PHY 610 QFT, Spring 2017
HW3 Solutions

1. (a)

$$\begin{aligned} S_0 &= \int d^dx - \frac{1}{2}\frac{\partial}{\partial x^\mu}\varphi(x)\frac{\partial}{\partial x_\mu}\varphi(x) \\ S_0' &= \int d^dx - \frac{1}{2}\frac{\partial}{\partial x^\mu}\varphi'(x)\frac{\partial}{\partial x_\mu}\varphi'(x) \\ &= \int d^dx - \frac{1}{2}\lambda^{-2\Delta}\frac{\partial}{\partial x^\mu}\varphi(x/\lambda)\frac{\partial}{\partial x_\mu}\varphi(x/\lambda) \\ &= \int d^dy - \frac{1}{2}\lambda^{d-2-2\Delta}\frac{\partial}{\partial y^\mu}\varphi(y)\frac{\partial}{\partial y_\mu}\varphi(y) \end{aligned}$$

Letting $y = x/\lambda$ has shown us that we get the same action if $\Delta = \frac{d-2}{2}$. This should not be surprising. It is the engineering dimension that φ needs to have in order for S to be unitless. Now that we know Δ, we may assume V to be a power law and determine the exponent.

$$\begin{aligned} S_1 &= \int d^dxg\varphi^n(x) \\ S_1' &= \int d^dxg\varphi'^n(x) \\ &= \int d^dx\lambda^{-n\Delta}\varphi^n(x/\lambda) \\ &= \int d^dy\lambda^{d-n\Delta}\varphi^n(y) \end{aligned}$$

We now have $n = \frac{d}{\Delta} = \frac{2d}{d-2}$. In two dimensions, the allowed potentials with classical scale invariance are essentially arbitrary.

(b) The identity transformation corresponds to $\lambda = 1$, so we perturb around this.

$$\begin{aligned} \varphi'(x) &= (1+\delta\lambda)^{-\Delta}\varphi\left(\frac{x}{1+\delta\lambda}\right) \\ &\approx (1-\Delta\delta\lambda)\varphi(x-\delta\lambda x) \\ &\approx \varphi - \delta\lambda\left(\Delta\varphi + x^\nu\partial_\nu\varphi\right) \end{aligned}$$

The part proportional to $\delta\lambda$ is of course $\delta\varphi$ so this is one thing that we plug into the conserved current expression. The other is the total derivative into which $\mathcal{L}_0$ varries. From above,

$$\begin{aligned} \mathcal{L}_0' &= -\frac{1}{2}(1+\delta\lambda)^{2-d}\partial_\mu\varphi(x(1-\delta\lambda))\partial^\mu\varphi(x(1-\delta\lambda)) \\ &= \mathcal{L}_0 + \frac{d-2}{2}\delta\lambda\partial_\mu\varphi\partial^\mu\varphi + \delta\lambda\partial_\mu(x^\nu\partial_\nu\varphi)\partial^\mu\varphi \\ &= \mathcal{L}_0 + \frac{d}{2}\delta\lambda\partial_\mu\varphi\partial^\mu\varphi + \delta\lambda x^\nu\partial_\mu\partial_\nu\varphi\partial^\mu\varphi \\ &= \mathcal{L}_0 + \delta\lambda\partial_\nu\left(\frac{1}{2}x^\nu\partial_\mu\varphi\partial^\mu\varphi\right) \end{aligned}$$

Now reading off K_μ, we have

$$\begin{aligned} J^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\frac{\delta\varphi}{\delta\lambda} - K^\mu \\ &= -\partial^\mu\varphi\frac{\delta\varphi}{\delta\lambda} - \frac{1}{2}x^\mu\partial_\nu\varphi\partial^\nu\varphi \\ &= \partial^\mu\varphi\left(\Delta\varphi + x^\nu\partial_\nu\varphi\right) - \frac{1}{2}x^\mu\partial_\nu\varphi\partial^\nu\varphi \end{aligned}$$

