PHY 610 QFT, Spring 2016

HW10 Solutions

1. (a) Relations derived in chapter 40 (based on general properties of momentum and angular momentum) reveal three transformation laws for Dirac spinors:

\[ P^{-1}\Psi(\vec{x}, t)P = i\beta\Psi(-\vec{x}, t) \]
\[ T^{-1}\Psi(\vec{x}, t)T = C\gamma_5\Psi(\vec{x}, -t) \]
\[ C^{-1}\Psi(\vec{x}, t)C = C\bar{\Psi}^T(\vec{x}, t) \]

From this, it follows that the \( \bar{\Psi}\Psi \) bilinear is even under C, P and T. The scalar \( \varphi \) must therefore be as well for the \( g\varphi\bar{\Psi}\Psi \) interaction to preserve these symmetries.

(b) The same transformations tell us that \( \bar{\Psi}i\gamma_5\Psi \) is even under C but odd under P and T. With a \( g\varphi\bar{\Psi}i\gamma_5\Psi \) interaction, we therefore need the field \( \varphi \) to be odd under P (pseudoscalar) and odd under T (possibly a property without a name).

2. The tree level graphs for \( e^+e^+ \rightarrow e^+e^+ \) (incoming momenta and spins \( p_1, s_1; p_2, s_2 \), outgoing momenta and spins \( p_3, s_3; p_4, s_4 \)) are the t and u diagrams

\[ iT = \]

\[ = (-ig)^2(\bar{v}_{s_3}(p_3))(\bar{v}_{s_2}(p_2)v_{s_4}(p_4)) \]
\[ \frac{-i}{-t + M^2 - i\epsilon} \]
\[ - (-ig)^2(\bar{v}_{s_1}(p_1)v_{s_4}(p_4))(\bar{v}_{s_2}(p_2)v_{s_3}(p_3)) \]
\[ \frac{-i}{-u + M^2 - i\epsilon} \]

Note the relative minus sign between the two diagrams, arising from anticommuting \( \Psi_3 \) and \( \Psi_4 \).

Similarly, for \( \varphi\varphi \rightarrow e^+e^- \) (incoming momenta \( p_1, p_2 \), outgoing momenta and spins \( p_3, s_3; p_4, s_4 \)), we have the t and u diagrams

\[ iT = \]

\[ = (-ig)^2\bar{u}_{s_3}(p_3) \]
\[ \left( \frac{-i(-p_3^1 + p_1^1 + m)}{-t + m^2 - i\epsilon} + \frac{-i(-p_3^2 + p_2^2 + m)}{-u + m^2 - i\epsilon} \right) v_{s_4}(p_4). \]

3. At tree level, \( e^+e^- \rightarrow \varphi\varphi \) proceeds via the t and u channels (writing \( u_i \) for \( u_{s_i}(p_i) \), etc. to reduce clutter):
\[ i \mathcal{T} = \quad + \quad \]

\[ = (-ig)^2 v_2 \left( \frac{-i(-(p_1^3 + p_4^3) + m)}{-t + m^2} + \frac{-i(-(p_1^4 + p_4^4) + m)}{-u + m^2} \right) u_1. \]

It helps simplify things if we use the on-shell condition \((p_1^3 + m)u_1 = 0\) at this point, so

\[ \mathcal{T} = g^4 v_2 \left( \frac{p_3^3 + 2m}{m^2 - t} + \frac{p_4^4 + 2m}{m^2 - u} \right) u_1. \]

Thus

\[ |\mathcal{T}|^2 = g^4 u_1 \left( \frac{p_3^3 + 2m}{m^2 - t} + \frac{p_4^4 + 2m}{m^2 - u} \right) v_2 v_2 \left( \frac{p_3^3 + 2m}{m^2 - t} + \frac{p_4^4 + 2m}{m^2 - u} \right) u_1. \]

