PHY 610 QFT, Spring 2017
HW1 Solutions

1. Note that the defining equation $\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho_\mu\Lambda^\sigma_\nu$ for a Lorentz transformation $\Lambda \in O(1,3)$ means the columns of Λ (and rows too, since Λ^T is also a Lorentz transformation) are orthonormal, with Λ^0_μ timelike and Λ^i_μ spacelike. In particular, $(\Lambda^0_0)^2 - \Lambda^0_i\Lambda^0_i = 1$ (which is one way to see that $\left|\Lambda^0_0\right| \geq 1$).

   Now suppose Λ is orthochronous, $\Lambda^0_0 \geq 1$, and let v^μ be a forward timelike or null vector, ie. $(v^0)^2 \geq (v^i)^2$ with $v^0 > 0$. Then, using the Cauchy-Schwarz inequality,

$$(\Lambda v)^0 = \Lambda^0_0 v^0 + \Lambda^0_i v^i \geq \Lambda^0_0 v^0 - \sqrt{(\Lambda^0_i)^2}\sqrt{(v^i)^2} \geq \Lambda^0_0 v^0 - v^0\sqrt{(\Lambda^0_0)^2 - 1} = v^0(\Lambda^0_0 - \sqrt{(\Lambda^0_0)^2 - 1}) > 0,$$

   so Λv is still a forward pointing vector. In particular, since $\tilde{\Lambda}^\mu_0$ is a forward timelike vector for orthochronous $\tilde{\Lambda}$, this shows that $(\Lambda\tilde{\Lambda})^0_0 > 0$, and by the above observation, it must in fact be at least 1, so $\Lambda\tilde{\Lambda}$ is once again orthochronous.

   Now suppose v^μ is a backward pointing spacelike vector, $(v^0)^2 < (v^i)^2$ and $v^0 < 0$. We shall show that there exists Lorentz transformations which can flip the temporal direction of v. Choose $\hat{\Lambda} \in SO(1,3)^+$ such that its first row $\hat{\Lambda}^0_\mu$ has its spatial components parallel to v, which is to say, $\hat{\Lambda}^0_j\delta^{ij} = \lambda v^i$, for some positive λ. Orthonormality requires that $\hat{\Lambda}^0_0 = \sqrt{1 + \lambda^2(v^i)^2}$. Then

$$(\hat{\Lambda}v)^0 = \hat{\Lambda}^0_0 v^0 + \hat{\Lambda}^0_i v^i = v^0\sqrt{1 + \lambda^2(v^i)^2} + \lambda(v^i)^2.$$

   Now, at small values of λ, this is approximately v^0, which is negative. Since $\sqrt{(v^i)^2} + v^0 > 0$, at large λ becomes positive. Hence there is some value of λ_0 where $(\hat{\Lambda}v)^0 = 0$, and for any $\lambda > \lambda_0$, $(\hat{\Lambda}v)^0 > 0$. Note that we did not have to demand that any of these Lorentz transformations were proper — just orthochronous.

2. (a) The equations of motion are

$$\pi_n = \frac{\partial H}{\partial \pi_n} = \dot{\varphi}_n,$$

$$\ddot{\varphi}_n = \dot{\pi}_n = -\frac{\partial H}{\partial \varphi_n} = -(\varphi_n - \varphi_{n-1}) - (\varphi_n - \varphi_{n+1}) - m^2\varphi_n.$$

   Equating the mode expansions of the left and right hand sides of the second equation yields the dispersion relation

$$\omega_k^2 = 2 + m^2 - e^{ik} - e^{-ik} = 2(1 - \cos k) + m^2.$$

   (Notice that ω_k depends only on $|k|$, as before.)

   (b) Since the positions of the atoms are discrete, the momentum is a periodic function. A more quantitative way of seeing this is as follows: the position n takes integer values, so $e^{i(-\omega_k t + kn)} = e^{i(-\omega_{k+2\pi}t + (k+2\pi m)n)}$ for any integer m, so k and $k + 2\pi m$ describe the same configuration. The interval $[-\pi, \pi]$ in which k takes its values is known as the Brillouin zone.

