The form factors in the finite volume

V. E. Korepin[†] and N. A. Slavnov[‡]

[†] *ITP, SUNY at Stony Brook, NY 11794-3840, USA.*

[‡] *Steklov Mathematical Institute, Gubkina 8, Moscow 117966, Russia.*

Abstract

The form factors of integrable models in finite volume are studied. We construct the explicite representations for the form factors in terms of determinants.

1 Introduction

In this paper we study form factors of quantum integrable models in finite volume. The possibility to obtain exact non-pertubative results in this field is of great importance. The Quantum Inverse Scattering Method created by L. D. Faddeev and his school [1, 2, 3] allows one to formulate general methods of solving this problem. Namely, in order to obtain an explicit representation for a form factor of an operator $\mathcal{O}$, it is sufficient to embed this operator into the algebra

$$R(\lambda,\mu)\Big(T(\lambda)\otimes T(\mu)\Big)=\Big(T(\mu)\otimes T(\lambda)\Big)R(\lambda,\mu). \tag{1.1}$$

After this is done, the problem is reduced to the standard Algebraic Bethe Ansatz procedures.

We would like to mention that our approach is drastically different from the ones, used in relativistic models [4]–[14]. We consider form factors between the states constructed on the bare vacuum, instead of the physical one. We also do not use the axioms, describing analytical properties of form factors. All our calculations are based on the algebra (1.1) only. In spite of this approach meets serious combinatorial difficulties, nevertheless it provides us with explicit determinant representations for form factors.

In the present paper we consider Quantum Nonlinear Schrödinger equation (QNLS) and XXZ (XXX) Heisenberg chains of spin 1/2. The QNLS model in a finite volume L with periodically boundary conditions is described by the Hamiltonian

$$H_{\rm QNLS}=\int_{-L/2}^{L/2}dx\left(\partial_x\Psi^\dagger\partial_x\Psi+c\Psi^\dagger\Psi^\dagger\Psi\Psi\right), \tag{1.2}$$

where $\Psi(x,t)$ and $\Psi^\dagger(x,t)$ are canonical Bose fields

$$[\Psi(x,t),\Psi^\dagger(y,t)] = \delta(x-y), \qquad \Psi|0\rangle = 0, \qquad \langle 0|\Psi^\dagger = 0, \tag{1.3}$$

and c is the coupling constant.

The Hamiltonian of the XXZ Heisenberg chain with M sites is

$$H_{XXZ} = -\sum_{m=1}^{M}\left(\sigma_-^{(m)}\sigma_+^{(m+1)} + \sigma_+^{(m)}\sigma_-^{(m+1)} + \Delta(\sigma_z^{(m)}\sigma_z^{(m+1)} - 1)\right). \tag{1.4}$$

Here $\sigma_a^{(m)}$ are local spin operators (Pauli matrices), associated to the m-th site, $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, $\sigma_a^{(M+1)} = \sigma_a^{(1)}$. Parameter Δ defines the anisotropy of the model. Particular choice $\Delta = 1$ corresponds to the XXX chain.

Let us describe now the content of the paper. In the section 2 we recall the basic definitions of the Algebraic Bethe Ansatz. In the section 3 the determinant representation for the form factor of the local field Ψ of QNLS is obtained. In the section 4 we consider different representations for form factors of QNLS and prove their equivalency. The last section is devoted to the form factors of Heisenberg chains.

2 Algebraic Bethe ansatz

We consider integrable models, which can be solved via the Algebraic Bethe Ansatz. Let the monodromy matrix is

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{2.1}$$

and the highest vector $|0\rangle$ (pseudovacuum) of this matrix has the following properties:

$$A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \quad C(\lambda)|0\rangle = 0. \tag{2.2}$$

The dual vector $\langle 0|$ is defined similarly

$$\langle 0|A(\lambda) = a(\lambda)\langle 0|, \quad \langle 0|D(\lambda) = d(\lambda)\langle 0|, \quad \langle 0|B(\lambda) = 0. \tag{2.3}$$

The explicit form of functions $a(\lambda)$ and $d(\lambda)$, entering Eqs. (2.2), (2.3), depends on specific model.