To simplify this, we think of \(|\mathcal{T}|^2\) as a trace, and use cyclicity to move the basis spinors next to each other, in the form \(u_1 v_1\), and so on, and then use the spin averaging

\[ \sum_{s_1 = \pm} u_{s_1}(p_1) \bar{u}_{s_1}(p_1) = -p_1^3 + m, \quad \sum_{s_2 = \pm} v_{s_2}(p_2) \bar{v}_{s_2}(p_2) = -p_2^4 - m \]

to eliminate the basis spinors. This yields

\[ \langle |\mathcal{T}|^2 \rangle = \frac{1}{4} \sum_{s_1, s_2 = \pm} g^4 \text{tr} u_1 \bar{u}_1 \left( \frac{p_3^3 + 2m}{m^2 - t} + \frac{p_4^4 + 2m}{m^2 - u} \right) v_2 v_2 \left( \frac{p_3^3 + 2m}{m^2 - t} + \frac{p_4^4 + 2m}{m^2 - u} \right)
\]

\[ = g^4 \left( \frac{\Phi_{tt}}{(m^2 - t)^2} + \frac{\Phi_{uu}}{(m^2 - u)^2} + \frac{\Phi_{tu} + \Phi_{ut}}{(m^2 - t)(m^2 - u)} \right), \]

and it remains to compute the gamma traces

\[ \begin{align*}
4\Phi_{tt} &= \text{tr}(-p_1^3 + m)(p_3^3 + 2m)(-p_2^2 - m)(p_4^4 + 2m), \\
4\Phi_{uu} &= \text{tr}(-p_1^3 + m)(p_4^4 + 2m)(-p_2^2 - m)(p_3^3 + 2m), \\
4\Phi_{tu} &= \text{tr}(-p_1^3 + m)(p_3^3 + 2m)(-p_2^2 - m)(p_4^4 + 2m), \\
4\Phi_{ut} &= \text{tr}(-p_1^3 + m)(p_4^4 + 2m)(-p_2^2 - m)(p_3^3 + 2m). 
\end{align*} \]

The \(t\) and \(u\) channels are related by \(p_3 \leftrightarrow p_4\), so it suffices to compute \(\Phi_{tt}\) and \(\Phi_{tu}\). To compute these traces, recall that the trace of an odd number of \(\gamma_5\) vanishes, and \(\text{tr} I = 4\), \(\text{tr} \gamma^\mu \gamma^\nu = -4g^{\mu\nu},\)
I will illustrate the process for $\Phi_{tt}$:

$$4\Phi_{tt} = \text{tr}(-\not p_1 + m)(\not p_4 + 2m)(-\not p_2 - m)(\not p_3 + 2m)$$

$$= -4m^4 \text{tr} I + m^2 \text{tr}(2\not p_1 \not p_3 + 4\not p_1 \not p_2 + 2\not p_1 \not p_2 - 2\not p_2 \not p_1 - \not p_3 \not p_1 - 2\not p_2 \not p_1) + \text{tr} \not p_1 \not p_3 \not p_2 \not p_3$$

$$= 4 \left(-4m^4 - m^2(2p_1 \cdot p_3 + 4p_1 \cdot p_2 - 4p_2 \cdot p_3 - p_3^2) + 2(p_1 \cdot p_3)(p_2 \cdot p_3) - (p_1 \cdot p_2)p_3^2\right)$$

In the fourth equality, we have converted to Mandelstam variables $s = -(p_1 + p_2)^2 = 2m^2 - 2p_1 \cdot p_2$, $t = -(p_1 - p_3)^2 = m^2 + 2p_1 \cdot p_3$, $u = m^2 + 2p_1 \cdot p_4$. ($M^2$ is the mass of the scalar.) In the fifth equality we collect terms and use that $s + t + u = 2m^2 + 2M^2$.