(c) First, invert the Fourier expansion,

$$\sum_n \varphi_n e^{ink} = \int \frac{dk'}{(2\pi)2\omega_{k'}} \sum_n (a_k e^{i(-\omega_{k'}t+(k+k')n)} + a_k^\dagger e^{-i(-\omega_{k'}t+(k'-k)n)})$$

$$= \frac{1}{2\omega_k}\left(a_{-k}e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right),$$

$$\sum_n \pi_n e^{ink} = \sum_n \dot{\varphi}_n e^{ink} = \frac{i}{2}(-a_{-k}e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}),$$

where we have used that $\sum_n e^{ink} = (2\pi)\delta(k)$. Thus

$$a_k = \sum_n (\omega_k \varphi_n + i\pi_n)e^{-(ikn - i\omega_k t)}, \qquad a_k^\dagger = \sum_n (\omega_k \varphi_n - i\pi_n)e^{ikn - i\omega_k t}.$$

It is then straightforward to show that $[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0$ and $[a_k, a_{k'}^\dagger] = (2\pi)(2\omega_k)\delta(k-k')$.

(d) We are to substitute the mode expansion of φ_n and $\pi_n = \dot{\varphi}_n$ into the hamiltonian. This gives us an "ω and m part" and "trig function part":

$$H = \frac{1}{2}\int_{-\pi}^{\pi} \frac{dk}{(2\pi)2\omega_k^2} \int_{-\pi}^{\pi} \frac{dk'}{(2\pi)2\omega_{k'}^2} \sum_n (\omega_k\omega_{k'} + m^2)$$
$$\left(a_k e^{-i\omega_k t + ikn} - a_k^\dagger e^{i\omega_k t - ikn}\right)\left(a_{k'}e^{-i\omega_{k'}t + ik'n} - a_{k'}^\dagger e^{i\omega_{k'}t - ik'n}\right)$$
$$+ \left(a_k e^{-i\omega_k t + ikn}(1 - e^{-ik}) + a_k^\dagger e^{i\omega_k t - ikn}(1 - e^{ik})\right)$$
$$\left(a_{k'}e^{-i\omega_{k'}t + ik'n}(1 - e^{-ik'}) + a_{k'}^\dagger e^{i\omega_{k'}t - ik'n}(1 - e^{ik'})\right)$$

Expanding this, we get exponents where n appears beside either $k - k'$ or $k + k'$. These naturally produce $\delta(k-k')$ and $\delta(k+k')$ factors when the sum is evaluated. Doing this,

$$H = \frac{1}{2}\int_{-\pi}^{\pi} \frac{dk}{(2\pi)2\omega_k^2} \int_{-\pi}^{\pi} \frac{dk'}{(2\pi)2\omega_{k'}^2}$$
$$\delta(k-k')\left[(\omega_k\omega_{k'} + m^2 + (1-e^{-ik})(1-e^{ik'}))a_k a_{k'}^\dagger e^{-it(\omega_k - \omega_{k'})} + h.c.\right]$$
$$+ \delta(k+k')\left[(-\omega_k\omega_{k'} + m^2 + (1-e^{-ik})(1-e^{-ik'}))a_k a_{-k'} e^{-it(\omega_k + \omega_{k'})} + h.c.\right]$$

Killing one of the integrals and recognizing that $\omega_k^2 = (1-e^{-ik})(1-e^{ik}) + m^2$, this simplifies to

$$H = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\frac{1}{2}(a_k a_k^\dagger + a_k^\dagger a_k).$$

Now, we normal order using the commutation relation in (c), to yield

$$H = \int_{-\pi}^{\pi} \frac{dk}{2\pi} a_k^\dagger a_k + \Omega_0 V,$$

where $V = 2\pi\delta(0)$ is the "volume of space" (really, the number of particles in this case), and $\Omega_0 = \int dk/2\pi\ \omega_k$ is the zero point energy.

This is the hamiltonian of non-interacting free scalar fields.

(e) Restoring factors of a, the hamiltonian is

$$H = \frac{1}{2}\sum_n \pi_n^2 + a^{-2}(\varphi_n - \varphi_{n-1})^2 + m^2\varphi_n^2.$$

In the continuum limit, this becomes (with $x = na$)

$$H \to \frac{1}{2} \int \frac{dx}{a} \pi^2(x) + (\partial_x \varphi(x))^2 + m^2 \varphi^2(x),$$

which is the hamiltonian of a free scalar field. Similarly, the dispersion relation

$$\omega_k^2 = 2(1 - \cos(ka))a^{-2} + m^2 \to k^2 + m^2$$

reproduces that of the free scalar in the small a limit. Note also that the Brillouin zone is $[-\pi/a, \pi/a]$, so in the continuum limit we recover that k is allowed to take any value.

