Notes on Finite Temperature effects to the Higgs Potential

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1 Introduction

We set out to calculate the full finite temperature potential for a scalar field theory with quartic coupling. The tree level 0 temperature potential is

\[ V_{\text{tree}} = -\frac{\mu^2}{2} h^2 + \frac{\lambda}{4} h^4 \]  

(1)

however in applications to phase transitions it is important to consider one-loop effects.

After summing over any number of vertices in one loop, we have

\[ V_{\text{CW}} = \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \text{Log}[k_E^2 + M^2] \]  

(2)

where

\[ M^2 = \frac{d^2 V_{\text{tree}}[h]}{dh^2} \]  

(3)

In finite temperature field theory we take

\[ \int \frac{dk^4}{2\pi^3} f(k^4) \rightarrow T \sum_n f(k^4 = i\omega_n), \omega_n = 2\pi nT \]  

(4)

we shall use p to represent the 3d part of k. substituting and differentiating w.r.t to \( M^2 \) to easily do the n sum we get

\[ \frac{dV_{\text{CW}}}{d(M^2)} = \frac{T}{2} \int \frac{d^3 p}{(2\pi)^3} \sum_n \frac{1}{p^2 + \omega_n^2 + M^2} \]  

(5)

after summing over n and integrating over \( M^2 \) we get,

\[ V_{\text{CW}} = V_{\text{CW}}^{T=0} + V_{\text{CW}}^{T=0} \]  

(6)

where

\[ V_{\text{CW}}^{T=0} = \frac{1}{(2\pi)^3} \int d^3 p \sqrt{p^2 + M^2} \]  

(7)
and

\[ V_{CW}^{T=0} = \frac{T}{2\pi^2} \int d\mathbf{p}^2 \log [1 - \exp[-\beta \sqrt{\mathbf{p}^2 + M^2}]] \]  \hspace{1cm} (8)

\( V_{CW}^{T=0} \) is the original T independent Coleman Weinberg Potential we know and love while \( V_{CW}^{T} \) is the finite temperature potential. While the former term has just one mass scale \( M^2 \), the latter has two; \( M^2 \) and \( T^2 \) and in regions of \( h \) space where \( M^2 \) is small compared to \( T^2 \) the one loop temperature dependent one-loop correction to the mass can be much larger than \( M^2 \) itself. Hence some terms in the two loop potential actually contribute comparable to the first loop potential and these terms can be taken into account by including the one-loop generated temperature dependent mass as a correction to the mass matrix that generates the 1-loop CW potential. ie replace \( M^2 \) with \( M^2 + \Pi_1(T) \) Towards this we compute the temperature dependent 1-loop mass correction in section 3. For this we would need certain artefacts of the 1-loop zero temperature potential which we shall compute in section 2.

2 0-temperature Coleman-Weinberg potential

\[ V_{CW}^{T=0} = \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \sqrt{\mathbf{p}^2 + \Lambda^2} \]  \hspace{1cm} (9)

In cut-off regularization,

\[ V_{CW}^{T=0} = \frac{M^4}{64\pi^2} \left( \log \left[ \frac{M^2}{\Lambda^2} \right] - \frac{1}{2} \right) + \frac{M^2\Lambda^2}{32\pi^2} \]  \hspace{1cm} (10)

we need counter terms to cancel divergences,

\[ V_{CT} = -\frac{\mu_{CT}^2}{2} \mathbf{h}^2 + \frac{\lambda_{CT}}{4} \mathbf{h}^4 \]  \hspace{1cm} (11)

with the renormalization conditions,

\[ \left( \frac{d^2(V_{CW}^{T=0} + V_{CT})}{dh^2} \right)_{h=v} = 0 \]  \hspace{1cm} (12)

and

\[ \left( \frac{d(V_{CW}^{T=0} + V_{CT})}{dh} \right)_{h=v} = 0 \]  \hspace{1cm} (13)

we get

\[ \mu_{CT}^2 = -\frac{3\lambda}{16\pi^2} \left( -\Lambda^2 + 3\mu^2 \log \left[ \frac{2\mu^2}{\Lambda^2} \right] \right) \]  \hspace{1cm} (14)

and

\[ \lambda_{CT} = -\frac{9}{16\pi^2} \left( \lambda^2 (1 + \log \left[ \frac{2\mu^2}{\Lambda^2} \right]) \right) \]  \hspace{1cm} (15)
3 Calculate the temperature dependent 1-loop polarization

Calculate \( \Pi_1(T) \) as But this is just

\[
\Pi_1(T) = \frac{d^2 V_{CW}^T}{dh^2}
\]  

(16)

substituting expression for \( V_{CW}^T \) we get \( \Pi_1(T) = R_1 + R_2 \) where

\[
R_1 = \frac{M^2(d_h^2 M^2)}{4\pi^2} \int_1^\infty dx \frac{\sqrt{x^2 - 1}}{\exp[\beta x \sqrt{M^2}] - 1}
\]  

(17)

and

\[
R_2 = -\frac{(d_h M^2)^2}{8\pi^2} \int_1^\infty dx \frac{\sqrt{x^2 - 1}(-1 + \exp[\beta x \sqrt{M^2}](1 + \beta x \sqrt{M^2}))}{(\exp[\beta x \sqrt{M^2}] - 1)^2 x^2}
\]  

(18)

