PHY 610 QFT, Spring 2016

HW7 Solutions

1. We will follow the steps from the path integral approach to free field theory correlators. After all, the integral given corresponds to a field theory that lives in a lattice with \( N \) sites instead of the continuum. Using the summation convention from 1 to \( N \), define

\[
Z(J) = \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_N \exp \left[ -\frac{1}{2} x_i A_{ij} x_j + x_i J_i \right]
\]

\[
= \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_N \exp \left[ -\frac{1}{2} (x_i - J_m A_{mi}^{-1}) A_{ij} (x_j - A_{jn}^{-1} J_n) + \frac{1}{2} J_i A_{ij} J_j \right]
\]

Of the two \( x \)s in the quadratic form, it looks like each has been shifted by a different amount. However, we are saved by the fact that \( A \) is a symmetric matrix. Now we know that \( Z(J) = e^{\frac{1}{2} J_i A_{ij} J_j} Z(0) \). This is enough to say

\[
\langle x_1 \ldots x_{2n} \rangle = \left. \frac{\partial}{\partial J_1} \ldots \frac{\partial}{\partial J_{2n}} e^{\frac{1}{2} J_i A_{ij} J_j} \right|_{J=0}
\]

\[
= \sum \text{pairings each pair} \prod A_{i_a i_b}^{-1}
\]

Applying this to the correlator with \( x \) and \( y \), we read off a matrix of

\[
A = \begin{bmatrix}
2 & 1 \\
1 & 4
\end{bmatrix}
\]

\[
A^{-1} = \frac{1}{7} \begin{bmatrix}
4 & -1 \\
-1 & 2
\end{bmatrix}
\]

One pairing we may construct is \((xx)(xx)(yy)\). It comes with a factor of 3 because the pair that is different may show up in three positions. The other pairing we have is \((xy)(xy)(xx)\). This also has the same factor of 3 but now it has additional factors of 2 for each \((xy)\).

\[
I = 3(A^{-1})_{11}^2 (A^{-1})_{22} + 12(A^{-1})_{12}^2 (A^{-1})_{11} = \frac{144}{343}
\]

2. (a) \( [\Psi] = (d - 1)/2 \)

(b) \( [g_n] = d - n(d - 1) = n - (n - 1)d \)

(c) A scalar field has dimension \( [\varphi] = d/2 - 1 \) from the kinetic term \( (\partial_\mu \varphi)^2 \), so \( [g_{m,n}] = d - n(d - 1) - m(d/2 - 1) = n + m - (n + m + 2d - 2) \).

(d) The only renormalizable interaction in \( d = 4 \) is \( g_{1,1} \varphi \bar{\Psi} \Psi \) (the Yukawa interaction).

3. The one-loop 2-to-2 scattering amplitude for \( \varphi^3 \) theory does not have a closed form in general, but chapter 20 solves for it in the limit of fixed angle scattering. This question asks us to solve for it in the threshold limit: \( s = 4m^2, t = u = 0 \). From the skeleton expansion, we may write this as

\[
\mathcal{T} = V_3^2(4m^2) \Delta(-4m^2) + 2V_3^2(0) \Delta(0) + V_4(4m^2, 0, 0)
\]
where each expression has an order $\alpha$ correction. Starting with the propagators, we may look up expressions from chapter 14 where one-loop two-point functions are found for general momenta.

$$\Pi(-4m^2) = \frac{1}{12}\alpha m^2 (9 - 2\sqrt{3})$$
$$\Pi(0) = \frac{1}{12}\alpha m^2 (11 - 2\sqrt{3})$$

Knowing that $\Pi$ describes the shift that must be performed on the denominator of the free Feynman propagator, we may write

$$\Delta(-4m^2) = -\frac{1}{3m^2} \left[ 1 + \frac{\alpha}{36} (9 - 2\sqrt{3}) \right]$$
$$\Delta(0) = \frac{1}{m^2} \left[ 1 + \frac{\alpha}{12} (11 - 2\sqrt{3}) \right].$$

The vertex functions are harder because they involve multiple integrals. However, for the easier one, there is a substitution that can turn the inner integral into the same one encountered during the computation of $\Pi$.