The R-matrix, defining the commutation relations of the monodromy matrix entries (1.1), can be written in the form:

$$R(\lambda,\mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix}. \tag{2.4}$$

For the models QNLS and XXX Heisenberg chain the R-matrix is rational:

$$f(\mu,\lambda)=\frac{\mu-\lambda+ic}{\mu-\lambda},\qquad g(\mu,\lambda)=\frac{ic}{\mu-\lambda}. \tag{2.5}$$

Here c is a constant. Below we shall use also two new functions

$$\begin{aligned} h(\mu,\lambda) &\equiv \frac{f(\mu,\lambda)}{g(\mu,\lambda)} &= \frac{\mu-\lambda+ic}{ic}, \\ t(\mu,\lambda) &\equiv \frac{g(\mu,\lambda)}{h(\mu,\lambda)} &= -\frac{c^2}{(\mu-\lambda)(\mu-\lambda+ic)}. \end{aligned} \tag{2.6}$$

For the XXZ chain the R-matrix has the same form (2.4), however the functions $g(\lambda,\mu)$ and $f(\lambda,\mu)$ become trigonometric ones

$$f(\mu,\lambda)=\frac{\sinh(\mu-\lambda+i\gamma)}{\sinh(\mu-\lambda)},\qquad g(\mu,\lambda)=\frac{i\sin\gamma}{\sinh(\mu-\lambda)}. \tag{2.7}$$

Respectively the functions $h(\lambda,\mu)$ and $t(\lambda,\mu)$ should be replaced by their trigonometric analogs according to the definition (2.6). The constant γ is related to the anisotropy of the model by $\cos\gamma=\Delta$.

The entries of the monodromy matrix (2.1) act in a space, consisting of states

$$|\Psi_N(\{\lambda\})\rangle=\prod_{j=1}^{N}B(\lambda_j)|0\rangle,\qquad N=0,1,\ldots \tag{2.8}$$

Here $\{\lambda\}$ are arbitrary complex parameters. Similarly dual states can be constructed by the operators $C(\lambda)$

$$\langle\Psi_N(\{\lambda\})|=\langle 0|\prod_{j=1}^{N}C(\lambda_j),\qquad N=0,1,\ldots \tag{2.9}$$

In what follows we shall refer to the states (2.8), (2.9) with arbitrary λ-s as 'BA-vectors' (Bethe Ansatz vectors).

The transfermatrix $\tau(\lambda)=\operatorname{tr}T(\lambda)=A(\lambda)+D(\lambda)$ generates the complete set of the conservation laws of the model. The eigenstates of the transfermatrix have the form (2.8), (2.9), however the parameters $\{\lambda\}$ are not arbitrary, but satisfy the system of Bethe equations

$$r(\lambda_j)\prod_{\substack{k=1\\k\neq j}}^{N}\frac{f(\lambda_j,\lambda_k)}{f(\lambda_k,\lambda_j)}=1,\qquad \text{where}\qquad r(\lambda)=\frac{a(\lambda)}{d(\lambda)}. \tag{2.10}$$

The eigenvalues of the transfermatrix are

$$\begin{aligned} &\tau(\mu)|\Psi_N(\{\lambda\})\rangle=\theta(\mu|\{\lambda\})|\Psi_N(\{\lambda\})\rangle,\\ &\theta(\mu|\{\lambda\})=a(\mu)\prod_{m=1}^{N}f(\mu,\lambda_m)+d(\mu)\prod_{m=1}^{N}f(\lambda_m,\mu). \end{aligned} \tag{2.11}$$

Let us focus now at the models, which will be considered in the next sections. For the QNLS the pseudovacuum (2.2) coincides with the Fock vacuum (1.3). The coupling constant c is equal to the constant in (2.5), the functions $a(\lambda)$ and $d(\lambda)$ are

$$a(\lambda) = \exp(-i\lambda L/2), \qquad d(\lambda) = \exp(i\lambda L/2). \tag{2.12}$$

In the XXZ chain with M sites the pseudovacuum vectors $|0\rangle$ and $\langle 0|$ are

$$|0\rangle = \bigotimes_{m=1}^{M} \uparrow_m, \qquad \langle 0| = (|0\rangle)^T. \tag{2.13}$$