3. We are to derive the canonical quantization conditions (3.29) for $a(\mathbf{k}), a^\dagger(\mathbf{k})$ from that of the fields φ and Π (3.28). With

$$a(\mathbf{k}) = \int d^3x\, e^{-ikx}(i\Pi(x) + \omega\varphi(x)), \qquad a^\dagger(\mathbf{k}) = \int d^3x\, e^{ikx}(-i\Pi(x) + \omega\varphi(x)),$$

we see that

$$\begin{aligned}
[a(\mathbf{k}), a(\mathbf{k'})] =& \int d^3x\, d^3y\, e^{-i(kx+k'y)}(i^2[\Pi(x), \Pi(y)] + \omega_\mathbf{k}\omega_{\mathbf{k'}}[\varphi(x), \varphi(y)] + i\omega_{\mathbf{k'}}[\Pi(x), \varphi(y)] + i\omega_\mathbf{k}[\varphi(x), \Pi(y)]) \\
=& \int d^3x\, d^3y\, e^{-i(kx+k'y)}(\omega_{\mathbf{k'}}\delta^3(x-y) - \omega_\mathbf{k}\delta^3(x-y)) \\
=& (\omega_{\mathbf{k'}} - \omega_\mathbf{k}) \int d^3x\, e^{-i(k+k')x}, \\
=& (\omega_{\mathbf{k'}} - \omega_\mathbf{k}) e^{2i\omega t}(2\pi)^3\delta(\mathbf{k} + \mathbf{k'}) = 0.
\end{aligned}$$

In the second last equality, the identity $\int d^3x\, e^{-i\mathbf{kx}} = (2\pi)^3\delta^3(\mathbf{k})$ is used (ie. the Fourier transform of unity is the delta function), and the last equality follows since ω is an even function of $\mathbf{k}$. Similarly, $[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k'})] = 0$. Meanwhile,

$$\begin{aligned}
[a(\mathbf{k}), a^\dagger(\mathbf{k'})] =& \int d^3x\, d^3y\, e^{-i(kx-k'y)}([\Pi(x), \Pi(y)] + \omega_\mathbf{k}\omega_{\mathbf{k'}}[\varphi(x), \varphi(y)] + i\omega_{\mathbf{k'}}[\Pi(x), \varphi(y)] - i\omega_\mathbf{k}[\varphi(x), \Pi(y)]) \\
=& (\omega_\mathbf{k} + \omega_{\mathbf{k'}}) \int d^3x\, d^3y\, e^{-i(kx-k'y)}\delta^3(x-y) \\
=& (2\omega_\mathbf{k})(2\pi)^3\delta^3(\mathbf{k} - \mathbf{k'}).
\end{aligned}$$

4. We are to show that $a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)\ldots a^\dagger(\mathbf{k}_n)|0\rangle$ is an eigenstate of $H = \int d^3k/2(2\pi)^3\, a^\dagger a$ with energy $\omega_1 + \ldots + \omega_n$. This is a straightforward calculation, using the commutation relations derived above. We have to move the annihilation operator $a(\mathbf{k})$ in H all the way to the right, where it annihilates the vacuum. For example, moving $a(\mathbf{k})$ past $a^\dagger(\mathbf{k}_1)$,

$$a(\mathbf{k})a^\dagger(\mathbf{k}_1) = [a(\mathbf{k}), a^\dagger(\mathbf{k}_1)] + a^\dagger(\mathbf{k}_1)a(\mathbf{k}) = (2\pi)^3 2\omega_1\delta^3(\mathbf{k} - \mathbf{k}_1) + a^\dagger(\mathbf{k}_1)a(\mathbf{k}),$$

picks up a factor of $(2\pi)^3 2\omega_1\delta^3(\mathbf{k} - \mathbf{k}_1)$. Moving $a(\mathbf{k})$ past each of the $a^\dagger(\mathbf{k}_j)$, we obtain

$$\begin{aligned}
H|k_1 \ldots k_n\rangle =& \int \frac{d^3k}{2(2\pi)^3}\, a^\dagger(\mathbf{k}) \left(\sum_{j=1}^n a^\dagger(\mathbf{k}_1)\ldots a^\dagger(\mathbf{k}_{j-1})(2\pi)^3 2\omega_j\delta^3(\mathbf{k} - \mathbf{k}_j)a^\dagger(\mathbf{k}_{j+1})\ldots a^\dagger(\mathbf{k}_n)|0\rangle \right) \\
=& \sum_{j=1}^n \omega_j\, |k_1 \ldots k_n\rangle.
\end{aligned}$$

(Note that we have used the fact that $a^\dagger$s commute in the last equality.)

5. (a) Up to boundary terms, we may integrate the kinetic term to write

$$-\int d^4x\ \partial^\mu\varphi^\dagger\partial_\mu\varphi = \int d^4x\ \varphi^\dagger\partial^\mu\partial_\mu\varphi,$$

so the Euler-Lagrange variation of $\varphi^\dagger$ yields the Klein-Gordon equation $(\partial^\mu\partial_\mu - m^2)\varphi = 0$.