2. We are to take the magnitude of (7.10). First, for real $f(t)$, note that $\tilde{f}(-E) = \tilde{f}(E)^*$, so

$$\begin{aligned} \left|\langle 0|0\rangle_f\right|^2 &= \left|\exp\left(\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\left|\tilde{f}(E)\right|^2}{-E^2+\omega^2-i\epsilon}\right)\right|^2 \\ &= \exp\left(-\operatorname{Im}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\left|\tilde{f}(E)\right|^2}{-E^2+\omega^2-i\epsilon}\right) \\ &= \exp\left(-\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left|\tilde{f}(E)\right|^2\frac{\epsilon}{(-E^2+\omega^2)^2+\epsilon^2}\right). \end{aligned}$$

Now, notice that as $\epsilon\to 0$, $\epsilon/((-E^2+\omega^2)^2+\epsilon^2)$ is an increasingly sharper function of E^2, peaked at $E^2=\omega^2$. The area under the graph is

$$\int dE^2\,\frac{\epsilon}{(-E^2+\omega^2)^2+\epsilon^2} = \pi,$$

independent of ϵ. This means that

$$\lim_{\epsilon\to 0}\frac{\epsilon}{(-E^2+\omega^2)^2+\epsilon^2} = \pi\delta(-E^2+\omega^2) = \frac{\pi}{2\omega}(\delta(E-\omega)+\delta(E+\omega)).$$

We therefore have

$$\left|\langle 0|0\rangle_f\right|^2 = \exp\left(-\frac{\left|\tilde{f}(\omega)\right|^2}{2\omega}\right).$$

3. We will act with $-\partial_x^2+m^2$ on the explicit expression for $\Delta(x-x')$. Since plane waves are annihilated by the Klein-Gordon operator, we only need to consider terms where at least one derivative acts on $\theta(t-t')$ or $\theta(t'-t)$. Therefore,

$$\begin{aligned} (-\partial_x^2+m^2)\Delta(x-x') =& -[i\partial_x^2\theta(t-t')]\int \tilde{dk}e^{ik(x-x')} - 2i[\partial_x\theta(t-t')]\int\tilde{dk}ike^{ik(x-x')} \\ & -[i\partial_x^2\theta(t'-t)]\int\tilde{dk}e^{-ik(x-x')} - 2i[\partial_x\theta(t'-t)]\int\tilde{dk}-ike^{-ik(x-x')} \\ =& i[\partial_t^2\theta(t-t')]\int\tilde{dk}e^{ik(x-x')} + i[\partial_t^2\theta(t'-t)]\int\tilde{dk}e^{-ik(x-x')} \\ & + 2[\partial_t\theta(t-t')]\int\tilde{dk}\omega e^{ik(x-x')} - 2[\partial_t\theta(t'-t)]\int\tilde{dk}\omega e^{-ik(x-x')} \end{aligned}$$

In the second step, we have used the fact that the functions being differentiated only depend on time. We may now use the identity $\partial_t\theta(t-t') = \delta(t-t') = -\partial_t\theta(t'-t)$. If there are two time derivatives on $\theta(t-t')$, the second one is handled by partially integrating. This tells us that

$$\partial_t^2\theta(\pm(t-t'))\int\tilde{dk}e^{\pm ik(x-x')} = i\delta(t-t')\int\tilde{dk}\omega e^{\pm ik(x-x')}$$

Substituting this into our expression,

$$\begin{aligned}(-\partial_x^2+m^2)\Delta(x-x') =& -\delta(t-t')\int \tilde{dk}\omega(e^{ik(x-x')}+e^{-ik(x-x')})+2\delta(t-t')\int \tilde{dk}\omega(e^{ik(x-x')}+e^{-ik(x-x')}) \\ =&\delta(t-t')\int \frac{d^3k}{2(2\pi)^3}e^{-i\omega(t-t')}e^{i\vec{k}(\vec{x}-\vec{x}')}+e^{i\omega(t-t')}e^{-i\vec{k}(\vec{x}-\vec{x}')} \\ =&\delta(t-t')\delta^3(\vec{x}-\vec{x}')\frac{1}{2}\left(e^{-i\omega(t-t')}+e^{i\omega(t-t')}\right) \\ =&\delta^4(x-x')\end{aligned}$$

4. We know how to evaluate $\langle 0|[\phi_0(x),\phi_0(y)]|0\rangle$ from the scalar field mode expansion. The exponential $e^{i\vec{k}(\vec{y}-\vec{x})}$ may be turned into $e^{i\vec{k}(\vec{x}-\vec{y})}$ with a $k\leftrightarrow -k$ substitution. Exponentials with time components however remain unaffected by this. This leads to the expression

$$\int \frac{d^3k}{2\omega_k(2\pi)^3}e^{i\vec{k}(\vec{x}-\vec{y})}\left(e^{i\omega_k(x^0-y^0)}+e^{-i\omega_k(x^0-y^0)}\right)$$