$$\left(1 - \frac{V_3(0)}{g}\right)/\alpha = \int_0^1 \int_0^{1-x} \log(1 - (x + y)(1 - x - y)) dy dx$$
$$= \int_0^1 \int_x^1 \log(1 - u(1 - u)) du dx$$
$$= \int_0^1 \left( \frac{1}{2} - x \right) \log(x^2 - x + 1) - 2(1 - x) + \sqrt{3} \left( \arctan \left( \frac{1}{\sqrt{3}} \right) - \arctan \left( \frac{2x - 1}{\sqrt{2}} \right) \right) dx$$

All of the terms above that naively look hard to integrate will vanish by symmetry, leaving us with $-1 + \frac{1}{6}\pi\sqrt{3}$ as the final value. When we evaluate $V_3$ at $4m^2$ instead of 0, the logarithm containing only $u$ above picks up another term with $u$ and $x$.

$$\left(1 - \frac{V_3(4m^2)}{g}\right)/\alpha = \int_0^1 \int_x^1 \log(1 - u(1 - u) - 4x(u - x)) du dx$$
$$= \int_0^1 \left( \frac{1}{2} - 2x \right) \log(4x^2 - 4x + 1) + \frac{1}{2} + x \right) \log(x^2 - x + 1) - 2(1 - x)$$
$$+ \sqrt{3 - 8x} \left[ \arctan \left( \frac{1 - 4x}{\sqrt{3 - 8x}} \right) - \arctan \left( \frac{-1 - 2x}{\sqrt{3 - 8x}} \right) \right] dx$$

The 0 to 1 integrals of the five pieces above may be found with a program like Mathematica. In the same order, they are $1, \frac{1}{6}\pi\sqrt{3} - 2, -1, \frac{1}{27}(k - 16 + \pi\sqrt{3})$ and $-\frac{1}{27}(k - 32 - 5\pi\sqrt{3})$. The numbers we have written as $k$ cancel out leading to $-\frac{4}{3} + \frac{1}{6}\pi\sqrt{3}$ as the final value. If we tried this with $V_4$, we would get even more tedious expressions because of its three integrals. Therefore, the only sensible thing is using Mathematica to do it all at once:

$$V_4(4m^2, 0, 0) = -\frac{\alpha g^2}{9m^2} \left( 3 - 2\pi\sqrt{3} \right).$$

Combining all of these, we see that

$$T = \frac{5g^2}{3m^2} \left( 1 + \frac{1}{180} (489 - 70\pi\sqrt{3})\alpha \right).$$
4. We should start by finding a relation between the regular and background versions of the generator of connected diagrams. With a shift of the path integral variable,

\[ e^{iW(J; \bar{\phi})} = \int D\phi e^{iS(\phi + \bar{\phi}) + i \int d^4x J \phi} \]

\[ = \int D\phi e^{iS(\phi) + i \int d^4x J(\phi - \bar{\phi})} \]

\[ = e^{-i \int d^4x J \bar{\phi} + iW(J;0)} . \]

This gives us an equation where we may take \( \frac{\delta}{\delta J(x)} \) of both sides.

\[ W(J;0) = W(J; \bar{\phi}) + \int d^4x J \bar{\phi} \]

\[ \frac{\delta}{\delta J(x)} W(J;0) = \frac{\delta}{\delta J(x)} W(J; \bar{\phi}) + \bar{\phi}(x) \]

\[ = \phi(x) + \bar{\phi}(x) \quad (1) \]

We will now start with the definition of the quantum action and replace \( \phi \) with \( \phi + \bar{\phi} \).

\[ \Gamma(\varphi; \bar{\varphi}) = W(J_{\varphi}; \bar{\varphi}) - \int d^4x J_{\varphi} \varphi \]

\[ \Gamma(\varphi + \bar{\varphi}; 0) = W(J_{\varphi + \bar{\varphi}}; 0) - \int d^4x J_{\varphi + \bar{\varphi}}(\varphi + \bar{\varphi}) \]

\[ = W(J_{\varphi + \bar{\varphi}}; \bar{\varphi}) - \int d^4x J_{\varphi + \bar{\varphi}} \varphi \quad (2) \]

where in the last step, we have used the first line of (1). The last line of (1) states that \( J_{\varphi} \) solves the same equation that \( J_{\varphi + \bar{\varphi}} \) is supposed to solve. We may therefore make this replacement in (2) and get

\[ \Gamma(\varphi + \bar{\varphi}; 0) = W(J_{\varphi}; \bar{\varphi}) - \int d^4x J_{\varphi} \varphi \]

\[ = \Gamma(\varphi; \bar{\varphi}) \]

as desired.