For intermediate calculations the inhomogeneous XXZ chain is suitable. For this case the monodromy matrix is given by

$$T(\lambda) = L_M(\lambda - \xi_M)\cdots L_1(\lambda - \xi_1), \tag{2.14}$$

where ξ_m are arbitrary complex numbers, and local L-operators are

$$L_m(\lambda) = i \begin{pmatrix} \cosh\Bigl(\lambda - i\frac{\gamma}{2}\sigma_z^{(m)}\Bigr) & \sin\gamma\sigma_-^{(m)} \\ \sin\gamma\sigma_+^{(m)} & \cosh\Bigl(\lambda + i\frac{\gamma}{2}\sigma_z^{(m)}\Bigr) \end{pmatrix} \tag{2.15}$$

The functions $a(\lambda)$ and $d(\lambda)$ are equal to

$$a(\lambda) = \prod_{m=1}^{M} \cosh(\lambda - \xi_m - i\gamma/2), \qquad d(\lambda) = \prod_{m=1}^{M} \cosh(\lambda - \xi_m + i\gamma/2). \tag{2.16}$$

In the limit $\xi_m \to 0$ we turn back to the homogeneous XXZ chain.

Particular case $\Delta = 1$ corresponds to XXX Heisenberg magnet. This model is described by rational R-matrix (2.4) with $c = 1$. The vacuum eigenvalues are

$$a(\lambda) = \prod_{m=1}^{M} (\lambda - \xi_m - i/2), \qquad d(\lambda) = \prod_{m=1}^{M} (\lambda - \xi_m + i/2). \tag{2.17}$$

In the conclusion of this section we would like to mention that the Hamiltonians, presented in the Introduction, are not of the most general form. One can consider, for instance, Heisenberg magnets in external magnetic field. Similarly it is possible to consider QNLS with a chemical potential. In order to do this, one need to add to the Hamiltonians (1.2), (1.4) the corresponding terms. However these additional terms do not play an essential role in the models in finite volume. They become significant only in the thermodynamic limit, in particular they define the structure of the ground state and the thermodynamic properties of the models. Since the thermodynamic limit is not considered in this paper, therefore we restrict our selves with the Hamiltonians (1.2) and (1.4) only.

3 Form factor of the local field in the QNLS

The calculation of form factors in the framework of the Algebraic Bethe Ansatz formally can be reduced to the calculation of scalar products of a special type:

$$S_N(\{\mu\},\{\lambda\}) = \langle 0| \left(\prod_{j=1}^{N} C(\mu_j)\right)\left(\prod_{j=1}^{N} B(\lambda_j)\right)|0\rangle. \tag{3.1}$$

In the last equation at least one of the states, say $\langle 0|\prod_{j=1}^{N} C(\mu_j)$, is an eigenfunction of the transfermatrix. In other words the parameters $\mu_1,\dots,\mu_N$ satisfy the system of the Bethe equations (2.10), while the parameters $\lambda_1,\dots,\lambda_N$ remain free, i.e. $\prod_{j=1}^{N} B(\lambda_j)$ is an arbitrary BA-vector.

Let us demonstrate the general idea of the method, which will be used. Consider a form factor of an operator $\mathcal{O}$

$$F_{\mathcal{O}} = \langle 0| \left(\prod_{j=1}^{N_\mu} C(\mu_j)\right)\mathcal{O}\left(\prod_{j=1}^{N_\nu} B(\nu_j)\right)|0\rangle. \tag{3.2}$$

Here both states $\langle 0|\prod_{j=1}^{N_\mu} C(\mu_j)$ and $\prod_{j=1}^{N_\nu} B(\nu_j)|0\rangle$ are eigenstates of the transfermatrix. The operator $\mathcal{O}$ acting on the vector $\prod_{j=1}^{N_\nu} B(\nu_j)|0\rangle$ gives a new state, which generally speaking is not an eigenstate of $\tau(\lambda)$. However, if this new state can be presented as a linear combination of BA-vectors with numerical coefficients c_k

$$\mathcal{O}\left(\prod_{j=1}^{N_\nu} B(\nu_j)\right)|0\rangle = \sum_k c_k \left(\prod_{j=1}^{N_k} B(\lambda_j^{(k)})\right)|0\rangle, \tag{3.3}$$

then evidently the form factor $F_{\mathcal{O}}$ is equal

$$F_{\mathcal{O}} = \sum_k c_k \langle 0| \left(\prod_{j=1}^{N_\mu} C(\mu_j)\right)\left(\prod_{j=1}^{N_k} B(\lambda_j^{(k)})\right)|0\rangle. \tag{3.4}$$

