PHY 610 QFT, Spring 2017

HW2 Solutions

1. We are to show that the Noether charge,

$$Q = \int d^3x\, j^0(x) = \int d^3x\, \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_a(x))} \delta\varphi_a(x) = \int d^3x\, \pi^a(x)\delta\varphi_a(x),$$

generates the symmetry transformation, $[Q, \varphi_a] = -i\delta\varphi_a$. This is a straightforward calculation, using the canonical commutation relation $[\varphi_a(x), \Pi^b(y)] = i\delta_a^b \delta^3(x-y)$,[1]

$$\begin{aligned}[Q, \varphi_a(x)] &= \int d^3y\, [\Pi^b(y)\delta\varphi_b(y), \varphi_a(x)] \\ &= \int d^3y\, \left(\pi^b(y)[\delta\varphi_b(y), \varphi_a(x)] + [\Pi^b(y), \varphi_a(x)]\delta\varphi_b(y)\right) \\ &= -\, i\delta\varphi_a(x),\end{aligned}$$

where we have assumed that $\delta\varphi_a$ is independent of Π^a in order to set the first term in the integrand to zero.

2. We are to verify that the Noether charge for translations,

$$P^\mu = \int d^3x\, T^{0\mu}(x) = \int d^3x\, (-\Pi^a(x)\partial^\mu\varphi_a(x) + g^{0\mu}\mathcal{L}(x)),$$

indeed generates infinitesimal translations $[P^\mu, \varphi_a(x)] = i\partial^\mu\varphi_a(x)$. For $\mu \neq 0$, the calculation proceeds exactly as in problem 2,

$$[P^i, \varphi_a(x)] = \int d^3y\, -[\Pi^b(y), \varphi_a(x)]\partial^i\varphi_b(y) = i\partial^i\varphi_a(x).$$

For $\mu = 0$, notice that the assumption in problem 2, that $\delta\varphi_a$ is independent of Π^a, no longer holds. In this case

$$\begin{aligned}[P^0, \varphi_a(x)] &= \int d^3y\, \frac{1}{2}[\Pi^b(y)\Pi^b(y) + \partial_i\varphi_b(y)\partial_i\varphi_b(y) + 2V(\varphi(y)), \varphi_a(x)] \\ &= -\, i\Pi^a(x) = i\partial^0\varphi_a(x).\end{aligned}$$

3. (a) We are to derive the algebra satisfied by the currents

$$\begin{cases} T^{00} = & \frac{1}{2}\Pi_a^2 + \frac{1}{2}(\partial_i\varphi_a)^2 + V(\varphi), \\ T^{0j} = & -\Pi_a\partial^j\varphi_a. \end{cases}$$

We begin with the less complicated one,[2]

$$\begin{aligned}[T^{0j}(x), T^{0k}(y)] =& [\Pi_a(x)\partial^j\varphi_a(x), \Pi_b(y)\partial^k\varphi_b(y)] \\ =& \Pi_a(x)[\partial^j\varphi_a(x), \Pi_b(y)]\partial^k\varphi_b(y) + \Pi_b(y)[\Pi_a(x), \partial^k\varphi_b(y)]\partial^j\varphi_a(x) \\ =& \Pi_a(x)\partial^{x_j}[\varphi_a(x), \Pi_b(y)]\partial^k\varphi_b(y) + \Pi_b(y)\partial^{y_k}[\Pi_a(x), \varphi_b(y)]\partial^j\varphi_a(x) \\ =& i\Pi_a(x)\partial^k\varphi_a(y)\partial^{x_j}\delta^3(x-y) - i\Pi_a(y)\partial^j\varphi_a(x)\partial^{y_k}\delta^3(y-x)\end{aligned}$$

[1] Strictly speaking, the canonical commutation relation holds only at equal times $x^0 = y^0$. However, because Q is a conserved charge, it is time independent, so we may choose y^0 to be equal to x^0 in the following calculation.

[2] Delta functions are even so the ∂^{x_j} and ∂^{y_k} could unambiguously be rewritten as ∂^j and ∂^k when acting on them.