It might be instructive to look at the origin of \( R_1 \) and \( R_2 \) before isolating the temperature dependent part and the sum over Matsubara frequencies ie. do

\[
\frac{d^2 V_{CW}}{dh^2} = S_1 + S_2
\]  

(19)

and remember that \( R_1 \) comes from ~1 and \( R_2 \) from \( S_2 ~

\[
S_1 = \frac{1}{2(2\pi)^4} \int d^4k \frac{(d_h^2 M^2)}{k^2 + M^2}
\]  

(20)

and

\[
S_2 = -\frac{1}{2(2\pi)^4} \int d^4k \frac{(d_h M^2)^2}{(k^2 + M^2)^2}
\]  

(21)

and hence \( R_1 \) and \( R_2 \) seem to contribute to the feynman diagrams A and B respectively

4 Calculate the temperature dependent 2-loop polarization

the 2-loop contributions to the polarization come from the following diagrams(ignoring for the moment, contributions from the 4-point function after vev-ing two external legs)

Display diagrams 2a 2b 2c and 2d

\[
\Sigma_{2a} = -\frac{(s_0 + s_\beta)(j_0 + j_\beta)}{18\lambda}
\]  

(22)

\[
\Sigma_{2b} = -\frac{2s_\beta j_0}{27\lambda}
\]  

(23)

\[
\Sigma_{2c} = \frac{(s_0 + s_\beta)\lambda_{CT}}{\lambda}
\]  

(24)

\[
\Sigma_{2d} = 18\lambda(j_0 + j_\beta)\mu_{CT}^2
\]  

(25)
5 Infrared Limit

now for small $u$,

$$F(u) = \int_{1}^{\infty} dx \frac{\sqrt{x^2 - 1}}{\exp[ux] - 1} = \frac{2\pi^2}{u^2} \left( \frac{1}{12} + O(u) \right)$$

(26)

hence in the $T^2 \gg M^2$ limit,

$$R_{1}^{IR} = \frac{M^2(d_h^2M^2)}{4\pi^2} \frac{2\pi^2}{12\beta^2 M^2} = \frac{T^2(d_h^2 M^2)}{24}$$

(27)

a similar computation in IR limit gives

$$G(u) = \int_{1}^{\infty} dx \frac{\sqrt{x^2 - 1}(\exp[ux](1 + ux) - 1)}{x^2(\exp[ux] - 1)^2} = \frac{\pi}{2u} + O(1)$$

(28)

and hence

$$R_{2}^{IR} = \frac{T(d_h M^2)^2}{16\pi \sqrt{M^2}}$$

(29)

Now, for the good old higgs theory, $d_h^2 M^2 = 6\lambda$ and $(d_h M^2) = 6\lambda h$. giving

$$R_{1}^{IR} = \frac{\lambda T^2}{4}$$

(30)

and

$$R_{2}^{IR} = -\frac{9T\lambda^2 h^2}{4\pi \sqrt{M^2}}$$

(31)

$R_1$ reproduces the Carrington et.al result. but it is not explained why $R_2$ is ignored, infact demanding $R_1 \gg |R_2|$, we get

$$T \gg \frac{9\lambda h^2}{\pi \sqrt{M^2}}$$

(32)

which is not true for arbitrary small $M^2$ keeping $T = T_c$ near the EW scale and $h$ also near electroweak scale.

6 Complete Π

Let us now compute the Potential after incorporating the ring terms. Simply replacing all $M^2$ with $M^2 + \pi^2$ in equation () will account for all the diagrams in (). Note that the right way to do this would be to replace $M^2$ with $M^2 + \pi^2$ both in $V_{CW}^{T=0}$ and $V_{CW}^{T}$. However the former gives temperature dependent singularities which are cancelled by additional diagrams and correctly taking into account 1-loop 0-temperature counter terms. After doing all this we find that the additional contribution by including the
thermal mass in the T=0 CW is small compared to non-zero temperature CW. Hence we proceed with replacing $M^2$ with $M^2 + \pi^2$ only in $V_{CW}^T$.

Now doing the above substitution and differentiating with respect to $h$ twice will produce a new $\Pi$ and this can be done iteratively. The end of such an iteration can be captured by the following equation:

$$\Pi = \frac{d^2 V_{CW}^T[M^2 \to M^2 + \Pi]}{dh^2}$$

(33)

This equation needs to be numerically solved. To compute this we write

$$V_{CW}^T = \sum_{i=\text{Bosons}} n_i T_i^4 J_B(y_i) + \sum_{j=\text{Fermions}} n_j T^4 J_F(y_j)$$

(34)

where $y_i = \frac{m_i^2}{T^2}$. I shall suppress Summation and boson fermion indices for brevity for the following discussion, now

$$\frac{d^2 V_{CW}^T}{d\phi_i^2} = \frac{d^2 V_{CW}^T}{dy^2} \left( \frac{dy}{d\phi_i} \right)^2 + \frac{dV_{CW}^T}{dy} \frac{d^2 y}{d\phi_i^2}$$

(35)

clearly it would be useful to compute the $J$ functions along with their derivatives with respect to $y$ which we shall call $J_B, J_{pB}, J_{ppB}$ and $J_F, J_{pF}, J_{ppF}$ for the bosons and fermions respectively. When summing over the different degrees of freedom this becomes a messy coupled equation. We iteratively compute $\Pi$ till the desired accuracy is reached.