$$

(d) First, we show that $\Theta^{\mu\nu}$ is symmetric:

$$\begin{aligned}\Theta^{[\mu\nu]} =& T^{[\mu\nu]} + \frac{1}{2}\partial_\rho(B^{\rho[\mu\nu]} - B^{[\mu|\rho|\nu]} - B^{[\nu|\rho|\mu]}) \\ =& T^{[\mu\nu]} + \partial_\rho B^{\rho\mu\nu} = 0,\end{aligned}$$

where we have used that $B^{\rho\mu\nu}$ is antisymmetric in $\mu, \nu$.

Next, the correction terms are divergenceless,

$$\partial_\mu(\Theta^{\mu\nu} - T^{\mu\nu}) = \frac{1}{2}\partial_\mu\partial_\rho(B^{\rho\mu\nu} - B^{\mu\rho\nu} - B^{\nu\rho\mu}) = 0,$$

so the conservation of $T^{\mu\nu}$ implies the conservation of $\Theta^{\mu\nu}$. In the second equality above we have used that $B^{\nu\rho\mu}$ is antisymmetric in $\rho, \mu$.

Finally, to show that $\int d^3x\, T^{0\nu} = \int d^3x\, \Theta^{0\nu}$, note that $\Theta^{0\nu} - T^{0\nu} = \partial_\rho(B^{\rho 0\nu} - B^{0\rho\nu} - B^{\nu\rho 0})$, and the $\rho = 0$ term does not contribute, since $B^{00\nu} - B^{00\nu} - B^{\nu 00} = 0$. Therefore, $\Theta^{0\nu} - T^{0\nu}$ is a spatial total divergence, and assuming that fields go to zero at infinity, its spatial integral vanishes.

(e) First, we show that $\Xi^{\mu\nu\rho}$ is conserved:

$$\partial_\mu \Xi^{\mu\nu\rho} = \partial_\mu(\Theta^{\mu[\rho}x^{\nu]}) = \Theta^{\mu[\rho}\delta_\mu^{\nu]} = \Theta^{[\nu\rho]} = 0,$$

where we have used that $\Theta^{\mu\nu}$ is conserved in the second equality and that it is symmetric in the last.

Next, we show that $\Xi^{0\nu\rho} - \mathcal{M}^{0\nu\rho}$ is a spatial total derivative:

$$\begin{aligned}
\Xi^{0\nu\rho} - \mathcal{M}^{0\nu\rho} =& (\Theta^{0[\rho} - T^{0[\rho})x^{\nu]} - B^{0\nu\rho} \\
=& \left(\frac{1}{2}\partial_\sigma(B^{\sigma 0\rho} - B^{0\sigma\rho} - B^{\rho\sigma 0})x^\nu - (\rho \leftrightarrow \nu)\right) - B^{0\nu\rho} \\
=& \frac{1}{2}\left(\partial_\sigma((B^{\sigma 0\rho} - B^{0\sigma\rho} - B^{\rho\sigma 0})x^\nu) - (B^{\nu 0\rho} - B^{0\nu\rho} - B^{\rho\nu 0}) - B^{0\nu\rho}\right) - (\rho \leftrightarrow \nu) \\
=& \frac{1}{2}\left(\partial_\sigma((B^{\sigma 0\rho} - B^{0\sigma\rho} - B^{\rho\sigma 0})x^\nu)\right) - (\rho \leftrightarrow \nu)
\end{aligned}$$

In the last equality, the extra $B$ terms on the right cancel due to antisymmetry in $\rho, \nu$ and in the second and third indices of $B$. Similarly to the previous part, the temporal component $\sigma = 0$ vanishes since $B^{00\rho} - B^{00\rho} - B^{\rho 00} = 0$, so we have shown that $\Xi^{0\nu\rho} - \mathcal{M}^{0\nu\rho}$ is a spatial total derivative (and hence their spatial integrals coincide).