Anyway the state $\langle 0|\prod_{j=1}^{N_\mu} C(\mu_j)$ remains the eigenstate of the transfermatrix. Thus, in order to find explicit expression for the form factor $\mathcal{O}$, one need to calculate the scalar products (3.1). In order to do this, it is enough to use the algebra (1.1) only. Moreover the specific form of the vacuum eigenvalues $a(\lambda)$ and $d(\lambda)$, as well as the structure of the pseudovacuum, are not essential. However, the straightforward calculation of the scalar products appears to be a complicated combinatorial problem. These difficulties had been partly overcame in [16], [17]. In the paper [18] the determinant representation for the scalar product of two arbitrary BA-vectors was obtained in terms of auxiliary quantum operators—dual fields. The explicit determinant representation for the scalar products (3.1) was found in [19] (see also [20]):

$$S_N(\{\mu\},\{\lambda\}) = \left\{\prod_{a=1}^{N} d(\mu_a)d(\lambda_a)\right\}\left\{\prod_{a,b=1}^{N} h(\mu_a,\lambda_b)\right\}$$
$$\times\left\{\prod_{N\geq a>b\geq 1} g(\lambda_a,\lambda_b)g(\mu_b,\mu_a)\right\}\det\left(M_{jk}\right), \tag{3.5}$$

where

$$M_{jk} = t(\mu_k, \lambda_j) - r(\lambda_j)t(\lambda_j, \mu_k) \prod_{m=1}^{N} \frac{f(\lambda_j, \mu_m)}{f(\mu_m, \lambda_j)}. \tag{3.6}$$

Recall that here the parameters $\{\mu\}$ satisfy the Bethe equations (2.10), while the parameters $\{\lambda\}$ are free. Before applying this formula to the calculations of form factors we would like to mention several features of the representation (3.5).

This representation is valid for arbitrary model, possessing rational R-matrix (2.4) or trigonometric one. The representation (3.5) also allows one to prove the orthogonality of the transfermatrix eigenstates directly. Indeed, the particular case of an BA-vector $\prod_{j=1}^{N} B(\lambda_j)|0\rangle$ is an eigenstate of $\tau(\lambda)$. Therefore, one can demand the parameters $\{\lambda\}$ to satisfy the Bethe equations (2.10), but to be different from the parameters $\{\mu\}$. In this case we obtain the scalar product of two different eigenstates, which must be equal to zero. It is easy to check, that for the such choice of $\{\lambda\}$ the determinant $\det M$ in (3.5) does vanishes. Indeed, the matrix (3.6) turns into

$$\tilde{M}_{jk} = t(\mu_k, \lambda_j) + V_j t(\lambda_j, \mu_k) \tag{3.7}$$

where

$$V_j = \prod_{m=1}^{N} \frac{h(\lambda_j, \mu_m)h(\lambda_m, \lambda_j)}{h(\mu_m, \lambda_j)h(\lambda_j, \lambda_m)}. \tag{3.8}$$

This matrix has the eigenvector ξ with zero eigenvalue:

$$\sum_{k=1}^{N} \tilde{M}_{jk}\xi_k = 0, \qquad \xi_k = \frac{\prod_{\substack{m=1\\ m\neq k}}^{N} g(\mu_k, \mu_m)}{\prod_{m=1}^{N} g(\mu_k, \lambda_m)}. \tag{3.9}$$

The proof of the (3.9) is given in Appendix 1. Hence $\det \tilde{M} = 0$.

Now let us apply the representation (3.5) to the calculation of specific form factors. Namely, consider the form factor of the local field $\Psi(x,t)$ in the QNLS

$$F_\Psi(x,t) = \langle 0| \left(\prod_{j=1}^{N} C(\mu_j)\right) \Psi(x,t) \left(\prod_{j=1}^{N+1} B(\lambda_j)\right) |0\rangle. \tag{3.10}$$

The dependency on time t and distance x evidently separated

$$F_\Psi(x,t) = \exp\left\{\sum_{k=1}^{N} \left(it\mu_k^2 - ix\mu_k\right) - \sum_{j=1}^{N+1} \left(it\lambda_j^2 - ix\lambda_j\right)\right\} F_\Psi(0,0). \tag{3.11}$$