To obtain the first equality, imagine pushing first $\partial^j\varphi_a$, and then Π_a, to the right. Now, what are derivatives of delta functions? Recall that delta functions only make sense inside an integral, $\int dx\ \delta(x-a)f(x) = f(a)$, so we can think of derivatives of delta functions as being defined by partial integration, ie. $\int dx\ \partial_x\delta(x-a)f(x) = -\int dx\ \delta(x-a)f'(x) = -f'(a)$. Hence we partially integrate to get rid of derivatives on the delta functions, yielding[3]

$$\boxed{[T^{0j}(x), T^{0k}(y)] = -i\delta^3(x-y)\left(\partial^{[j}\Pi_a(x)\partial^{k]}\varphi_a(x) + \Pi_a(x)\partial^k\varphi_a(x)\partial^{x_j} - \Pi_a(x)\partial^j\varphi_a(x)\partial^{y_k}\right)}.$$

Note that the last two terms in the parentheses on the right are important if the commutator is multiplying another term inside the integral (the open-ended derivatives will act on the other term).

Similarly,

$$[T^{00}(x), T^{0j}(y)] = -\frac{1}{2}\Pi_b(y)[\Pi^2(x), \partial^j\varphi_b(y)] - \frac{1}{2}[(\partial^k\varphi_a)^2(x), \Pi_b(y)]\partial^j\varphi_b(y) - [V(\varphi(x)), \Pi_b(y)]\partial^j\varphi_b(y).$$

We now know how to deal with the first two terms, but how about $[V(\varphi(x)), \Pi_b(y)]$? The trick is to think of the commutator as a derivative, and use the chain rule to bring down powers of φ from $V(\varphi)$.[4] Therefore

$$[V(\varphi(x)), \Pi_b(y)] = i\frac{\partial V}{\partial\varphi_b(x)}\delta^3(x-y).$$

(Alternatively, this can be seen by Taylor expanding V.) Therefore,

$$[T^{00}(x), T^{0j}(y)] = i\Pi_a(x)\Pi_a(y)\partial^{y_j}\delta^3(y-x) - i\partial^k\varphi_a(x)\partial^{x_k}\delta^3(x-y)\partial^j\varphi_a(y) - i\frac{\partial V}{\partial\varphi_b(x)}\delta^3(x-y)\partial^j\varphi_b(y).$$

This commutator may be written in a neat way as follows. Replace $\partial^{x_k}\delta^3(x-y)$ with $-\partial^{y_k}\delta^3(x-y)$ in the second term, and then partial integrate the first two terms in y, to yield

$$\begin{aligned}[T^{00}(x), T^{0j}(y)] = \delta^3(x-y)&\left(-i\Pi_a(x)\partial^j\Pi_a(y) - i\partial^k\varphi_a(x)\partial^k\partial^j\varphi_a(y) - i\frac{\partial V}{\partial\varphi_b(x)}\partial^j\varphi_b(y)\right)\\ &- i\delta^3(x-y)\left(\Pi_a(x)\Pi_a(y)\partial^{y_j} + \partial^k\varphi_a(x)\partial^j\varphi_a(y)\partial^{y_k}\right).\end{aligned}$$

Due to the delta function, we may replace y with x in the expression above. We then recognize that the first term in parentheses as $-i\partial^j T^{00}(x)$, so

$$\boxed{[T^{00}(x), T^{0j}(y)] = -i\delta^3(x-y)\partial^j T^{00}(x) - i\delta^3(x-y)(\Pi^2(x)\partial^{y_j} + \partial^k\varphi_a(x)\partial^j\varphi_a(y)\partial^{y_k})}.$$

[3] $A^{[j}B^{k]}$ is shorthand for $A^jB^k - A^kB^j$.

[4] The operator $[A,\cdot]$ satisfies the Leibniz rule, which is to say, $[A, BC] = B[A,C] + [A,B]C$, so the commutator (with some field A) is in fact a derivation.

Finally, a similar calculation shows

$$\begin{aligned}
[T^{00}(x), T^{00}(y)] =& \frac{1}{4}[\Pi^2(x), (\partial^j\varphi(y))^2] + \frac{1}{2}[\Pi^2(x), V(\varphi(y))] - (x \leftrightarrow y) \\
=& \frac{1}{4}[\Pi^2(x), \partial^j\varphi_a(y)]\partial^j\varphi_a(y) + \frac{1}{4}\partial^j\varphi_a(y)[\Pi^2(x), \partial^j\varphi_a(y)] + \frac{1}{2}[\Pi^2(x), V(\varphi(y))] - (x \leftrightarrow y) \\
=& -\frac{i}{2}\Pi_a(x)\partial^j\varphi_a(y)\partial^{y_j}\delta^3(x-y) - \frac{i}{2}\partial^j\varphi_a(y)\Pi_a(x)\partial^{y_j}\delta^3(x-y) \\
& \quad - i\Pi_a(x)\frac{\partial V}{\partial\varphi_a(y)}\delta^3(x-y) - (x \leftrightarrow y) \\
=& \frac{i}{2}\delta^3(x-y)\left(\Pi_a(x)\partial^j\varphi_a(y)\partial^{y_j} + \partial^j\varphi_a(y)\Pi_a(x)\partial^{y_j} - (x \leftrightarrow y)\right) \\
=& \boxed{-\frac{i}{2}\delta^3(x-y)\left(\Pi_a(x)\partial^j\varphi_a(x) + \partial^j\varphi_a(x)\Pi_a(x)\right)\left(\partial^{x_j} - \partial^{y_j}\right)}.
\end{aligned}$$