(b) The conjugate momenta are

$$\Pi_\varphi(x) = \frac{\partial\mathcal{L}}{\partial\partial_0\varphi(x)} = \partial_0\varphi^\dagger(x), \qquad \Pi_{\varphi^\dagger}(x) = \partial_0\varphi(x).$$

The hamiltonian density is

$$\mathcal{H} = \Pi_\varphi\partial_0\varphi + \Pi_{\varphi^\dagger}\partial_0\varphi^\dagger - \mathcal{L} = \Pi_{\varphi^\dagger}\Pi_\varphi + \partial^i\varphi^\dagger\partial_i\varphi + m^2\varphi^\dagger\varphi - \Omega_0.$$

(Note the ordering of the $\Pi_{\varphi^\dagger}\Pi_\varphi$ term; this will be important in part (e).)

(c) Inverting the Fourier transform (following (3.20)),

$$\int d^3x\ e^{-ikx}\varphi(x) = \frac{1}{2\omega}(a(\mathbf{k}) + e^{2i\omega t}b^\dagger(-\mathbf{k})).$$

(This is because

$$\begin{aligned}\int d^3x\ e^{-ikx}\varphi(x) &= \int d^3x\ e^{-ikx}\frac{d^3k'}{(2\pi)^3 2\omega_{\mathbf{k}'}}(a(\mathbf{k}')e^{ik'x} + b^\dagger(\mathbf{k}')e^{-ik'x})\\ &= \int \frac{d^3k'}{2\omega_{\mathbf{k}'}}(a(\mathbf{k}')e^{-i(\omega_{\mathbf{k}'}-\omega_{\mathbf{k}})t}\delta^3(\mathbf{k}-\mathbf{k}') + b^\dagger(\mathbf{k}')e^{i(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}})t}\ \delta^3(\mathbf{k}+\mathbf{k}')).)\end{aligned}$$

Similarly,

$$\int d^3x\ e^{-ikx}\partial_0\varphi(x) = \frac{i}{2}(-a(\mathbf{k}) + e^{2i\omega t}b^\dagger(-\mathbf{k})),$$

so that

$$\begin{aligned}a(\mathbf{k}) &= \int d^3x\ e^{-ikx}(\omega\varphi(x) + i\partial_0\varphi(x)) = \int d^3x\ e^{-ikx}(\omega\varphi(x) + i\Pi_{\varphi^\dagger}),\\ b^\dagger(-\mathbf{k}) &= \int d^3x\ e^{-ikx-2i\omega t}(\omega\varphi(x) - i\partial_0\varphi(x)).\end{aligned}$$

To obtain $b(\mathbf{k})$, take the conjugate and relabel $\mathbf{k} \mapsto -\mathbf{k}$ (so $e^{ikx+2i\omega t} = e^{i\omega t+i\mathbf{kx}} \mapsto e^{i\omega t-i\mathbf{kx}} = e^{-ikx}$), yielding

$$b(\mathbf{k}) = \int d^3x\ e^{-ikx}(\omega\varphi^\dagger(x) + i\partial_0\varphi^\dagger(x)) = \int d^3x\ e^{-ikx}(\omega\varphi^\dagger(x) + i\Pi_\varphi(x)).$$

(d) The canonical commutation relations are

$$[\varphi(t,\mathbf{x}), \Pi_\varphi(t,\mathbf{y})] = [\varphi^\dagger(t,\mathbf{x}), \Pi_{\varphi^\dagger}(t,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}),$$

with all other commutators vanishing. The nonvanishing commutators between the creation/annihilation operators are

$$\begin{aligned}[a(\mathbf{k}), a^\dagger(\mathbf{k}')] &= \int d^3x\ d^3y\ e^{-i(kx-k'y)}(-i\omega_{\mathbf{k}}[\varphi(x), \Pi_\varphi(y)] + i\omega_{\mathbf{k}'}[\Pi_{\varphi^\dagger}(x), \varphi^\dagger(y)])\\ &= 2\omega(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}'),\end{aligned}$$

and similarly,

$$[b(\mathbf{k}), b^\dagger(\mathbf{k}')] = 2\omega(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}').$$

It is clear that the $[a, a]$ (and therefore also $[a^\dagger, a^\dagger]$, $[b, b]$ and $[b^\dagger, b^\dagger]$) commutators vanish, since φ and $\Pi_{\varphi^\dagger}$ mutually commute. For $[a, b]$ (and $[a^\dagger, b^\dagger]$), the contributions from $[\varphi, \Pi_\varphi]$ and $[\Pi_{\varphi^\dagger}, \varphi^\dagger]$ are equal and opposite.