5. We have computed the one loop corrections to the propagator and vertex in \( \varphi^4 \) theory in the previous homework; they are, in the \( \overline{\text{MS}} \) scheme,

\[
\begin{cases}
Z_\varphi = 1 + O[\lambda]^2, \\
Z_m = 1 + \frac{\alpha}{\epsilon} + O[\lambda]^2, \\
Z_\lambda = 1 + \frac{3\alpha}{\epsilon} + O[\lambda]^2,
\end{cases}
\]

where \( \alpha = \lambda/(4\pi)^2 \).

Recall that the beta function is defined as the dependence of the coupling \( \alpha \) on the renormalization scale \( \mu \). In terms of bare parameters, \( Z_\varphi^2 \varphi^4 = \lambda_0 \varphi_0^4 \), with \( \varphi_0 = Z_\varphi^{1/2} \varphi \), so the bare coupling is \( \alpha_0 = Z_{\lambda} Z_{\varphi}^{-2} \lambda_0 \varphi_0 \). The bare coupling is \( \mu \)-independent, so

\[ 0 = \frac{d}{d \log \mu} \log \alpha_0 = \left( \frac{\partial \log Z_\lambda}{\partial \alpha} - 2 \frac{\partial \log Z_\varphi}{\partial \alpha} + \frac{1}{\alpha} \right) \frac{d \log \alpha}{d \log \mu} + \epsilon. \]
Multiplying both sides by $\alpha$, we have

\[
\left( \alpha \frac{\partial \log Z_\lambda}{\partial \alpha} - 2\alpha \frac{\partial \log Z_\phi}{\partial \alpha} + 1 \right) \frac{d\alpha}{d\log \mu} + \epsilon \alpha = 0.
\]

In a renormalizable theory, $d\alpha/d\log \mu$ is an affine function of $\epsilon$. The linear piece is fixed by the above renormalization group equation to be $-\epsilon \alpha$; while the $\epsilon$-independent piece is the $\beta$ function:

\[
\left( \alpha \frac{\partial \log Z_\lambda}{\partial \alpha} - 2\alpha \frac{\partial \log Z_\phi}{\partial \alpha} + 1 \right) (-\epsilon \alpha + \beta(\alpha)) + \epsilon \alpha = 0
\]

Comparing both sides at order $\epsilon^0$ gives the one loop beta function

\[
\beta(\alpha) = 3\alpha^2 + O[\alpha]^3.
\]

(If you had worked with $\lambda$ instead of $\alpha = \lambda/(4\pi)^2$, you would have obtained $\beta(\lambda) = 3\lambda^2/(4\pi)^2 + O[\lambda]^3$.)

Similarly, the anomalous dimension of $m$ is defined as $\gamma_m = d \log m / d \log \mu$, the dependence of the mass on scale $\mu$. This may be computed using the $\mu$-independence of the bare mass $m_0^2 = Z_m Z^{-1}_\phi m^2$, so that

\[
0 = \frac{d}{d \log \mu} \log m_0 = \left( \frac{1}{2} \frac{\partial \log Z_m}{\partial \alpha} - \frac{1}{2} \frac{\partial \log Z_\phi}{\partial \alpha} \right) \frac{d\alpha}{d\log \mu} + \frac{d \log m}{d \log \mu}.
\]

Hence

\[
\gamma_m(\alpha) = \frac{1}{2} \alpha + O[\alpha]^2.
\]

Finally, the anomalous dimension of $\phi$ is

\[
\gamma_\phi(\alpha) = \frac{d \log Z^{1/2}_\phi}{d \log \mu} = \frac{1}{2} \frac{\partial \log Z_\phi}{\partial \alpha} (-\epsilon \alpha + \beta(\alpha)) = 0 + O[\alpha]^2.
\]