PHY 610 QFT, Spring 2017

HW4 Solutions

1. This problem is a toy model of the QFT free field path integral. Since the matrix A is real symmetric with positive eigenvalues, we could find a $O(N)$ transformation M to diagonalize A without changing the integral measure $dx_1 \cdots dx_N$. Then the Gaussian integral itself would become:

$$Z = \prod_{i=1}^{N} \int dx'_i e^{-\frac{1}{2} x'_i A_{ii} x'_i}$$

with A_{ii} labeling the ith eigenvalue of A and $x' = Mx$.
Now, we could add a "source term" $J'_i x'_i$, then integrate out all the x variables. Note that since all eigenvalues of A are positive, we do not need to worry if the integral is ill-defined. (Yet in general, in QFT things might be a little more tricky. For example, we may have zero modes, from which we can extract meaningful data.) Now we have only quadric J terms left. By $O(N)$ transforming back, we have now

$$Z(J) = (2\pi)^{\frac{N}{2}} \sqrt{\prod_{i=1}^{N} A_{ii}^{-1}} e^{\frac{1}{2} J^T \frac{1}{A} J} = (2\pi)^{\frac{N}{2}} (\det A)^{-\frac{1}{2}} e^{\frac{1}{2} J^T \frac{1}{A} J}$$

Then for a "correlator",

$$< \prod_{k=1}^{2n} x_{i_k} >= \frac{1}{Z} \prod_{k=1}^{2n} \frac{\partial}{\partial J_{i_k}} Z(J=0) = \prod_{k=1}^{2n} \frac{\partial}{\partial J_{i_k}} e^{\frac{1}{2} J_m A^{-1}_{mn} J_n} = \sum_{pairings} \prod_{eachpair} A^{-1}_{i_a i_b}$$

Here we have

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

and

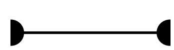
$$A^{-1} = \frac{1}{7} \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

Then $< x^4 y^2 >= 3(A^{-1}_{11})^2 A^{-1}_{22} + 12(A^{-1}_{12})^2 A^{-1}_{11} = \frac{144}{343}$

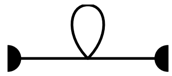
2. The solutions are in the text.

3. (a) $\mathcal{L}_1 = -Z_\lambda \lambda \varphi^4/4!$, so there is a ϕ^4 vertex with factor $-iZ_\lambda \lambda$.

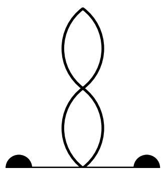
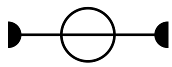
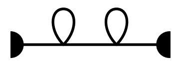
   (b) A diagram with E external lines and V vertices has $E + 4V$ lines to connect, and P propagators connect two lines. Therefore, $2P = E + 4V$ for each diagram. In particular, $E = 1$ and $E = 3$ are not possible. For $E = 2$, we have the following diagrams:

$V = 0$:  ,

$S = 2$

$V = 1$: 

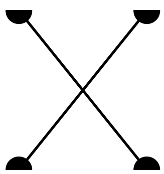
$S = 4$

$V = 2$:   

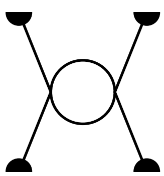
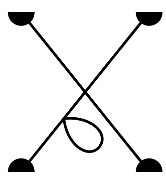
$S = 8$ $\quad S = 12$ $\quad S = 8$

For $E = 4$:

$V = 0$: no connected graphs

$V = 1$: 

$S = 4!$

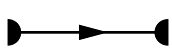
$V = 2$:  

$S = 16$ $\quad S = 12$

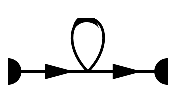
(c) As we saw in (b), there are no graphs with $E = 1$, so $\langle \varphi \rangle$ is zero.

4. (a) The vertex is $-iZ_\lambda \lambda$, involving two $\varphi^\dagger$s and two φs (ie. two inward arrows and two outward). Note that this implies that arrows cannot end or begin except at an external source. (The charge associated with the $U(1)$ complex phase symmetry is conserved.)

