PHY 610 QFT, Spring 2017

HW6 Solutions

1. (a) With incoming and outgoing electron labeled by p and p' and the incoming and outgoing photon by k and k', we could set, in the FT frame:

$$\begin{aligned} p =&(m,0,0,0) \\ k =&(\omega,0,0,\omega) \\ k' =&(\omega', \omega' sin\theta, 0, \omega' cos\theta) \\ p' =&p+k-k' \end{aligned}$$

Then we have $s=-(p+k)^2=m^2+2m\omega$ and $u=-(p-k')^2=m^2-2m\omega'$.

(b) Again we could start from $p'=p+k-k'$ and may end up with:

$$\begin{aligned} (p')^2=-m^2 =&(p+k-k')^2 \\ =&p^2+k^2+k'^2+2pk-2pk'-2kk' \\ =&-m^2-2m\omega+2m\omega'-2\omega\omega'(cos\theta-1) \end{aligned}$$

Then the relationship is

$$1-cos\theta=m(\frac{1}{\omega'}-\frac{1}{\omega})$$

(c) From the textbook we may have an expression of $\mathcal{T}$, and with the help of what we found in part (b), we could recast it as:

$$\begin{aligned} |\mathcal{T}|^2 =&32\pi^2\alpha^2[\frac{m^2+m\omega+\omega\omega'}{\omega^2}+\frac{m^2-m\omega'+\omega\omega'}{\omega'^2}-\frac{2m^2+m\omega-m\omega'}{\omega\omega'}] \\ =&32\pi^2\alpha^2[m^2(\frac{1}{\omega^2}+\frac{1}{\omega'^2}-\frac{2}{\omega\omega'})+2m(\frac{1}{\omega}-\frac{1}{\omega'})+\frac{\omega}{\omega'}+\frac{\omega'}{\omega}] \\ =&32\pi^2\alpha^2(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-sin^2\theta) \end{aligned}$$

Now we could take a look at ω'. Expressing ω' as a function of ω and θ and then take d differential, we may have:

$$d\omega'=\frac{m\omega^2}{[m+\omega(1-cos\theta)]^2}dcos\theta=\frac{\omega'^2}{m}dcos\theta$$

With $t=2m^2-s-u=2m(\omega'-\omega)$, we find:

$$dt=2md\omega'=2\omega'^2dcos\theta=\frac{\omega'^2}{\pi}d\Omega_{FT}$$

Finally, by putting all these components together, we get:

$$\frac{d\sigma}{d\Omega_{FT}}=\frac{\omega'^2\alpha^2}{2m^2\omega^2}(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-sin^2\theta)$$

2. (a) We could get the answer immediately by putting (11.53) and (11.23) together.

   (b) Taking care that on the left-hand side, we have a two-index tensor and the only vector building block would be k. Then all possible terms we could write down would be $g^{\mu\nu}$ and $k^\mu k^\nu$. Then since A and B could only depend on some scalar variables, yet the only candidate, $k^2 = -m^2$ is not dimensionless, we know that A and B must be pure numbers.

   (c) With $m = 0$, we have $|k_1'| = \frac{\sqrt{s}}{2}$. Integrating over $d\Omega$ may give us a factor 4π.

   (d) by projecting out the $g^{\mu\nu}$ or $k^\mu k^\mu$, we may have :

$$\int (k_1' k_2') dLPS_2(k) = (4A + B)k^2$$
$$\int (kk_1')(kk_2') dLPS_2(k) = (A + B)(k^2)^2$$

Noting that we have δ function in $dLIPS_2(k)$, enforcing $k_1\prime + k_2' = k$. Also the with the help of on-shell condition, we may make a replacement: $(kk_1')(kk_2') \to \frac{(k^2)^2}{4}$ and $(k_1' k_2') \to \frac{k^2}{2}$. Equations above are simplified to:

$$4A + B = \frac{1}{16\pi}$$
$$A + B = \frac{1}{32\pi}$$

which yields $A = \frac{1}{48\pi}$ and $B = \frac{1}{96\pi}$

3. (a) There is a $U(1)$ symmetry acting as $\varphi_1 \mapsto e^{i\theta}\varphi_1$, $\varphi_1^* \mapsto e^{-i\theta}\varphi_1^*$, $\varphi_2 \mapsto e^{-i\theta}\varphi_2$, $\varphi_2^* \mapsto e^{i\theta}\varphi_2^*$.