A similar calculation may be done for $\Phi_{tu}$, and $\Phi_{uu}, \Phi_{ut}$ may be obtained by exchanging $t \leftrightarrow u$ (in fact, $\Phi_{tu} = \Phi_{ut}$). The results are

$$\begin{align*}
\Phi_{tt} &= \frac{1}{2} (tu - m^2(9t + u) - 7m^4 + 8m^2M^2 - M^4), \\
\Phi_{uu} &= \frac{1}{2} (tu - m^2(9u + t) - 7m^4 + 8m^2M^2 - M^4), \\
\Phi_{tu} &= \frac{1}{2} (-tu - 3m^2(t + u) - 9m^4 + 8m^2M^2 + M^4) = \Phi_{ut}.
\end{align*}$$

Comparing with the result from $e^- \phi \rightarrow e^- \phi$ (48.26-29), we see that the amplitude is related by exchanging $s$ with $t$, and multiplying by $-1/2$. At a diagrammatic level, this can be seen by the fact that the $e^+ e^- \rightarrow \phi \phi$ diagrams are those of $e^- \phi \rightarrow e^- \phi$, but rotated by $\pi/2$. (The minus sign is due to moving a fermion from the initial to final state, and the $1/2$ is due to the fact that in $e^- \phi \rightarrow e^- \phi$ we are summing over the final spin states of the electron, rather than averaging (see (46.9)).)

4. The calculation is virtually identical to $e^+ e^- \rightarrow \phi \phi$ in Yukawa theory from the previous problem, and indeed a very similar example of $e^+ e^- \rightarrow \gamma \gamma$ is done in the text. I will therefore be brief.

The relevant tree level diagrams for $e^- \gamma \rightarrow e^- \gamma$ are (once again, with $u_1$ short for $u_{s_1}(p_1)$, and $\epsilon'_l$ short for $\epsilon'_{\lambda_1}(p_1)$ and so on)

$$iT = \begin{array}{c}
p_2 \\
p_3 \\
p_1 \\
p_4
\end{array} \begin{array}{c}
p_2 \\
p_1 \\
p_3 \\
p_4
\end{array}$$

$$= \epsilon'^e_{\mu_1} \epsilon'^e_{\mu_2} \epsilon'^e_{\rho} \epsilon'^e_{\lambda_1} \epsilon_3 \bar{u}_3 A_{\sigma \rho} u_3 \bar{u}_3 A_{\mu \nu} u_1,$$

so, using $(\gamma^\mu \gamma^\nu \ldots \gamma^\lambda)^1 = \beta \gamma^\rho \ldots \gamma^\lambda \gamma^\mu \beta$,

$$|T|^2 = e^{4\epsilon'^e_{\mu_1} \epsilon'^e_{\mu_2} \epsilon'^e_{\rho} \epsilon'^e_{\lambda_1} \epsilon_3} \bar{u}_3 A_{\sigma \rho} u_3 \bar{u}_3 A_{\mu \nu} u_1.$$

\(^1\text{The index structures of these gamma traces are fixed by the cyclicity property of the trace. Alternatively, these identities are derived in section 47.}\)
where
\[
A_{\mu\nu} = \frac{\gamma_{\nu}(-p_1 - p_2 + m)\gamma_{\mu}}{-s + m^2} + \frac{\gamma_{\mu}(-p_1 + k_2 + m)\gamma_{\nu}}{-u + m^2}.
\]

Averaging over initial, and summing over final states, with \(\sum_s u_s\bar{u}_s = -p + m\), \(\sum_\lambda \epsilon^\mu_{\lambda} \epsilon'^\nu_\lambda = g^{\mu\nu}\), we arrive at
\[
\langle |T|^2 \rangle = e^4 \left( \frac{\langle \Phi_{ss} \rangle}{(s - m^2)^2} + \frac{\langle \Phi_{su} \rangle + \langle \Phi_{us} \rangle}{(s - m^2)(u - m^2)} + \frac{\langle \Phi_{uu} \rangle}{(u - m^2)^2} \right),
\]
where
\[
\langle \Phi_{ss} \rangle = \frac{1}{4} \mathrm{tr} \gamma_\nu(-p_1 - p_2 + m)\gamma_\mu(-p_1 + m)\gamma_\mu(-p_2 + m) \gamma_\nu(-p_2 + m),
\]
\[
\langle \Phi_{su} \rangle = \frac{1}{4} \mathrm{tr} \gamma_\nu(-p_1 - p_2 + m)\gamma_\mu(-p_1 + m)\gamma_\nu(-p_2 + m) \gamma_\mu(-p_2 + m),
\]
and \(\langle \Phi_{uu} \rangle\) and \(\langle \Phi_{us} \rangle\) are obtained from \(\langle \Phi_{ss} \rangle\) and \(\langle \Phi_{su} \rangle\) respectively by \(s \leftrightarrow u\), i.e. \(p_2 \leftrightarrow -p_4\).