The action of the operator $\Psi(0,0)$ on the BA-vector had been found in [15]:

$$\Psi(0,0) \prod_{j=1}^{N+1} B(\lambda_j)|0\rangle = -i\sqrt{c} \sum_{\ell=1}^{N+1} a(\lambda_\ell) \left(\prod_{\substack{m=1\\ m\neq\ell}}^{N+1} f(\lambda_\ell, \lambda_m)\right) \prod_{\substack{m=1\\ m\neq\ell}}^{N+1} B(\lambda_m)|0\rangle. \tag{3.12}$$

In spite of the parameters $\lambda_1, \dots, \lambda_{N+1}$ satisfy the Bethe equations

$$r(\lambda_j) = \prod_{p=1}^{N+1} \frac{h(\lambda_p, \lambda_j)}{h(\lambda_j, \lambda_p)}, \tag{3.13}$$

the states entering the r.h.s. of (3.12) are not eigenstates of the transfermatrix, since each of them depends only on N parameters λ. Nevertheless all these states are BA-vectors, therefore we can apply (3.5) for calculation of the scalar products

$$S_N^{(\ell)}(\{\mu\}, \{\lambda\}) = \langle 0| \left(\prod_{j=1}^{N} C(\mu_j) \right) \left(\prod_{\substack{j=1 \\ j\neq \ell}}^{N+1} B(\lambda_j) \right) |0\rangle, \tag{3.14}$$

since the state $\langle 0| \prod_{j=1}^{N} C(\mu_j)$ remains an eigenstate. After simple algebra [21] we obtain for the form factor $F_\Psi(0,0)$:

$$F_\Psi(0,0) = -i\sqrt{c} \prod_{a>b}^{N} g(\mu_a, \mu_b) \prod_{a>b}^{N+1} g(\lambda_b, \lambda_a) \prod_{a,b=1}^{N+1} h(\lambda_a, \lambda_b)$$

$$\times \prod_{a=1}^{N} d(\mu_a) \prod_{b=1}^{N+1} d(\lambda_b) \left(\sum_{\ell=1}^{N+1} (-1)^{\ell+1} \det S_{jk}^{(\ell)} \right). \tag{3.15}$$

Here we introduce new matrix S:

$$S_{jk} = t(\mu_k, \lambda_j) \frac{\prod_{m=1}^{N} h(\mu_m, \lambda_j)}{\prod_{p=1}^{N+1} h(\lambda_p, \lambda_j)} - t(\lambda_j, \mu_k) \frac{\prod_{m=1}^{N} h(\lambda_j, \mu_m)}{\prod_{p=1}^{N+1} h(\lambda_j, \lambda_p)}, \qquad j,k = 1, \dots, N. \tag{3.16}$$

The matrix $S^{(\ell)}$ can be obtained from the matrix S via the replacement of the ℓ-th row by $S_{N+1,k}$:

$$S_{jk}^{(\ell)} = \begin{cases} S_{jk}, & j \neq \ell, \\ S_{N+1,k}, & j = \ell. \end{cases} \tag{3.17}$$

The sum with respect to ℓ in (3.16) gives a single determinant:

$$\sum_{\ell=1}^{N+1} (-1)^{\ell+1} \det S_{jk}^{(\ell)} = \det(S_{jk} - S_{N+1,k}). \tag{3.18}$$

In order to prove (3.18) one can use the Laplace formula for determinant of sum of two matrices, and to take into account that $S_{N+1,k}$ is the first-rank matrix. Thus we arrive at the determinant representation for the form factor of the local field ib QNLS [21]:

$$F_\Psi(0,0) = -i\sqrt{c} \prod_{a>b}^{N} g(\mu_a, \mu_b) \prod_{a>b}^{N+1} g(\lambda_b, \lambda_a) \prod_{a,b=1}^{N+1} h(\lambda_a, \lambda_b)$$

$$\times \prod_{a=1}^{N} d(\mu_a) \prod_{b=1}^{N+1} d(\lambda_b) \det(S_{jk} - S_{N+1,k}). \tag{3.19}$$

Here S_{jk} is given by (3.16), and $S_{N+1,k}$ can be obtained from S_{jk} via the replacement $\lambda_j \to \lambda_{N+1}$.

In the conclusion of this section we would like to emphasize once more that the most serious problem of the described method is to embed the operator $\mathcal{O}$, whose form factor wanted to be found, into the algebra (1.1). However, after the commutation relations between $\mathcal{O}$ and the monodromy matrix are defined, then the problem can be solved via standard methods of the Algebraic Bethe Ansatz.