(f) For a left handed Weyl spinor, $S_L^{\mu\nu} = (i/2)\sigma^{\mu\nu}$, where the notation $\sigma^{\mu\nu} := (1/2)(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$, $\bar{\sigma}^{\mu\nu} := (1/2)(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$ is used (spinor indices $a, b = 1, 2$ suppressed). The lagrangian is $\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - (m/2)(\psi\psi + \psi^\dagger\psi^\dagger)$. From the above results, we have[1]

$$T^{\mu\nu} = g^{\mu\nu}\mathcal{L} - i\psi^\dagger\bar{\sigma}^\mu\partial^\nu\psi,$$

$$B^{\mu\nu\rho} = \frac{i}{2}\psi^\dagger\bar{\sigma}^\mu\sigma^{\nu\rho}\psi.$$

The improved energy momentum tensor is therefore

$$\Theta^{\mu\nu} = g^{\mu\nu}\mathcal{L} - i\psi^\dagger\bar{\sigma}^\mu\partial^\nu\psi + \frac{i}{4}\partial_\rho\left(\psi^\dagger(\bar{\sigma}^\rho\sigma^{\mu\nu} - \bar{\sigma}^\mu\sigma^{\rho\nu} - \bar{\sigma}^\nu\sigma^{\rho\mu})\psi\right).$$

Let us check that $\Theta^{\mu\nu}$ is manifestly symmetric. We will need the equations of motion, $i\partial_\rho\psi^\dagger\bar{\sigma}^\rho = m\psi, i\bar{\sigma}^\rho\partial_\rho\psi = m\psi$, as well as the identities $\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = -2g^{\mu\nu} = \bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu$. Keeping this

[1] Since fermions anticommute, it actually matters which direction we take the derivative with respect to $\partial_\mu\psi$. We should take the derivative from the side where we multiply $\partial^\nu\psi$, ie. either $\frac{\overleftarrow{\partial}\mathcal{L}}{\partial(\partial_\mu\psi)}\partial^\nu\psi$ or $\partial^\nu\psi\frac{\overrightarrow{\partial}\mathcal{L}}{\partial(\partial_\mu\psi)}$.

in mind, we simplify $\Theta^{\mu\nu}$:

$$\begin{aligned}\Theta^{\mu\nu} =& g^{\mu\nu}\mathcal{L} - i\psi^\dagger\bar\sigma^\mu\partial^\nu\psi + \frac{m}{4}\psi\sigma^{\mu\nu}\psi + \frac{i}{4}\partial_\rho\psi^\dagger(\bar\sigma^\mu(g^{\rho\nu}+\sigma^\nu\bar\sigma^\rho) + \bar\sigma^\nu(g^{\rho\mu}+\sigma^\mu\bar\sigma^\rho))\psi \\ &+ \frac{i}{4}\psi^\dagger(\bar\sigma^\rho\sigma^{\mu\nu} + \bar\sigma^\mu(g^{\rho\nu}+\sigma^\nu\bar\sigma^\rho) + \bar\sigma^\nu(g^{\rho\mu}+\sigma^\mu\bar\sigma^\rho))\partial_\rho\psi \\ =& g^{\mu\nu}\mathcal{L} - i\psi^\dagger\bar\sigma^\mu\partial^\nu\psi + \frac{m}{4}\psi\sigma^{\mu\nu}\psi + \frac{i}{4}\partial^{(\mu}\psi^\dagger\bar\sigma^{\nu)}\psi - \frac{m}{2}\psi\psi g^{\mu\nu} \\ &+ \frac{i}{4}\psi^\dagger(\bar\sigma^{\mu\nu}\bar\sigma^\rho - 2g^{\rho[\mu}\bar\sigma^{\nu]})\partial_\rho\psi + \frac{i}{4}\psi^\dagger\bar\sigma^{(\mu}\partial^{\nu)}\psi - \frac{m}{2}\psi^\dagger\psi^\dagger g^{\mu\nu} \\ =& g^{\mu\nu}\mathcal{L} + \frac{m}{4}(\psi\sigma^{\mu\nu}\psi + \psi^\dagger\bar\sigma^{\mu\nu}\psi^\dagger + 2g^{\mu\nu}(\psi\psi+\psi^\dagger\psi^\dagger)) + \frac{i}{4}\partial^{(\mu}\psi^\dagger\bar\sigma^{\nu)}\psi - \frac{i}{4}\psi^\dagger\bar\sigma^{(\mu}\partial^{\nu)}\psi.\end{aligned}$$

