1. (a) Compute the tree-level scattering amplitude $\mathcal{T}$ for $e^- e^- \rightarrow e^- e^-$ in QED. Draw the relevant Feynman diagrams and write the corresponding expressions. Denote the momenta of the incoming electrons as $p_1$ and $p_2$ and of the outgoing electrons as $p'_1$ and $p'_2$, and express your answer in terms of Mandelstam invariants

\begin{align*}
  s &= -(p_1 + p_2)^2, \\
  t &= -(p_1 - p'_1)^2, \\
  u &= -(p_1 - p'_2)^2.
\end{align*}

(b) Repeat the problem for $e^- e^+ \rightarrow e^- e^+$, denoting the momentum of the incoming electron as $p_1$, of the incoming positron as $p_2$, of the outgoing electron as $p'_1$ and of the outgoing positron as $p'_2$.

2. Consider the same model as you encountered in the midterm, the QFT of two real scalar fields $A$ and $B$ with Lagrangian density

\begin{align*}
  \mathcal{L} &= -\frac{1}{2} Z_A \partial_\mu A \partial^\mu A - \frac{1}{2} Z^A m_A^2 A^2 - \frac{1}{2} Z_B \partial_\mu B \partial^\mu B - \frac{1}{2} Z^B m_B^2 B^2 \\
  &\quad + Y B + \frac{1}{2} Z_\alpha \bar{\mu}^{\epsilon/2} \alpha \lambda A^2 B - \frac{1}{4!} Z_\beta \bar{\mu}^{\epsilon} \beta \lambda^2 B^4.
\end{align*}

Compared to the midterm:

- We have introduced renormalizing $Z$ factors for all the terms in the Lagrangian. We have also added a tadpole counterterm for $B$. We impose vanishing of tadpoles as a normalization condition.
- We have allowed the masses of $A$ and $B$ to be different, as we should since there is no symmetry enforcing their equality in the quantum theory.
- We are working in $d = 4 - \varepsilon$ dimensions. The factors of $\bar{\mu}$ has been introduced to keep the couplings $\beta$ and $\lambda$ dimensionless and the coupling $\alpha$ to have mass dimension one. (There are really only two independent couplings, for example we could set $\lambda$ to one by a redefinition of $\alpha$ and $\beta$. I’ve kept $\lambda$ for notational consistency with the midterm.)
You are asked to compute the one-loop corrections to the $A$ and $B$ propagators in the $\overline{\text{MS}}$ scheme. Recall that the momentum space propagator of a scalar field takes the form

$$\frac{1}{k^2 + m^2 - \Pi(k^2) - i\epsilon}$$

where $i\Pi(k^2)$ is the sum of 1PI diagrams.

(a) Draw the one-loop diagrams that contribute to $i\Pi_A(k^2)$ and $i\Pi_B(k^2)$.

(b) Evaluate the diagrams in dimensional regularization. You may find the following formulas useful (the bar on $\bar{p}$ means Euclidean)

$$\frac{1}{A_1 A_2} = \int_0^1 dx (xA_1 + (1 - x)A_2)^{-2}$$

$$\int \frac{d^d\bar{p}}{(2\pi)^d} \frac{(\bar{p}^2)^a}{(\bar{p}^2 + D)^b} = \frac{\Gamma(b - a - \frac{d}{2})\Gamma(a + \frac{d}{2})}{(4\pi)^{d/2}\Gamma(b)\Gamma(d/2)} D^{-(b-a-d/2)}$$

(c) Renormalize in the $\overline{\text{MS}}$ scheme and give expressions for the renormalized $\Pi(k^2)$. Recall

$$\mu^2 = 4\pi e^{-\gamma} \tilde{\mu}^2$$

$$\Gamma(-n + x) = \frac{(-1)^n}{n!} \left[ \frac{1}{x} - \gamma + \sum_{k=1}^{n} \frac{1}{k} + O(x) \right]$$

3. Finally, a more conceptual problem that requires little calculation but a good understanding of symmetries (spacetime and global, continuous and discrete). Below is a list of QFT models in various spacetime dimensions (check the measure $d^d x$ to know the value of $d$ in each case). The models are naively renormalizable by power counting. However, some of the Lagrangians are missing renormalizable terms allowed by symmetries – such terms will be generated by loop corrections, so to make the models renormalizable we must add them from the start. For each model, you are asked to determine which additional terms are needed (if any). Briefly explain your reasoning.

(a)

$$S = \int d^4 x \left( -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \bar{\Psi}(i\gamma_\mu \partial^\mu - m)\Psi + g\phi \bar{\Psi} \Psi \right)$$
(b) \[ S = \int d^3x \left( -\frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{2} m^2 \phi_i \phi_i - \frac{g}{6!} (\phi_i \phi_i)^3 \right) \]

where \( i \) is a \( O(N) \) index (summed over when repeated).

(c) \[ S = \int d^3x \left( -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!} \phi^4 \right) \]

(d) \[ S = \int d^4x \left( -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!} \phi^4 \right) \]

(e) \[ S = \int d^4x \left( -\frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} m_1^2 \phi_1^2 - \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 + Y \phi_2 + g \phi_1^2 \phi_2 \right) \]