(b) Integrating the commutators we just derived over x and y will yield the algebra for the translation generators, $[H, H]$, $[P^j, H]$ and $[P^j, P^k]$. It is clear that $[H, H] = 0$, since $[T^{00}, T^{00}]$ only involves derivative operators. Next, $[P^i, P^j] = -i\int d^3x\, \partial^{[j}\Pi_a\partial^{k]}\varphi_a$, and partially integrating in x^j, the integrand becomes $\Pi_a\partial^{[j}\partial^{k]}\varphi_a = 0$. Finally, $[H, P^j] = -i\int d^3x\, \partial^j T^{00} = 0$, since $\partial^j T^{00}$ is a total derivative.

For the Lorentz generators $M^{\mu\nu} = \int d^3x\, T^{0[\nu}x^{\mu]}$, we have to integrate the commutators we derived, multiplied by x and y:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \int d^3x\, d^3y\, x^{[\mu}[T^{0\nu]}(x), T^{0[\sigma}(y)]y^{\rho]}.$$

With the expressions for the commutators we have derived, and some tedious but straightforward algebra, which I will not include here, we should be able to verify the algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}),$$

from which $[K^i, K^j] = -i\epsilon_{ijk}J^k, [J^i, J^j] = i\epsilon_{ijk}J^k, [J^i, K^j] = i\epsilon_{ijk}K^k$ follows.

Finally, for the commutators between translation and Lorentz generators, we could integrate the commutators we derived, multiplied by x. However, I will describe a simpler method, using what we proved in problem 3, that P^μ generates translations, ie. $[P^\mu, \varphi_a(x)] = i\partial^i\varphi_a(x)$ and $[P^\mu, \Pi_a(x)] = i\partial^i\Pi_a(x)$. (We did not prove the second statement, but the proof proceeds exactly analogously.) Together with the fact that the commutator is a derivative, it means that for any local operator $f(\varphi(x), \Pi(x))$, we have $[P^\mu, f(\varphi(x), \Pi(x))] = i\partial^\mu f(\varphi(x), \Pi(x))$. In particular, we take f to be $T^{0\mu}$, so

$$\begin{aligned}
[P^\mu, M^{\nu\rho}] = \int d^3x\, [P^\mu, T^{0[\rho}]x^{\nu]} = i\int d^3x\, \partial^\mu T^{0[\rho}x^{\nu]} \\
= -i\int d^3x\, T^{0[\rho|}\partial^\mu x^{|\nu]} = -i\int d^3x\, T^{0[\rho}\delta^{\nu]\mu}
\end{aligned}$$

Therefore, $[P^\mu, M^{\nu\rho}]$ vanishes unless $\mu = \nu$ or ρ. For $\mu = 0 = \rho$, we obtain $[H, K^n] = iP^n$. For $\mu = m = \rho$ and $\nu = n$, we obtain $[P^m, N^{nm}] = iP^n$, ie. $[P^m, J^j] = -i\epsilon_{jnm}P^n$. For $\mu = m = \rho$ and $\nu = 0$, we obtain $[P^j, K^k] = -i\delta^{jk}H$.

4. (a) We are to find the Noether current corresponding to infinitesimal $SO(N)$ transformation, $\delta\varphi_i = \theta_{ij}\varphi_j = -i\theta^a(T_a)_{ij}\varphi_j$. One may simply use the formula (22.27) (with $K^\mu = 0$), but let us use a different method (which is often easier). Vary the action with respect to a position dependent $SO(N)$ transformation $\theta^a = \theta^a(x)$ (which is no longer a symmetry!). Since the action has only single derivative terms, the variation will contain two terms, one multiplying θ^a, and one multiplying the first derivative, $\partial_\mu\theta^a$. We expect the coefficient of the θ term to be a total derivative, because the transformation with constant θ is indeed a symmetry. Working it out, we find

$$\delta\mathcal{L} = i\partial^\mu\varphi_i(T_a)_{ij}\varphi_j\partial_\mu\theta^a,$$

so the θ term vanishes. Let $j^\mu_a = i\partial^\mu\varphi_i(T_a)_{ij}\varphi_j$ be the coefficient of the $\partial_\mu\theta^a$ term. Partially integrating, we see that $\delta S = -\int dx\, \partial_\mu j^\mu_a\theta^a$. But, once again, this has to vanish for constant θ^a, so we must have $\partial_\mu j^\mu_a = 0$, and j^μ_a is the conserved Noether current.