(b) As before, we have $2P = E + 4V$. In general, due to the arrows, the symmetry factors in this case are much smaller. For $E_\varphi = 1, E_{\varphi^\dagger} = 1$:

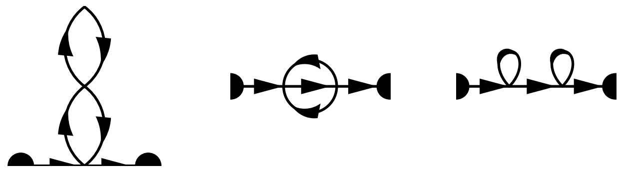
$V = 0$:  ,

$S = 1$

$V = 1$: 

$S = 1$

$V = 2$:

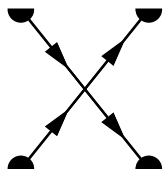


$S = 1$ $\quad$ $S = 2$ $\quad$ $S = 1$

For $E_\varphi = 2, E_{\varphi^\dagger} = 2$:

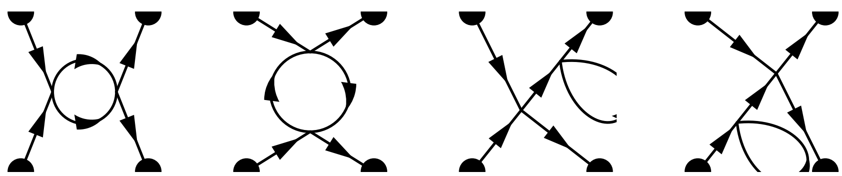
$V = 0$: no connected graphs

$V = 1$:



$S = 4$

$V = 2$:



$S = 8$ $\quad$ $S = 2$ $\quad$ $S = 2$ $\quad$ $S = 2$

5. We are asked to develop perturbation theory for $\langle 0|T\varphi(x_n)\dots\varphi(x_1)|0\rangle$ time-ordered products using the canonical operator formalism instead of path integrals. This involves decomposing the Hamiltonian into $H = H_0 + H_1$ and time-evolving the field with respect to the free part only:

$$\varphi_I(\vec{x}, t) = e^{iH_0t}\varphi(\vec{x}, 0)e^{-iH_0t}.$$

(a) There are two commutators we need which follow from the CCR. One of them is:

$$\begin{aligned}
[H_0, \varphi(\vec{x}, 0)] &= \int d^3x' [\frac{1}{2}\Pi(\vec{x}')^2, \varphi(\vec{x}, 0)] \\
&= \int d^3x' \Pi(\vec{x}')[\Pi(\vec{x}'), \varphi(\vec{x}, 0)] \\
&= - i\Pi(\vec{x})
\end{aligned}$$

The next is:

$$\begin{aligned}
[H_0, \Pi(\vec{,}x)] &= \int d^3x' \left\{\frac{1}{2}[(\nabla\varphi(\vec{x}'))^2, \Pi(\vec{x}, 0)] + \frac{1}{2}m^2[\varphi(\vec{x}')^2, \Pi(\vec{x}, 0)]\right\} \\
&= im^2\varphi(\vec{x}, 0) + i\int d^3x' \nabla\varphi(\vec{x}')\nabla\delta(\vec{x} - \vec{x}') \\
&= i(m^2 - \nabla^2)\varphi(\vec{x}, 0)
\end{aligned}$$

We may now start differentiating the interaction picture field. Using the product rule,

$$\partial_t\varphi_I(x) = ie^{iH_0t}[H_0, \varphi(\vec{x}, 0)]e^{-iH_0t} = e^{iH_0t}\Pi(\vec{x}, 0)e^{-iH_0t}.$$

Clearly, another derivative results in

$$\partial_t^2\varphi_I(x) = ie^{iH_0t}[H_0, \Pi(\vec{x}, 0)]e^{-iH_0t} = -e^{iH_0t}(m^2 - \nabla^2)\varphi(\vec{x}, 0)e^{-iH_0t}.$$

Bringing the exponentials inside again, we have

$$\partial_t^2 \varphi_I(x) = \nabla^2 \varphi_I(x) - m^2 \varphi_I(x)$$

which rearranges to Klein-Gordon.

(b) This follows immediately from the fact that the free and interacting Hamiltonians are separately Hermitian.

(c) The boundary condition is trivial. For the differential equation,

$$\begin{aligned} i\frac{d}{dt}U(t) =& -\left[H_0 e^{iH_0 t} e^{-iHt} - e^{iH_0 t} H e^{-iH_0 t}\right] \\ =& e^{iH_0 t} H_1 e^{-iHt} \\ =& e^{iH_0 t} H_1 e^{-iH_0 t} e^{iH_0 t} e^{-iHt} \\ \equiv& H_I(t) U(t) \end{aligned}$$

(d) This is the familiar fact from linear algebra that the act of taking sums and products of matrices commutes with the act of changing from one basis to another.