To evaluate these traces, the following identities are useful:
\[
\begin{align*}
\gamma^\mu \gamma_\mu &= g_{\mu\nu} \gamma^\nu = -g_{\mu\nu} g^{\mu\nu} = -4, \\
\gamma^\mu \gamma_\nu \gamma_\mu &= \gamma^\mu (-2g^{\mu\nu} - \gamma_\mu \gamma_\nu) = 2\gamma_\nu, \\
\mathrm{tr} \gamma_\nu \gamma_\nu &= -g_{\mu\nu} \mathrm{tr} I = -4g_{\mu\nu}, \mathrm{tr} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma = 4(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}).
\end{align*}
\]

I will quote the result after performing the gamma matrix algebra:
\[
\langle \Phi_{ss} \rangle = 8(p_1 + p_2) \cdot (p_1 + p_2) p_3 \cdot (p_1 + p_2) - 4(p_1 + p_2)^2 p_1 \cdot p_3 - 16m^2(p_1 + p_2)^2 \\
+ 16m^2(p_1 + p_3) \cdot (p_1 + p_2) - 4m^2 p_1 \cdot p_3 + 16m^4 \\
= 2(-su + m^2(3s + u) + m^4)
\]
\[
\langle \Phi_{su} \rangle = -8(p_1 + p_2) \cdot (p_1 - p_4) p_1 \cdot p_3 - 4m^2((p_1 - p_4) \cdot (p_1 + p_2) + (p_1 + p_3) \cdot (p_1 - p_4)) \\
+ (p_1 + p_3) \cdot (p_1 + p_2) + p_1 \cdot p_3 - 8m^4 \\
= 2m^2(-t + 4m^2) = \langle \Phi_{us} \rangle
\]
\[
\langle \Phi_{uu} \rangle = 2(-su + m^2(3s + u) + m^4).
\]

This is related to the \(e^+ e^- \rightarrow \gamma \gamma\) cross section (59.22-25) by crossing symmetry \(s \leftrightarrow t\), as remarked in the problem.

(a) Converting the above \(\langle |T|^2 \rangle\) into a fixed target frame cross section was done in homework 4 so I will just repeat the result here.
\[
\frac{d\sigma}{d\Omega_{FT}} = \frac{\alpha^2}{2m^2} \frac{\omega'^2}{\omega^2} \left[ \frac{\omega}{\omega'} + \frac{\omega'}{\omega} \sin^2 \theta \right]
\]

The non-relativistic limit of the Klein-Nishina formula may be found if we square both sides of \(p_3 = p_1 + p_2 - p_4\). Using the mass shell condition and the fact that \(p_1 = (m, 0)\), we arrive at
\[
-m^2 = -m^2 - 2m\omega + 2m\omega' + 2\omega' \omega(1 - \cos \theta)
\]
\[
\omega - \omega' = \frac{2m}{m} \omega' (1 - \cos \theta)
\]
\[
\omega \approx \omega'.
\]
Plugging this into our differential cross section, we have

\[
\frac{d\sigma}{d\Omega_{\text{FT}}} \approx \frac{\alpha^2}{2m^2} (2 - \sin^2 \theta)
\]

which is the formula for Thomson scattering.

