Temperature Correlations of Quantum Spins

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Abstract. We consider isotropic version of XY model in transverse magnetic field [in one space dimension]. We evaluate of asymptotics of temperature correlations and explain the physical meaning of our result. To do this we represent the quantum correlation function as a tau function of a completely integrable differential equation. This is the well-known Ablowitz-Ladik lattice version of nonlinear Schroedinger differential equation.

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1. Introduction

The XY model was introduced and studied by E. Lieb, T. Schultz and D. Mattis [1]. It describes the interaction of spins 1/2 situated on a 1-dimensional periodic lattice. The Hamiltonian of the model is

$$H = -\sum_{n} \left[\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + h\sigma_n^z \right]. \tag{1}$$

Here σ are Pauli matrices, h is transverse magnetic field and n enumerates the sites of the lattice. At zero temperature the problem of evaluation of asymptotics of correlation functions was solved in [2,3]. Here we consider the temperature correlation function.

$$g(n,t) = \frac{Tr\{e^{-\frac{H}{T}} \sigma_{n_2}^+(t_2) \sigma_{n_1}^-(t_1)\}}{Tre^{-\frac{H}{T}}}, \qquad n = n_2 - n_1, \qquad t = t_2 - t_1 \tag{2}$$

for the infinite lattice. We consider finite temperature $0 < T < \infty$ and a moderate magnetic field $0 \leq h < 2$.

We evaluated the asymptotics in cases where both space and time separation go to infinity $n \to \infty$, $t \to \infty$, in some direction φ

$$\frac{n}{4t} = \cot\varphi, \qquad 0 \leq \varphi \leq \frac{\pi}{2}. \tag{3}$$

In accordance with our calculations, correlation function $g(n,t)$ decays exponentially in any direction, but the rate of decay depends on the direction. In the space like direction, $0 \leq \varphi < \frac{\pi}{4}$, the asymptotics are

$$g(n,t) \to C \exp\left\{ \frac{n}{2\pi} \int_{-\pi}^{\pi} dp \ln \left| \tanh(\frac{h - 2\cos p}{T}) \right| \right\}. \tag{4}$$

In the time like direction $\frac{\pi}{4} < \varphi \leq \frac{\pi}{2}$, the asymptotics are different:

$$g(n,t) \to Ct^{(2\nu_+^2+2\nu_-^2)} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} dp\,|n-4t\sin p| \ln\left|\tanh(\frac{h-2\cos p}{T})\right|\right\}. \quad (5)$$

The values $\nu_\pm$, which define the pre-exponent, are

$$\nu_+ = \frac{1}{2\pi}\ln\left|\tanh\left(\frac{h-2\cos p_0}{T}\right)\right|$$

$$\nu_- = \frac{1}{2\pi}\ln\left|\tanh\left(\frac{h+2\cos \rho_0}{T}\right)\right| \quad (6)$$

where $\frac{n}{4t} = \sin p_0$. Equation (5) is valid in the whole time like cone, with exception of one direction $h = 2\cos p_0$. Higher asymptotic corrections will modify formulae by a factor of $(1+c(t,x))$ (c decays exponentially in the space-like region and as $t^{-1/2}$ in the time-like region). Also, it should be mentioned that the constant factor C in (4) does not depend on the direction φ, but does depend on φ in (5). We want to emphasize that for the pure time direction, $\varphi = \pi/2$, the leading factor in the asymptotics (exponent in (5)) was first obtained in [10].

To derive these formulae we went through a few steps.

The first step: The explicit expression for eigenfunctions of the Hamiltonian (1) (see [1]) was used to represent the correlation function as a determinant of an integral operator (of Fredholm type) [8]. In order to explain we need to introduce some notation. Let us consider the integral operator $\hat{V}$. Its kernel is equal to

$$V(\lambda\mu) = \frac{e_+(\lambda)e_-(\mu) - e_-(\lambda)e_+(\mu)}{\pi(\lambda-\mu)}. \quad (8)$$

Here λ and μ are complex variables, which go along the circle $|\lambda| = |\mu| = 1$ in the positive direction. The functions $e_\pm$ are

$$e_-(\lambda) = \lambda^{-n/2} \cdot e^{-it(\lambda+1/\lambda)} \sqrt{v(\lambda)}, \tag{9}$$

where

$$v(\lambda) = \left\{ 1 + \exp\left[\frac{2h - 2(\lambda + \frac{1}{\lambda})}{T} \right] \right\}^{-1}, \tag{10}$$

and

$$e_+(\lambda) = e_-(\lambda) E(x, t, \lambda). \tag{11}$$

Here E is defined as an integral

$$E(n, t, \lambda) = \frac{1}{\pi} v.p. \int \exp\{2it(\mu + \frac{1}{\mu})\} \cdot \frac{\mu^n d\mu}{\mu - \lambda}. \tag{12}$$

It is convenient to define functions $f_\pm(\lambda)$ as solutions of the following integral equations:

$$(I + \hat{V}) f_k = e_k. \tag{13}$$

Here I is the identity operator and $k = \pm$. Next we define the potentials $B_{kj}(k, j = \pm)$:

$$B_{kj}(n, t) = \frac{1}{2\pi i} \int f_k(\lambda) e_j(\lambda) \frac{d\lambda}{\lambda}. \tag{14}$$