(b) We are to show that the Noether charge Q_a generates the symmetry transformation, $[Q_a, \varphi_i] = -(T_a)_{ij}\varphi_j$. This is a straightforward calculation, with $Q_a = \int d^3x\, j^0 = -\int d^3x\, i\Pi_i(T_a)_{ij}\varphi_j$ and the canonical commutation relations, yielding

$$[Q_a, \varphi_i(x)] = \int d^3y\, i(T_a)_{jk}[\Pi_j(y), \varphi_i(x)]\varphi_k(y) = -(T_a)_{ij}\varphi_j(x).$$

(This is a special case of problem 2.)

(c) Consider the commutator of two symmetry transformations on φ_i, which may be evaluated using the jacobi identity,

$$\begin{aligned}
[[Q_a, Q_b], \varphi_i] =& - [[\varphi_i, Q_a], Q_b] - [[Q_b, \varphi_i], Q_a] \\
=& - (T_a)_{ik}(T_b)_{kj}\varphi_j + (T_b)_{ik}(T_a)_{kj}\varphi_j \\
=& - if_{abc}(T_c)_{ij}\varphi_j = if_{abc}[Q_c, \varphi_i].
\end{aligned}$$

(In technical terms, this shows that the φ_i is a representation of the algebra generated by the Q_a, which, of course, is the algebra of $SO(n)$.) The action of $[Q_a, Q_b]$ and $if_{abc}Q_c$ on φ_i coincide. We wish to show that $[Q_a, Q_b]$ in fact is equal to $if_{abc}Q_c$. (In technical terms, we want to show that this representation is faithful.) The most straightforward way of doing this is to note that since $[Q_a, Q_b] - if_{abc}Q_c$ commutes with φ_i and $\Pi_i = \partial_0\varphi_i$ (since charges are time independent), $[Q_a, Q_b] - if_{abc}Q_c$ must in fact be a constant. But there are no $SO(n)$ invariant tensors with two antisymmetric indices for $n > 2$, so in fact $[Q_a, Q_b] - if_{abc}Q_c = 0$.

5. (a) In general, we would have $[A_i(x), \pi_j(y)] = i(\delta_{ij} - \frac{\nabla_i\nabla_j}{\nabla^2})\delta^3(x-y)$, yet for physical states, we also have $\epsilon_\lambda k = 0$. Thus we could replace the commutator by $[A_i(x), \pi_j(y)] = i\delta_{ij}\delta^3(x-y)$.

$$\begin{aligned}
[a_\lambda(k), a_{\lambda'}(k')] =& \epsilon^i_\lambda(k)\epsilon^j_{\lambda'}(k') \int d^3x d^3y [e^{-ikx}(\pi_i(x) - i\omega A_i(x)), e^{-ik'y}(\pi_j(y) - i\omega' A_j(y))] \\
=& \epsilon^i_\lambda(k)\epsilon^j_{\lambda'}(k') \int d^3x d^3y e^{-ikx-ik'y}(\omega'\delta_{ij}\delta^3(x-y) - \omega\delta_{ij}\delta^3(x-y)) = 0
\end{aligned}$$

$$[a^\dagger_\lambda(k), a^\dagger_{\lambda'}(k')] = -[a_\lambda(k), a_{\lambda'}(k')]^\dagger = 0$$

$$
\begin{aligned}
[a_\lambda(k), a^\dagger_{\lambda'}(k')] =& \epsilon^i_\lambda(k)\epsilon^{*j}_{\lambda'}(k') \int d^3x d^3y [e^{-ikx}(\pi_i(x) - i\omega A_i(x)), e^{ik'y}(\pi_j(y) + i\omega' A_j(y))] \\
=& \epsilon^i_\lambda(k)\epsilon^{*j}_{\lambda'}(k') \int d^3x d^3y e^{-ikx-ik'y}(i\omega'\delta_{ij}\delta^3(x-y) + i\omega\delta_{ij}\delta^3(x-y)) \\
=& \epsilon^i_\lambda(k)\epsilon^{*j}_{\lambda'}(k')(2\pi)^3\delta^3(k-k')(2\omega)\delta_{ij} = (2\pi)^3\delta^3(k-k')(2\omega)\delta_{\lambda\lambda'}
\end{aligned}
$$