Multipyling this by a Heaviside function determines which propagator we are evaluating. We will consider the advanced propagator $\theta(x^0-y^0)\langle 0|[\phi_0(x),\phi_0(y)]|0\rangle$. We know that a function with $e^{\pm i\omega_k(x^0-y^0)}/2\omega_k$ residues is $f(\omega)=\frac{e^{-i\omega(x^0-y^0)}}{\omega^2-\omega_k^2}$. If C is a contour encircling the two poles,

$$\begin{aligned}\theta(x^0-y^0)\langle 0|[\phi_0(x),\phi_0(y)]|0\rangle =&\theta(x^0-y^0)\frac{1}{2\pi i}\int_C\int \frac{d\omega d^3k}{(2\pi)^3}e^{-i\omega_k(x^0-y^0)+i\vec{k}(\vec{x}-\vec{y})}\frac{1}{\omega^2-\omega_k^2} \\ =&\theta(x^0-y^0)\int_C\int \frac{d\omega d^3k}{(2\pi)^4}e^{ik(x-y)}\frac{-i}{\omega^2-\omega_k^2} \\ =&\theta(x^0-y^0)\int_C\int \frac{dk_0 d^3k}{(2\pi)^4}e^{ik(x-y)}\frac{i}{-k_0^2+\vec{k}^2+m^2}\end{aligned}$$

To replace the C integral with a $-\infty$ to ∞ integral, we need to close the contour with a path where $f(\omega)$ vanishes: we need to decide whether to go up or down. Since $x^0-y^0>0$, negative imaginary values of ω are what cause $f(\omega)$ to decay. We therefore close down which allows us to change the integral and drop the Heaviside function. Also, we notice that this picks up a sign for being a clockwise integral. We arrive at

$$\int \frac{d^4k}{(2\pi)^4}e^{ik(x-y)}\frac{-i}{-(k_0+i\epsilon)^2+\vec{k}^2+m^2}$$

The k_0 poles have been shifted into the lower half plane so that they remain strictly within the contour we have drawn. The retarded propagator calculation goes the same way and results in a $-i\epsilon$ in the final step.

5. Writing a path integral with the suggested source term,

$$\begin{aligned}Z[J,J^\dagger] &= \int \mathcal{D}\phi \exp\left[i\int d^4x\partial_\mu\phi^\dagger\partial^\mu\phi-m^2\phi^\dagger\phi+J^\dagger\phi+J\phi^\dagger\right] \\ &= \int \mathcal{D}\phi \exp\left[i\int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}d^4x-(kk'+m^2)\tilde{\phi}^\dagger(k)\tilde{\phi}(k')+\tilde{J}^\dagger(k)\tilde{\phi}(k')+\tilde{J}(k)\tilde{\phi}^\dagger(k')\right] \\ &= \int \mathcal{D}\phi \exp\left[i\int \frac{d^4k}{(2\pi)^4}-\tilde{\phi}^\dagger(k^2+m^2)\tilde{\phi}+\tilde{J}^\dagger\tilde{\phi}+\tilde{J}\tilde{\phi}^\dagger\right]\end{aligned}$$

We must now complete the square using the following change of variables that has a Jacobian of 1.

$$\chi(k) = \tilde{\phi}(k) - \frac{\tilde{J}(k)}{k^2+m^2}, \quad \chi^\dagger(k) = \tilde{\phi}^\dagger(k) - \frac{\tilde{J}^\dagger(k)}{k^2+m^2}$$

Plugging this in,

$$\begin{aligned} Z[J,J^\dagger] =& e^{i\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}^\dagger \tilde{J}}{k^2+m^2}} \int \mathcal{D}\chi \exp\left[i\int \frac{d^4k}{(2\pi)^4} - \chi^\dagger(k^2+m^2)\chi\right] \\ =& e^{i\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}^\dagger \tilde{J}}{k^2+m^2}} \end{aligned}$$

The path integral that this multiplies has been set to 1 because it is nothing but the vacuum to vacuum amplitude. Inserting an $i\epsilon$ for causality and Fourier transforming back, $Z[J,J^\dagger] = e^{i\int d^4x d^4y J^\dagger(x)\Delta(x-y)J(y)}$.