For the $\psi^\dagger\bar\sigma^\rho\sigma^{\mu\nu}\psi$ term in the first line, we have used the identity $\bar\sigma^\rho\sigma^{\mu\nu} - \bar\sigma^{\mu\nu}\bar\sigma^\rho = -2g^{\rho[\mu}\bar\sigma^{\nu]}$. Now, note that (35.17) $(\sigma^{\mu\nu})_{ac} = (1/2)\epsilon^{\dot a\dot c}(\sigma^\mu_{a\dot a}\sigma^\nu_{c\dot c} - \sigma^\nu_{a\dot a}\sigma^\mu_{c\dot c})$ is symmetric in $a, c$, so $\psi\sigma^{\mu\nu}\psi = \psi^a(\sigma^{\mu\nu})_{ac}\psi^c = 0$ (recall that the $\psi$s anticommute). Similarly, $\psi^\dagger\bar\sigma^{\mu\nu}\psi^\dagger = 0$, so we obtain

$$\begin{aligned}\Theta^{\mu\nu} =& g^{\mu\nu}\mathcal{L} + \frac{m}{2}g^{\mu\nu}(\psi\psi+\psi^\dagger\psi^\dagger) + \frac{i}{4}(\partial^{(\mu}\psi^\dagger\bar\sigma^{\nu)}\psi - \psi^\dagger\bar\sigma^{(\mu}\partial^{\nu)}\psi) \\ =& g^{\mu\nu}m\psi^\dagger\psi^\dagger + \frac{i}{4}(\partial^{(\mu}\psi^\dagger\bar\sigma^{\nu)}\psi - \psi^\dagger\bar\sigma^{(\mu}\partial^{\nu)}\psi),\end{aligned}$$

which is manifestly symmetric. Notice that these currents are not hermitian, because the lagrangian is hermitian only up to a total derivative. If we had started with the hermitian lagrangian $\mathcal{L} = (i/2)(\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - \partial_\mu\psi^\dagger\bar\sigma^\mu\psi) - (m/2)(\psi\psi + \psi^\dagger\psi^\dagger)$, we would obtain hermitian currents.

For a Dirac spinor, $S^{\mu\nu} = (i/2)\gamma^{\mu\nu}$, where $\gamma^{\mu\nu} := (1/2)[\gamma^\mu,\gamma^\nu]$. The lagrangian is $\mathcal{L} = i\bar\Psi\gamma^\mu\partial_\mu\Psi - m\bar\Psi\Psi$. From the above results we get

$$T^{\mu\nu} = g^{\mu\nu}\mathcal{L} - i\bar\Psi\gamma^\mu\partial^\nu\Psi,$$

$$B^{\mu\nu\rho} = \frac{i}{2}\bar\Psi\gamma^\mu\gamma^{\nu\rho}\Psi,$$

yielding

$$\Theta^{\mu\nu} = g^{\mu\nu}\mathcal{L} - i\bar\Psi\gamma^\mu\partial^\nu\Psi + \frac{i}{4}\partial_\rho(\bar\Psi(\gamma^\rho\gamma^{\mu\nu} - \gamma^\mu\gamma^{\rho\nu} - \gamma^\nu\gamma^{\rho\mu})\Psi).$$

A similar simplification as in the Weyl spinor case, with the equations of motion $(i\gamma^\mu\partial_\mu - m)\Psi = 0$, $i\partial_\mu\bar\Psi\gamma^\mu + m\bar\Psi = 0$, and the identity $\{\gamma^\mu,\gamma^\nu\} = -2g^{\mu\nu}$ yields

$$\Theta^{\mu\nu} = -\frac{i}{4}\bar\Psi\gamma^{(\mu}\partial^{\nu)}\Psi + \frac{i}{4}\partial^{(\mu}\bar\Psi\gamma^{\nu)}\Psi.$$

(Notice that $\mathcal{L} = 0$ on shell, since $\mathcal{L}$ is linear in $\partial_\mu\Psi$.)