(e) We are to rewrite the hamiltonian $H = \int d^3x\, \mathcal{H}$ in terms of the creation and annihilation operators, by substituting the mode expansions (c) into the expression for the hamiltonian density we obtained in (b). Expanding out the terms yields

$$\begin{aligned}
H = \int d^3x\, &\frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \frac{d^3k'}{(2\pi)^3 2\omega_{\mathbf{k}'}} \\
&\Big(\omega_{\mathbf{k}}\omega_{\mathbf{k}'}(-a(k)a^\dagger(k')e^{-i(k'-k)x} + b^\dagger(k)b(k')e^{i(k'-k)x} - a(k)b(k')e^{i(k'+k)x} + b^\dagger(k)a^\dagger(k')e^{-i(k+k')x}) \\
&+ (\mathbf{k}\mathbf{k}' + m^2)(a^\dagger(k)a(k')e^{i(k'-k)x} + b(k)b^\dagger(k')e^{-i(k'-k)x}) \\
&+ (-\mathbf{k}\mathbf{k}' + m^2)(b(k)a(k')e^{i(k'+k)x} - a^\dagger(k)b^\dagger(k')e^{-i(k+k')x})\Big) - \Omega_0 V
\end{aligned}$$

The dx integral may be performed, yielding either $\delta^3(\mathbf{k}-\mathbf{k}')$ or $\delta^3(\mathbf{k}+\mathbf{k}')$ for each term, which then cancels one of the dk integrals. After simplification, one arrives at

$$\begin{aligned}
H =& \frac{1}{2}\int \frac{d^3k}{2(2\pi)^3}(a(k)a^\dagger(k) + b^\dagger(k)b(k) - a(k)b(-k)e^{-2i\omega t} - b^\dagger(k)a^\dagger(-k)e^{2i\omega t}) \\
&+ \frac{1}{2}(k^2+m^2)\int \frac{d^3k}{(2\pi)^3 2\omega^2}(a^\dagger(k)a(k) + b(k)b^\dagger(k) + b(k)a(-k)e^{-2i\omega t} + a^\dagger(k)b^\dagger(-k)e^{2i\omega t}) - \Omega_0 V.
\end{aligned}$$

Now, we use the commutation relations derived in (d). First, we consider the terms involving ab and $a^\dagger b^\dagger$. Commuting b past a (and $a^\dagger$ past $b^\dagger$), and relabeling $k \to -k$, we see that they add to

$$\int \frac{d^3k}{(2\pi)^3(2\omega)^2}(-\omega^2 + k^2 + m^2)(a(k)b(-k)e^{-2i\omega t} + a^\dagger(k)b^\dagger(-k)e^{2i\omega t}) = 0,$$

which vanishes due to the mass shell relation. Meanwhile, for the other terms, we use the commutation relations to write all the terms in the order $a^\dagger a$ and $b^\dagger b$ (which annihilates the ground state). This yields

$$\begin{aligned}
H &= \int \frac{d^3k}{(2\pi)^3(2\omega)^2}\Big((\omega^2+k^2+m^2)(2\omega)\delta^3(0) + (\omega^2+k^2+m^2)(a^\dagger(k)a(k) + b^\dagger(k)b(k))\Big) - \Omega_0 V \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega}\omega(a^\dagger(k)a(k) + b^\dagger(k)b(k)) + 2\mathcal{E}_0 V - \Omega_0 V,
\end{aligned}$$

so $\Omega_0 = 2\mathcal{E}_0$ for zero ground state energy. Therefore, as expected, a complex scalar field has two sets of oscillators, $a(k)$ and $b(k)$, as opposed to real scalar fields which have just one. Both sets of oscillators contribute to the zero point energy.

6. This is a straightforward evaluation of the integral

$$\mathcal{E}_0 = \int_{|k|<\Lambda} \frac{d^3k}{(2\pi)^3 2\omega}\omega^2 = \frac{1}{2(2\pi)^3}\int_{|k|<\Lambda} d^3k\,\sqrt{k^2+m^2} = \frac{\Lambda^4}{2(2\pi)^3}\int_{|k'|<1} d^3k'\,\sqrt{k'^2 + m^2/\Lambda^2},$$

where in the last equality we have rescaled $k' = k/\Lambda$. For $m/\Lambda \ll 1$, the integrand may be approximated by $|k'|$, so

$$\mathcal{E}_0 = \frac{\Lambda^4}{2(2\pi)^3} \int_0^1 d\,|k| \;\; 4\pi\,|k|^3 = \frac{\Lambda^4}{16\pi^2}.$$