(b)

$$
\mathcal{H} = \frac{1}{2}(\nabla_j A_i \nabla_j A_i + \pi_i\pi_i) - J_i A_i + \mathcal{H}_c = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3
$$

For the first term, we would have:

$$
\begin{aligned}
H_1 =& \frac{1}{2}\int d^3x d^3k d^3p \frac{1}{(2\pi)^3\sqrt{\omega_k\omega_p}}(-i\omega_k)\epsilon^*_\lambda a_\lambda e^{ikx} + (i\omega_k)\epsilon_\lambda a^\dagger_\lambda e^{-ikx}][(-i\omega_p)\epsilon^*_{\lambda'} a_{\lambda'} e^{ipx} + (i\omega_p)\epsilon_{\lambda'} a^\dagger_{\lambda'} e^{-ipx}] \\
&+ \frac{1}{2}\int d^3x d^3k d^3p \frac{1}{(2\pi)^3\sqrt{\omega_k\omega_p}}[k_j\epsilon^*_\lambda a_\lambda e^{ikx} + (-k_j)\epsilon_\lambda a^\dagger_\lambda e^{-ikx}][p_j\epsilon^*_{\lambda'} a_{\lambda'} e^{ipx} + (-p_j)\epsilon_{\lambda'} a^\dagger_{\lambda'} e^{-ipx}] \\
=& \frac{1}{2}\int d^3x d^3k d^3p \frac{1}{(2\pi)^3\sqrt{\omega_k\omega_p}}(\omega_k\omega_p + kp)e^{i(k+p)x}\epsilon^*_\lambda\epsilon^*_{\lambda'} a_\lambda a_{\lambda'} \\
&+ \frac{1}{2}\int d^3x d^3k d^3p \frac{1}{(2\pi)^3\sqrt{\omega_k\omega_p}}(\omega_k\omega_p + kp)e^{-i(k+p)x}\epsilon_\lambda\epsilon_{\lambda'} a^\dagger_\lambda a^\dagger_{\lambda'} \\
&- \frac{1}{2}\int d^3x d^3k d^3p \frac{1}{(2\pi)^3\sqrt{\omega_k\omega_p}}(\omega_k\omega_p + kp)e^{i(k-p)x}\epsilon^*_\lambda\epsilon_{\lambda'} a_\lambda a^\dagger_{\lambda'} \\
&- \frac{1}{2}\int d^3x d^3k d^3p \frac{1}{(2\pi)^3\sqrt{\omega_k\omega_p}}(\omega_k\omega_p + kp)e^{-i(k-p)x}\epsilon_\lambda\epsilon^*_{\lambda'} a^\dagger_\lambda a_{\lambda'}
\end{aligned}
$$

After integrating over x, we would get either $k = -p$ for the first two terms or $k = p$ for the last two terms. Then with $\omega^2 = k^2$, we know that the first two terms will make no contribution. Then

$$
\begin{aligned}
H_1 =& \frac{1}{2}\int d^3k\omega(\epsilon_\lambda\epsilon^*_{\lambda'} a^\dagger_\lambda a_{\lambda'} + \epsilon^*_\lambda\epsilon_{\lambda'} a_\lambda a^\dagger_{\lambda'}) \\
=& \int d^3k\omega a^\dagger_\lambda a_\lambda + 2V\epsilon_0
\end{aligned}
$$

6. (a) By taking a map between x^i and $A = x^i\sigma^i$ where and considering a group element g of SU(2) acting on A, transforming it to $g^\dagger A g$, we see $det(g^\dagger A g) = det(A) = -x^2 = -(Rx)^2$ with some R in SO(3), which means we could build a map between SU(2) and SO(3). Then noticing that both $\pm I$ of SU(2) would be mapped to I of SO(3), we know there should be Z_2 kernel.

   (b) Now forming a map between (x^0, x^1, x^2) and $A = x^1\sigma^3 + x^2\sigma^1 + x^0(i\sigma^2)$, the action of a group element g of SL(2,R) on A would be resulted as $g^{-1}Ag$. Again, by $det(g^{-1}Ag) = det(A) = -x^2 = -(Rx)^2$ with metric (-,+,+) and R belonging to SO(2,1), we could map SL(2,R) to SO(2,1). The Z_2 could be seen from that both $\pm I$ of SL(2,R) being mapped to I of SO(2,1).