(e) If $U(t) = T \exp\left[-i\int_0^t H_I(t')dt'\right]$, it clearly satisfies $U(0) = 1$. Check that it satisfies the evolution equation by expanding

$$U(t) = 1 - i\int_0^t H_I(t')dt' - \int_0^t \int_{t'}^t H_I(t')H_I(t'')dt''dt' + i\int_0^t \int_{t'}^t \int_{t''}^t H_I(t')H_I(t'')H_I(t''')dt'''dt''dt' + \dots$$

Acting with $\frac{d}{dt}$ will kill the outermost integral and replace t' with t. Since $H_I(t')$ is always on the left with this ordering, the result is a left-multiplied $H_I(t)U(t)$ as desired. This assumes $t > 0$. If $t < 0$, we must change the time-ordering above to *reverse* time-ordering.

(f) If H_I commuted with itself at all times, we would clearly have

$$U(t_2)U^\dagger(t_1) = \exp\left(-i\int_0^{t_2} H_I(t')dt'\right)\exp\left(i\int_0^{t_1} H_I(t')dt'\right) = \exp\left(-i\int_{t_1}^{t_2} H_I(t')dt'\right).$$

What allows us to still do this for more general Hamiltonians is the time-ordering symbol. Any "would-be mistake" of writing a product in the wrong order is undone by T because this prescribes a specific order for all the operators anyway.

(g) The integral representation above, and the equivalent for reverse time-ordering, are not needed to show these basic properties. Simply writing $U(t_2, t_1) = U(t_2)U^\dagger(t_1)$ is enough to show that $U(t_3, t_1) = U(t_3, t_2)U(t_2, t_1) = U^\dagger(t_1, t_3)$.

(h) Using part (b), we may put $U(t_j)U^\dagger(t_{j-1})$ between pairs of fields $\varphi(x_j)\varphi(x_{j-1})$ and then use part (f). For the $U^\dagger(t_n)$ and $U(t_1)$ that remain on the outside, we may freely substitute 0 as the second argument.

(i) We may get this by substituting 0 and ∞ into the identities of part (g) because this question only asks us to prove a special case of them.

(j) We know that $e^{-iHt}|0\rangle = |0\rangle$ after renormalization. Therefore $U(-\infty, 0)|0\rangle = \lim_{t\to-\infty} e^{iH_0 t}|0\rangle$. We will invoke the $i\epsilon$ prescription for H_0 and expand $|0\rangle$ in its eigenbasis $|\not{n}\rangle$.

$$U(\infty, 0)|0\rangle = \lim_{t\to-\infty} \sum_n e^{i(1-i\epsilon)H_0 t}|\not{n}\rangle\langle\not{n}|0\rangle = |\not{0}\rangle\langle\not{0}|0\rangle.$$

We have used the fact that $\epsilon > 0$ to say that only the $n = 0$ term survives. The other identity follows from Hermitian conjugation.

(k) In the result of part (h), we want to write the first $U^\dagger(t_n, 0)$ and the last $U(t_1, 0)$ in the form suggested by part (i). Then it is clear that our two extra unitary operators (those depending on ∞) turn $|0\rangle$ into $|\emptyset\rangle$ and $\langle 0|$ into $\langle\emptyset|$ while giving us $\langle\emptyset|0\rangle$ twice.

(l) The magic of applying T to both sides of

$$\langle 0|\varphi(x_n)\ldots\varphi(x_1)|0\rangle = \langle\emptyset|U(\infty, t_n)\varphi_I(x_n)U(t_n, t_{n-1})\varphi_I(x_{n-1})\ldots U(t_2, t_1)\varphi_I(x_1)U(t_1, -\infty)|\emptyset\rangle| \langle 0|\emptyset\rangle |^2$$

is again the fact that it prescribes a specific ordering. So we may blithely rearrange operators after doing so. This allows us to put all of the Us together giving us $U(\infty, t_n)\ldots U(t_1, -\infty) = U(\infty, -\infty) = e^{-i\int_{-\infty}^{\infty} H_I(t)dt} = e^{-i\int d^4x\mathcal{H}_I}$.

(m) Since all states are normalized, we should get $| \langle 0|\emptyset\rangle |^2$ by setting all $\varphi(x_j)$ to the identity in our expression.