Using the fact that $J^\dagger$ sources ϕ and vice versa,

$$\langle 0|T\phi(x_1)\ldots\phi^\dagger(y_1)\ldots|0\rangle = \frac{\delta}{i\delta J^\dagger(x_1)}\cdots\frac{\delta}{i\delta J(y_1)}\cdots Z[J^\dagger,J]\bigg|_{J=J^\dagger=0}$$

This is one of the nice things about path integrals. They give us time ordering for free. From this it follows that

$$\begin{aligned} \langle 0|T\phi(x_1)\phi(x_2)\ldots|0\rangle =& 0 \\ \langle 0|T\phi^\dagger(x_1)\phi^\dagger(x_2)\ldots|0\rangle =& 0 \\ \langle 0|T\phi(x_1)\phi^\dagger(x_2)\ldots|0\rangle =& -i\Delta(x_1-x_2) \end{aligned}$$

After all, if we only differentiate with respect to one source, the other one will still be multiplying everything at the end. Because of this, any correlation function of the charged scalar fields needs to have an equal number of ϕ and $\phi^\dagger$ insertions to survive. Decomposing into two-point functions where there is always one of each, Wick's theorem becomes

$$\langle 0|T\phi(x_1)\ldots\phi(x_n)\phi^\dagger(y_1)\ldots\phi^\dagger(y_n)|0\rangle = (-i)^n \sum_{\sigma\in S_n}\prod_{i=1}^n \Delta(x_i - y_{\sigma(i)})$$

6. We are to explicitly evaluate the Feynman propagator in position space, which in our metric convention is

$$D_F(x,0) = \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+m^2-i\epsilon}e^{-ikx}.$$

Let us first perform the k^0 integral. (The following discussion is essentially identical to that of Schwartz, p. 76, but going backwards.) Writing the denominator as $-k_0^2 + \vec{k}^2 + m^2 - i\epsilon = -(k_0 + \sqrt{\vec{k}^2+m^2-i\epsilon})(k_0 - \sqrt{\vec{k}^2+m^2-i\epsilon})$, we see that there are two simple poles at $\pm\sqrt{\vec{k}^2+m^2-i\epsilon}$, one with positive real part and displaced slightly below the real axis, and one with negative real part displaced slightly above the real axis (see fig 6.1, p. 76). We wish to use the residue theorem of complex analysis to perform this integral. For $x^0 > 0$, the exponential factor e^{-ikx} becomes small when k_0 has large imaginary part, so we can close the integral in the upper half plane. This means we pick up the residue of the pole at $k_0 = -\sqrt{\vec{k}^2+m^2-i\epsilon}$. Conversely, for $x^0 < 0$, we have to close the integral in the lower half plane, picking up the residue of the other pole (as well as a minus sign for a clockwise contour). Thus

$$\begin{aligned}
D_F(x,0) &= \int \frac{d^3k}{(2\pi)^3}\left(\theta(x^0) i \frac{ie^{-i\vec{k}\vec{x}-ix^0\sqrt{\vec{k}^2+m^2-i\epsilon}}}{-2\sqrt{\vec{k}^2+m^2-i\epsilon}} + \theta(-x^0)(-i)\frac{ie^{-i\vec{k}\vec{x}+ix^0\sqrt{\vec{k}^2+m^2-i\epsilon}}}{2\sqrt{\vec{k}^2+m^2-i\epsilon}}\right) \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{e^{-i\vec{k}\vec{x}-i|x^0|\sqrt{\vec{k}^2+m^2-i\epsilon}}}{2\sqrt{\vec{k}^2+m^2-i\epsilon}}.
\end{aligned}$$

In 3+1 dimensions, the angular integral can be performed easily, since there is conveniently a $\sin\theta$ in the measure to go with the $e^{-ikr\cos\theta}$ in the integrand:

$$\begin{aligned}
D_F(x,0) &= \int_0^\infty k^2\, dk\, \frac{1}{8\pi^2}\int_{-1}^{1} d\cos\theta\, \frac{e^{-ikx\cos\theta - i|x^0|\sqrt{\vec{k}^2+m^2-i\epsilon}}}{\sqrt{\vec{k}^2+m^2-i\epsilon}} \\
&= \int_0^\infty dk\, \frac{k^2 e^{-i|x^0|\sqrt{\vec{k}^2+m^2-i\epsilon}}}{4\pi^2\sqrt{\vec{k}^2+m^2-i\epsilon}}\frac{\sin(kx)}{kx}.
\end{aligned}$$