5. (a) Unlike a Dirac mass term, a Majorana mass term $m\psi\psi + \mathrm{hc}$ breaks charge conservation; it is only invariant under real unitary transformations, ie. orthogonal transformations $O(N)$. (Explicitly, $\psi_j\psi_j \mapsto U_{jk}\psi_k U_{jl}\psi_l$, so $U^T_{kj}U_{jl} = \delta_{kl}$ for invariance.)

   (b) Recall that, in $d = 1+3$, a Majorana spinor is constructed out of a Weyl spinor and its conjugate, $\binom{\psi_a}{\psi^{\dagger\dot a}}$. Therefore, the lagrangian for a Majorana spinor may be written, up to a total derivative, as that of a Weyl spinor (suppressing flavor indices):

$$
\begin{aligned}
\mathcal{L} =& \frac{i}{2}\Psi^T \mathcal{C}\gamma^\mu \partial_\mu \Psi \\
=& \frac{i}{2}\begin{pmatrix}\psi_a & \psi^{\dagger\dot{a}}\end{pmatrix}\begin{pmatrix}-\epsilon^{ab} & \\ & -\epsilon^{\dot{a}\dot{b}}\end{pmatrix}\begin{pmatrix} & \sigma^\mu_{b\dot{c}} \\ \bar{\sigma}^{\mu\dot{b}c} & \end{pmatrix}\partial_\mu \begin{pmatrix}\psi_c \\ \psi^{\dagger\dot{c}}\end{pmatrix} \\
=& \frac{i}{2}\left(\psi^a \sigma^\mu_{a\dot{c}}\partial_\mu \psi^{\dagger\dot{c}} + \psi^{\dagger\dot{a}}\bar{\sigma}^{\mu\dot{a}c}\partial_\mu \psi_c\right) \\
=& \frac{i}{2}\left(\partial_\mu(\psi\sigma^\mu\psi^\dagger) - \partial_\mu\psi^a\sigma^\mu_{a\dot{c}}\psi^{\dagger\dot{c}} + \psi^{\dagger\dot{a}}\bar{\sigma}^{\mu\dot{a}c}\partial_\mu\psi_c\right) \\
=& \frac{i}{2}\partial_\mu(\psi\sigma^\mu\psi^\dagger) + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi,
\end{aligned}
$$

where in the last equality we have used that $-\partial_\mu\psi^a\sigma^\mu_{a\dot{c}}\psi^{\dagger\dot{c}} = \psi^{\dagger\dot{c}}\sigma^\mu_{a\dot{c}}\partial_\mu\psi^a = \psi^\dagger_{\dot{c}}\bar{\sigma}^\mu_{\dot{c}a}\partial_\mu\psi_a$, owing to the anticommutivity of fermions, and that $\epsilon^{ab}\sigma^\mu_{b\dot{c}}\epsilon^{\dot{c}\dot{d}} = \bar{\sigma}^{\mu\dot{d}a}$.

Hence, the invariance group is the same as that of the Weyl group, $U(N)$.

(c) Using the same procedure as above, the Majorana mass term may be written as a Weyl mass term. Therefore, the invariance group is the same as that in part a, $O(N)$.

(d) Recall that a Dirac field consists of two Weyl spinors, which we may choose to be left handed, and write as $\Psi = \binom{\chi}{\xi^\dagger}$. The Dirac action then may be expanded, up to a total derivative, as the sum of the two Weyl actions. The invariance group is therefore $U(2N)$. (The $\chi$ and $\xi$ fields are allowed to interchange. The subgroup which does not mix $\chi$ and $\xi$ is $U(N) \times U(N)$.)

(e) By the same reasoning, the massive Dirac action has invariance group $O(2N)$. This $O(2N)$ symmetry does not seem manifest looking at the mass terms $m\bar{\Psi}\Psi = m(\chi\xi + \xi^\dagger\chi^\dagger)$, but by going back to the real basis (36.14-15), $\psi_1 = (\chi+\xi)/\sqrt{2}, \psi_2 = -i(\chi-\xi)/\sqrt{2}$, we may write the mass term as $(m/2)(\psi_1\psi_1 + \psi_2\psi_2 + \psi_1^\dagger\psi_1^\dagger + \psi_2^\dagger\psi_2^\dagger)$, making the $O(2N)$ symmetry manifest.