(b) The expression

\[
\frac{d\sigma}{dt} = \frac{1}{64\pi s |\vec{p}_1|^2_{\text{CM}}} |T|^2
\]

allows us to find the differential cross section in any frame if we know what \(\frac{dt}{d\Omega_{\text{CM}}} / \frac{dt}{d\Omega_{\text{FT}}}\) is for that frame. It is therefore tempting to take our previous result and multiply it by \(\frac{dt}{d\Omega_{\text{CM}}} / \frac{dt}{d\Omega_{\text{FT}}}\). While technically correct, this would be misleading because anyone reading the result would expect \(\omega\) and \(\omega'\) to be suitably redefined. In part (a), \(\omega\) should represent the incoming photon energy in the FT frame. In part (b), \(\omega\) should represent the incoming photon energy in the CM frame, which is of course different. Let us therefore avoid shortcuts and write down the momenta

\[
\begin{align*}
p_1 &= (\sqrt{\omega^2 + m^2}, 0, 0, -\omega) \\
p_2 &= (\omega, 0, 0, \omega) \\
p_3 &= (\sqrt{\omega^2 + m^2}, -\omega \sin \theta, 0, -\omega \cos \theta) \\
p_4 &= (\omega, \omega \sin \theta, 0, \omega \cos \theta)
\end{align*}
\]

Neglecting mass, this leads to the Mandelstam variables

\[
\begin{align*}
s &= 4\omega^2 \\
t &= -2\omega^2 (1 - \cos \theta) \\
u &= -2\omega^2 (1 + \cos \theta)
\end{align*}
\]

This also yields the quantities

\[
\frac{dt}{d\Omega} = \frac{1}{2\pi} \frac{d\cos \theta}{d\Omega} = \frac{\omega^2}{\pi}
\]

\[
|\vec{p}_1|^2_{\text{CM}} = \omega^2.
\]

Inserting these into the squared amplitude is now straightforward and we arrive at

\[
\frac{d\sigma}{d\Omega} = \frac{1}{256\pi^2 \omega^2} |T|^2 = \frac{\epsilon^4}{256\pi^2 \omega^2} \frac{5 + 2 \cos \theta + \cos^2 \theta}{1 + \cos \theta}.
\]

5. The tree level diagrams for \(e^+e^- \rightarrow e^+e^-\) are

\[
\tau = \frac{\bar{v}_2 (-ie\gamma^\mu) u_1}{-s} \bar{u}_3 (-ie\gamma^\nu) v_4 - \bar{u}_3 (-ie\gamma^\mu) u_1 \frac{-ig_{\mu\nu}}{-t} \bar{v}_2 (-ie\gamma^\nu) v_4.
\]
Note the relative minus sign owing to anticommuting $u_1$ past $\tilde{u}_3$. A similar calculation to that of the above yields

$$|T|^2 = e^4 \left( \frac{\Phi_{ss}}{s^2} + \frac{\Phi_{st} + \Phi_{ts}}{st} + \frac{\Phi_{tt}}{t^2} \right),$$

where, upon averaging over initial and summing over final spins,

$$\begin{cases}
\langle \Phi_{ss} \rangle = \frac{1}{4} \text{tr}(\mathbf{p}_1 + m)^\mu (\mathbf{p}_2 - m)^\nu \text{tr}(\mathbf{p}_3 + m)^\sigma (\mathbf{p}_4 - m)^\tau) \\
\langle \Phi_{st} \rangle = \frac{1}{4} \text{tr}(\mathbf{p}_1 + m)^\mu (\mathbf{p}_3 - m)^\nu \text{tr}(\mathbf{p}_2 - m)^\sigma (\mathbf{p}_4 - m)^\tau)
\end{cases}$$

and $\langle \Phi_{tt} \rangle, \langle \Phi_{ts} \rangle$ are obtained via exchanging $p_2 \leftrightarrow -p_3$. Performing the gamma matrix algebra yields

$$\begin{cases}
\langle \Phi_{ss} \rangle = 2(s^2 + 2st + t^2 - 8m^2t + 8m^4) \\
\langle \Phi_{tt} \rangle = 2(t^2 + 2st + 2s^2 - 8m^2s + 8m^4) \\
\langle \Phi_{st} \rangle = -2(u^2 - 8m^2u + 12m^4) = \langle \Phi_{ts} \rangle.
\end{cases}$$