This expression is actually a Bessel function. (Bessel functions are solutions to Laplace's equation.) One way to see this is to invoke Lorentz invariance to simplify the integral. For timelike separations x, rotate to a frame where $x^\mu = (x^0, 0)$, so that the $\sin(kx)/kx$ factor in the integrand becomes unity. Then,

$$\begin{aligned}
D_F(x,0) &= \int_0^\infty dk\, \frac{k^2 e^{-i|x^0|\sqrt{\vec{k}^2+m^2-i\epsilon}}}{4\pi^2\sqrt{\vec{k}^2+m^2-i\epsilon}} \\
&= \frac{m^2}{4\pi^2}\int_{1-i\epsilon}^{\infty-i\epsilon} dy\, \sqrt{y^2-1+i\epsilon}\, e^{-im|x^0|y} \\
&= \frac{im}{8\pi|x^0|}H_1^{(1)}(m|x^0|), \qquad \text{(timelike)}
\end{aligned}$$

where on the second line we have changed variables to $y = \sqrt{k^2+m^2-i\epsilon}/m$, and on the last line we have used the following integral representation for the Hankel function $H_\nu^{(1)}$,[1]

$$H_\nu^{(1)} = -i\frac{2(-a/2)^\nu}{\pi^{1/2}\Gamma(\nu+1/2)}\int_1^{\infty-i\epsilon} e^{-iat}(t^2-1)^{\nu-1/2}\, dt, \qquad \Re(\nu) > 1/2, a > 0.$$

Meanwhile, for spacelike separations x, rotate to a frame where $x^\mu = (0, \vec{x})$, then

$$\begin{aligned}
D_F(x,0) &= \int_0^\infty dk\, \frac{k\sin(kx)}{4\pi^2 x\sqrt{\vec{k}^2+m^2-i\epsilon}} \\
&= \frac{m}{4\pi^2 x}K_1(mx), \qquad \text{(spacelike)}
\end{aligned}$$

where we have used another integral representation of the modified Hankel function $K_\nu(a) = i^{\nu+1}(\pi/2)H_\nu^{(1)}(ia)$. Note that K_1 is simply H_1 at imaginary arguments (up to some constant factors), so this spacelike case could have been obtained from the timelike case by analytic continuation.

Finally, for lightlike x, notice that, writing $2i\sin(kx) = e^{ikx} - e^{-ikx}$, the numerator of the integrand consists of the exponentials $e^{-i\omega_{\vec{k}}|x^0| \pm ikx}$, which, since $x^\mu x_\mu = 0$, approaches 1 as $\left|\vec{k}\right| \to \infty$. Therefore,

[1] Bessel functions satisfy a wealth of identities. For more information, see eg. Arfken and Weber's Mathematical Methods for Physicists.

at large k, the integrand is of order 1, so $D_F(x,0)$ diverges. Since this is an ultraviolet divergence, we can compute it in the massless limit (ie. consider the region with $m/k \ll 1$). Then

$$\begin{aligned} D_F(x,0)|_{m=0} =& \frac{-i}{8\pi^2 x}\int_0^\infty dk\ (e^{-ik(|x^0|-x)} - e^{-ik(|x^0|+x)}) \\ =& \frac{-i}{8\pi x}(\delta(x-|x^0|) - \delta(x+|x^0|)) \\ =& \frac{-i}{4\pi}\delta(x^\mu x_\mu). \end{aligned} \qquad \text{(lightlike)}$$

Putting this all together, and restoring Lorentz invariance, the final result for the Feynman propagator is

$$D_F(x,0) = \theta(-x^2)\frac{im}{8\pi\sqrt{-x^2}}H_1^{(1)}(m\sqrt{-x^2}) + \theta(x^2)\frac{m}{4\pi^2\sqrt{x^2}}K_1(m\sqrt{x^2}) + \frac{-i}{4\pi}\delta(x^2).$$

In the timelike region, $H_1^{(1)}$ is an outgoing wave, while in the spacelike region, K_1 is exponentially decaying.[2] This is indeed what we expect for a solution to the wave equation.