   (b) Using either $j^\mu = (\partial\mathcal{L}/\partial\partial_\mu\varphi)\delta\varphi$, or varying the action under a spacetime dependent symmetry transformation $\theta(x)$, and collecting the coefficient of $\partial_\mu\theta$, we get

$$j^\mu = i\left(\varphi_1\partial^\mu\varphi_1^* - \partial^\mu\varphi_1\varphi_1^* - \varphi_2\partial^\mu\varphi_2^* + \partial^\mu\varphi_2\varphi_2^*\right).$$

The Noether charge is

$$Q = \int d^3x\, i\left(\varphi_1\partial^0\varphi_1^* - \partial^0\varphi_1\varphi_1^* - \varphi_2\partial^0\varphi_2^* + \partial^0\varphi_2\varphi_2^*\right).$$

Substituting the mode expansions

$$\varphi_j(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}}(a_j(\mathbf{k})e^{ikx} + b_j^\dagger(\mathbf{k})e^{-ikx}), \qquad j = 1, 2$$

yields

$$Q = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}}\left(a_1^\dagger(\mathbf{k})a_1(\mathbf{k}) - b_1(\mathbf{k})b_1^\dagger(\mathbf{k}) - a_2^\dagger(\mathbf{k})a_2(\mathbf{k}) + b_2(\mathbf{k})b_2^\dagger(\mathbf{k})\right),$$

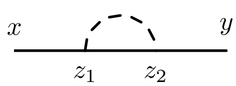
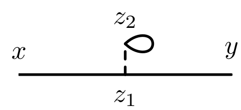
that is, the Noether charge counts the difference between the number of φ_1 and φ_2 particles (with antiparticles contributing opposite signs).

   (c) At $\mu = 0$ there is a $U(1) \times U(1)$ symmetry, acting as $\varphi_1 \mapsto e^{i\theta_1}\varphi_1$, $\varphi_2 \mapsto e^{i\theta_2}\varphi_2$.

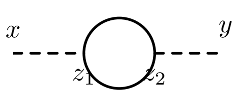
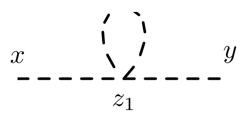
(d) When $\mu = 0$, $m_1 = m_2$, the symmetry is enhanced to an $SO(4)$ symmetry rotating $\varphi_j :=(\Re\varphi_1, \Im\varphi_1, \Re\varphi_2, \Im\varphi_2)$. The Noether current is

$$j^\mu_a = i\partial^\mu \varphi_i (T_a)_{ij}\varphi_j,$$

where $(T_a)_{ij}$ are the (hermitian) generators of $SO(4)$ in the defining representation.

4. (a) $\langle 0|\mathrm{T}A(x)A(y)|0\rangle =$  $+$  $+$  $+O[\lambda^3]$

$$= \Delta_{xy} - \lambda^2\alpha^2 \int d^4z_1 d^4z_2 (\Delta_{x1}\Delta_{12}\Delta_{12}\Delta_{2y} + \frac{1}{2}\Delta_{x1}\Delta_{12}\Delta_{22}\Delta_{1y}) + O[\lambda^3]$$

(b) $\langle 0|\mathrm{T}B(x)B(y)|0\rangle =$  $+$  $+$  $+O[\lambda^3]$

$$= \Delta_{xy} - \lambda^2\alpha^2 \int d^4z_1 d^4z_2 \, \frac{1}{2}\Delta_{x1}\Delta_{12}\Delta_{12}\Delta_{2y} + i\lambda^2\beta \int d^4z_1 \frac{1}{2}\Delta_{x1}\Delta_{11}\Delta_{1y} + O[\lambda^3]$$