   (c) Following the same argument, yet this time we map x_μ to $A = x_\mu\sigma^\mu$ with $\sigma^\mu = (1, \sigma^i)$. We have $det(g^{-1}Ag) = det(A) = -x^2 = -(Rx)^2$ with metric (-,+,+,+) and R belonging to SO(3,1), g to SL(2,C). The Z_2 could be seen from that both $\pm I$ of SL(2,C) being mapped to I of SO(3,1).

7. Let us vary the action with respect to a local translation $\delta A_\mu = -a^\nu(x)\partial_\nu A_\mu$, with a infiniestimal. We obtain

$$\begin{aligned}\delta S &= \int d^4x \ -\frac{1}{2}F^{\mu\nu}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) + J^\mu \delta A_\mu \\ &= \int d^4x \ \frac{1}{4}a^\rho \partial_\rho (F^{\mu\nu}F_{\mu\nu}) + \partial_\mu a^\rho F^{\mu\nu}\partial_\rho A_\nu - a^\rho \partial_\rho (J^\mu A_\mu) \\ &= \int d^4x \ \partial_\mu a^\nu \left(-\frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}\delta^\mu_\nu + F^{\mu\rho}\partial_\nu A_\rho + J^\rho A_\rho \delta^\mu_\nu \right),\end{aligned}$$

where the assumption that J^μ is a conserved current, $\partial_\mu J^\mu = 0$, has been used in the second equality. According to the prescription in question 5, we can identify the coefficient of $\partial_\mu a^\nu$ as the Noether current,

$$T^{\mu\nu} = -\frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + F^{\mu\rho}\partial^\nu A_\rho + J^\rho A_\rho g^{\mu\nu}.$$

The same result may be obtained using (22.27), noting that $K^\mu = a^\mu(-\frac{1}{4}(F_{\rho\sigma})^2 + J^\rho A_\rho)$.

Notice that this stress tensor, obtained via the Noether method, is neither symmetric (due to $F^{\mu\rho}\partial^\nu A_\rho$ term), nor gauge invariant. The first term is gauge invariant, and the third term is gauge invariant up to a total derivative $\partial_\rho(J^\rho \lambda g^{\mu\nu})$, but the second term is not, since $\delta(F^{\mu\rho}\partial^\nu A_\rho) = F^{\mu\rho}\partial^\nu\partial_\rho\lambda$.

Remark

We can add improvement terms to the Noether stress tensor, of the form $\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}$, where $B^{\rho\mu\nu}$ is some tensor antisymmetric in its first two indices, $B^{\rho\mu\nu} = -B^{\mu\rho\nu}$. Notice that $\tilde{T}^{\mu\nu}$ is still conserved, since $\partial_\rho\partial_\mu B^{\rho\mu\nu} = 0$, and the momenta are unchanged as the improvement term is a total derivative. By considering the spin transformation of the fields in the action, a $B^{\rho\mu\nu}$ can always be found such that $\tilde{T}^{\mu\nu}$ is symmetric.

In the case of the Maxwell field, the improvement term is given by $B^{\rho\mu\nu} = F^{\rho\mu}A^\nu$. Indeed, we have

$$\begin{aligned}\tilde{T}^{\mu\nu} &= -\frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + F^{\mu\rho}(\partial^\nu A_\rho - \partial_\rho A^\nu) + \partial_\rho F^{\rho\mu}A^\nu \\ &= -\frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + F^{\mu\rho}F^\nu{}_\rho + \partial_\rho F^{\rho\mu}A^\nu\end{aligned}$$

On shell, the last term vanishes (recall that we have set $J^\mu = 0$), so $\tilde{T}^{\mu\nu}$ is both symmetric and gauge invariant.

Finally, note that there is an alternative definition of the stress tensor, used in general relativity. We place the theory in a curved background, restoring explicit factors of the metric,

$$S = \int d^4x\sqrt{-g} \ -\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + A_\mu J^\mu,$$

and define the stress tensor as the variation with respect to the metric,

$$T_g^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}.$$

Since the metric is symmetric, this produces an off-shell symmetric stress tensor, which coincides with the improved stress tensor on shell. (Try deriving $T_g^{\mu\nu}$! The identity $\delta\sqrt{-g} = -(1/2)\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ might be useful.)