In the $m^2 \to 0$ limit, using the Bessel function asymptotics $K_1(z) = 1/z + O[z]$, $H_1^{(1)}(z) = -2i/\pi z + O[z]$, or by computing the integral form of D_F directly, we obtain the conformal propagator

$$D_F(x,0) = \frac{1}{4\pi^2 x^2} + \frac{-i}{4\pi}\delta(x^2) = \frac{1}{4\pi^2(x^2+i\epsilon)},$$

where the useful identity $1/(a+i\epsilon) = \mathrm{p.\,v.}\,1/a - i\pi\delta(a)$ of distributions has been used.

For the advanced Green function with $m=0$, we have $\Delta_{adv}(x) = 0$ if $x_0 < 0$.

For $x_0 > 0$,

$$\begin{aligned} D_{adv}(x)|_{m=0} &= \int \frac{d^3k}{(2\pi)^3}e^{ikx}\int\frac{dk^0}{2\pi}\frac{e^{-ik_0x_0}}{(-k_0+\omega_k+i\epsilon)(-k_0-\omega_k+i\epsilon)} \\ &= \int \frac{d^3k}{(2\pi)^3}e^{ikx}i[\frac{e^{-i\omega_k x_0} - e^{i\omega_k x_0}}{2\omega_k}] \\ &= \int \frac{d^3k}{(2\pi)^3}e^{ikx}\frac{\sin(kx_0)}{k} \\ &= \int_0^\infty \frac{k^2dk}{2\pi^2}\frac{-\sin(k|x|)}{k|x|}\frac{\sin(kx_0)}{k} \\ &= \frac{1}{8\pi^2 x}\int_0^\infty dk(e^{ik|x|} - e^{-ik|x|})(e^{ikx_0} - e^{-ikx_0}) \\ &= \frac{\delta(|x|+x_0) - \delta(|x|-x_0)}{4\pi x} \\ &= -\frac{1}{2\pi}\delta(x^\mu x_\mu). \end{aligned}$$

7. The idea is to compare two methods of computing the commutator $[\varphi(x), \dot\varphi(y)]$; using canonical quantization and the Lehmann-Källén exact propagator. It is straightforward to show that, using canonical quantization, we have at equal times

$$Z_\varphi[\varphi(x), \dot\varphi(y)] = [\varphi(x), \Pi(y)] = i\delta^3(\mathbf{x}-\mathbf{y}).$$

[2] A useful mnemonic for the asymptotic behaviors of the Bessel functions is that J behaves like cos, N like sin, $H^{(1)} = J + iN$ like e^{iz} and $H^{(2)} = J - iN$ like e^{-iz}. Then $K(z) \sim H^{(1)}(iz)$ is exponentially damped and $I(z) \sim H^{(2)}(iz)$ grows exponentially.

Meanwhile, take the y^0 derivatives of (13.12), (13.13) to obtain, at equal times,

$$\begin{aligned}\langle 0|\varphi(x)\dot{\varphi}(y)|0\rangle =& \int \frac{d^3k}{(2\pi)^3 2\sqrt{\vec{k}^2+m^2}} i\sqrt{\vec{k}^2+m^2} e^{ik(x-y)} + \int_{4m^2}^{\infty} \rho(s) ds \int \frac{d^3k}{(2\pi)^3 2\sqrt{\vec{k}^2+s}} i\sqrt{\vec{k}^2+s} e^{ik(x-y)} \\ =& \frac{i}{2}\delta^3(\mathbf{x}-\mathbf{y})\left(1+\int_{4m^2}^{\infty}\rho(s)\, ds\right).\end{aligned}$$

Hence

$$\langle 0|[\varphi(x), \dot{\varphi}(y)]|0\rangle = i\delta^3(\mathbf{x}-\mathbf{y})\left(1+\int_{4m^2}^{\infty}\rho(s)\, ds\right),$$

and comparing the two expressions, we conclude that

$$Z_\varphi = \left(1+\int_{4m^2}^{\infty}\rho(s)\, ds\right)^{-1}.$$

The Lehmann-Källén exact propagator yields an easy way to compute the wavefunction normalization.