They depend on space and time variables n, t. These we shall use to define new potentials b_{kj}:

$$\begin{aligned} b_{--}(n, t) &= B_{--}(n, t), \\ b_{++}(n, t) &= B_{++}(n, t) - 2iG(n, t) B_{+-}(n, t) - G(n, t). \end{aligned} \tag{15}$$

Here we used the function

$$G(n,t) = \frac{1}{2\pi i}\int \lambda^{n-1}\exp\{2it(\lambda + \frac{1}{\lambda})\}d\lambda. \tag{16}$$

Now all the notation is ready to write a determinant formula for the correlation function $g(n,t)$ (see (4)):

$$g(n,t) = e^{-2iht} b_{++}(n,t)\exp\{\sigma(n,t)\}. \tag{17}$$

Here e^{σ} is a determinant of the integral operator

$$\exp\{\sigma(n,t)\} = \det(1+\hat{V}). \tag{18}$$

Second step: Formulae (8)-(15) can be used to show that the potentials b_{++} and b_{--} satisfy a system of nonlinear differential equations.

$$\frac{i}{2}\frac{\partial}{\partial t}b_{--}(n,t) = (1+4b_{--}(n,t)b_{++}(n,t))(b_{--}(n+1,t)+b_{--}(n-1,t))$$

$$-\frac{i}{2}\frac{\partial}{\partial t}b_{++}(n,t) = (1+4b_{--}(n,t)b_{++}(n,t))(b_{++}(n+1,t)+b_{++}(n-1,t)). \tag{19}$$

The derivation of these equations is similar to [4,5,7]. Equations (19) are completely integrable differential equations. They were first discovered by Ablowitz and Ladik [9] as an integrable discretization of the nonlinear Schroedinger equation. The logarithmic derivatives of $\sigma(x,t)$ (see (18)) can be expressed in terms of solutions of the system (19):

$$
\begin{aligned}
\frac{\partial^2\sigma(n,t)}{16\partial t^2} &= 2b_{--}(n,t)b_{++}(n,t) - b_{++}(n-1,t)b_{--}(n+1,t) - \\
&-b_{--}(n-1,t)b_{++}(n+1,t) - 4b_{++}(n,t)b_{--}(n,t)[b_{++}(n-1,t)b_{--}(n+1,t) + \\
&b_{--}(n-1,t)b_{++}(n+1,t)]
\end{aligned}
\tag{20}
$$

$$
\sigma(n+1,t) + \sigma(n-1,t) - 2\sigma(n,t) = \ln[1 + 4b_{--}(n,t)b_{++}(n,t)] \tag{21}
$$

$$
\frac{\partial}{\partial t}[\sigma(n+1,t) - \sigma(n,t)] = 8i[b_{++}(n+1,t)b_{--}(n,t) - b_{++}(n,t)b_{--}(n+1,t)]. \tag{22}
$$

This shows that the quantum correlation function g (2) can be expressed in terms of the solution of the system (19). The meaning of all these formulae is that the correlation function of the XY model is the τ- function (in a sense of the well-known works [11,12]) of Ablowitz-Ladik's differential-difference equations. In the papers [5,6,7] the relation between the τ functions of the classical partial differential equations and quantum correlation functions, together with the history of the question, is explained in more detail. It is also worth mentioning that the idea to connect quantum correlation functions and classical completely integrable systems goes back to the work [13] and was first applied to the XY model in [14].

Third step: In order to evaluate the asymptotics one should solve Ablowitz-Ladik's differential equation. Initial data can be extracted from the integral representations (8)-(18). We use the Riemann-Hilbert problem in order to evaluate the asymptotics of the solution of equation (19). It is quite similar to the nonlinear Shrodinger case [6,7].

Finally, let us explain the physical meaning of our asymptotic formula (4). We start from the expression for the free energy [1]:

$$f(h) = -h - \frac{T}{2\pi}\int_{-\pi}^{\pi} dp \ln\left(1 + \exp[\frac{4\cos p - 2h}{T}]\right). \tag{23}$$

We emphasize the dependence on the magnetic field h. The definition of $f(h)$ is standard:

$$Tre^{-\frac{H}{T}} = \exp\{-\frac{L}{T}f(h)\}. \tag{24}$$

Here L is the length of the box. Let us use Jordan-Wigner transformation to transform correlator (2) (in the equal time case):

$$\sigma^+_{n_2}(0)\sigma^-_{n_1}(0) = \psi_{n_2}\exp\left\{i\pi\sum_{k=n_1+1}^{n_2-1}\psi^+_k\psi_k\right\}\psi^+_{n_1}, \qquad \psi^+_k\psi_k = \frac{1}{2}(1-\sigma^z_k). \tag{25}$$

Here ψ_k is a canonical Fermi field. We note that numerator in (2) differs from the denominator by replacement of the magnetic field $h \to h - i\pi T/2$ on the space interval $[n_1+1, n_2-1]$. This leads us to the following asymptotic expression for correlator $g(n,0)$

$$g(n,0) \to \exp\left\{Re\frac{n}{T}\left[f(h) - f(h - \frac{i\pi T}{2})\right]\right\}. \tag{26}$$

The reason we wrote Re is that $i\pi$ in (25) can be replaced by $-i\pi$. It is remarkable that (26) coincides with the correct answer (4). It is also worth mentioning that to go to the exponent in (5) one should replace the differential $d(np)$ by the expression $|d(np - t\varepsilon(p))|$, where $\varepsilon(p) = -4\cos p + 2h$ is the energy of the quasiparticle of the model.

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