4 Other representations for form factors

As we have mentioned already, the method used for calculations of form factors of relativistic models is essentially different from the method, presented in the previous section. Nevertheless a series of the results, obtained in this domain for the quantum field theory models, also can be applied for the models in finite volume. Thus, for instant, the form factors of $O(3)$-invariant σ-model unsuspectingly had been found closely related to the form factors of the QNLS [22]. In the present section we consider several examples.

The approach, developed in papers [16, 17], allows to express form factors of the QNLS in terms of some rational functions Σ^α (originally they were denoted as σ^α), which depend on N_μ arguments $\{\mu_a\}$ and N_λ arguments $\{\lambda_b\}$. The only singularities of these functions are simple poles at $\lambda_j = \mu_k$, and the residues in these points can be expressed in terms of Σ^α, depending on the arguments $\{\mu_{a\neq k}\}$ and $\{\lambda_{b\neq j}\}$. This property allows one to construct Σ^α via a recurrence, however during a long time this recurrence have not been solved in general form.

On the other hand the same functions Σ^α appeared to be useful for the analysis of the form factors of $O(3)$-invariant σ-model. The series of the explicit determinant representations for Σ^α, depending on elementary symmetric polynomials $\sigma_k^{(n)}$, were found in [23]:

$$\sigma_k^{(n)}(x_1,\dots,x_n) = \frac{1}{(n-k)!}\,\frac{\partial^{(n-k)}}{\partial x^{(n-k)}}\prod_{m=1}^{n}(x+x_m)\Big|_{x=0}. \tag{4.1}$$

The explicit solution for the functions Σ^α has the form:

$$\Sigma^\alpha(\{\mu\},\{\lambda\}) = \frac{\det\Omega}{\Delta(\mu)\Delta(\lambda)\prod_{a,b}(\mu_a-\lambda_b)}. \tag{4.2}$$

Here $(N_\mu+N_\lambda)\times(N_\mu+N_\lambda)$ matrix Ω has the following entries:

$$\Omega_{j,k} \equiv M_{jk} = \sigma_j^{(N_\lambda+N_\mu-1)}\left(\{\mu_{a\neq k}-\frac{ic}{2}\},\{\lambda_a+\frac{ic}{2}\}\right)$$

$$-e^\alpha\sigma_j^{(N_\lambda+N_\mu-1)}\left(\{\mu_{a\neq k}+\frac{ic}{2}\},\{\lambda_a-\frac{ic}{2}\}\right), \qquad k=1,\dots,N_\mu, \tag{4.3}$$

$$\Omega_{j,k+N_\mu} \equiv \Lambda_{jk} = \sigma_j^{(N_\lambda+N_\mu-1)}\left(\{\mu_a+\frac{ic}{2}\},\{\lambda_{a\neq k}-\frac{ic}{2}\}\right)$$

$$-\sigma_j^{(N_\lambda+N_\mu-1)}\left(\{\mu_a-\frac{ic}{2}\},\{\lambda_{a\neq k}+\frac{ic}{2}\}\right),\qquad k=1,\dots,N_\lambda. \tag{4.4}$$

Index j runs through $0,1,\dots,N_\mu+N_\lambda-1$. By $\Delta(\cdot)$ we denote the Wandermonde determinants.

Form factors of the QNLS can be expressed in terms of the functions Σ^α. In particular the form factor of the local field $F_\Psi(0,0)$, considered in the previous section, is proportional to the function Σ^0 with $N_\mu=N_\lambda-1=N$:

$$F_\Psi(0,0)=-i\sqrt{c}\prod_{a=1}^{N}d(\mu_a)\prod_{b=1}^{N+1}d(\lambda_b)\;\Sigma^\alpha(\{\mu\}_N,\{\lambda\}_{N+1})\Big|_{\alpha=0}. \tag{4.5}$$

It is wort mentioning, that in this case Ω is $(2N+1)\times(2N+1)$ matrix. On the other hand, as we have seen above, the form factor of the local field is proportional to the determinant of the $N\times N$ matrix (3.6). There exists a nice method to reduce the determinant of the matrix Ω (4.3), (4.4) to the determinant of (3.6). We shall demonstrate this method for more simple form factor.

Consider an operator Q_1 of the number of particles in the interval $[0,x]$. In the QNLS this operator is equal to

$$Q_1=\int_0^x\Psi^\dagger(y)\Psi(y)\,dy. \tag{4.6}$$

The form factor of this operator was calculated in [17] in terms of the function Σ^α with $N_\mu=N_\lambda=N$:

$$\begin{aligned}F_{Q_1}&\equiv\langle 0|\left(\prod_{j=1}^{N}C(\mu_j)\right)Q_1\left(\prod_{j=1}^{N}B(\lambda_j)\right)|0\rangle\\
&=\left[\exp\left\{ix\sum_{a=1}^{N}(\lambda_a-\mu_a)\right\}-1\right]\frac{\partial}{\partial\alpha}\Sigma^\alpha(\{\mu\}_N,\{\lambda\}_N)\Big|_{\alpha=0}.\end{aligned} \tag{4.7}$$

Let us show, how one can reduce the determinant of $2N\times 2N$ matrix (4.2) to a determinant of $N\times N$ matrix.

Theorem 4.1 *The function $\Sigma^\alpha(\{\mu\}_N,\{\lambda\}_N)$ is proportional to the determinant of $N\times N$ matrix*

$$\begin{aligned}\Sigma^\alpha(\{\mu\}_N,\{\lambda\}_N)&=\prod_{a>b}^{N}g(\lambda_j,\lambda_k)g(\mu_k,\mu_j)\prod_{a,b=1}^{N}h(\lambda_a,\mu_b)\\
&\times\det\left(e^\alpha V_j t(\mu_k,\lambda_j)+t(\lambda_j,\mu_k)\right),\end{aligned} \tag{4.8}$$

where

$$V_j=\prod_{m=1}^{N}\frac{h(\mu_m,\lambda_j)h(\lambda_j,\lambda_m)}{h(\lambda_j,\mu_m)h(\lambda_m,\lambda_j)}. \tag{4.9}$$

(compare with (3.8)).

Proof. The determinant representation for the function $\Sigma^\alpha(\{\mu\}_N, \{\lambda\}_N)$ is given by (4.2)–(4.4) with $N_\mu = N_\lambda = N$. Let us multiply the matrix Ω from the left by matrix U

$$U_{j,k} \equiv P_{jk} = p_j^{2N-k-1}, \qquad j = 1, \ldots, N, \tag{4.10}$$

$$U_{j+N,k} \equiv Q_{jk} = q_j^{2N-k-1}, \qquad j = 1, \ldots, N. \tag{4.11}$$

Index k runs through the set $0, 1, \ldots, 2N-1$, parameters p_j and q_j are some complex numbers, which will be fixed later. Obviously

$$\det \Omega = \frac{\det(U\Omega)}{\det U}, \tag{4.12}$$

and

$$\det U = \Delta(p)\Delta(q) \prod_{a,b=1}^{N} (p_a - q_b). \tag{4.13}$$

The product $U\Omega$ can be written as 2×2 block-matrix

$$U\Omega = \begin{pmatrix} PM & P\Lambda \\ QM & Q\Lambda \end{pmatrix}, \tag{4.14}$$

where each of these blocks is $N \times N$ matrix. Using the representation for elementary symmetric polynomials (4.1), one can easily find the entries of these blocks, for example:

$$\begin{aligned} (PM)_{jk}(\{p\}, \{\mu\}, \{\lambda\}) = &\prod_{\substack{a=1\\a\neq k}}^{N} \left(p_j + \mu_a - \frac{ic}{2}\right) \prod_{a=1}^{N} \left(p_j + \lambda_a + \frac{ic}{2}\right) \\ &- e^\alpha \prod_{\substack{a=1\\a\neq k}}^{N} \left(p_j + \mu_a + \frac{ic}{2}\right) \prod_{a=1}^{N} \left(p_j + \lambda_a - \frac{ic}{2}\right), \end{aligned} \tag{4.15}$$

Other blocks have similar form. Namely, replacing p_j by q_j in (4.15) we obtain the block QM. As for the right blocks, they can be obtained from the left ones, in which one should replace $\{\mu\}$ by $\{\lambda\}$ and put $\alpha = 0$. Thus we obtain the representation for the function $\Sigma^\alpha(\{\mu\}_N, \{\lambda\}_N)$, containing $2N$ arbitrary complex parameters p_j and q_j, but does not depending on their specific choice.

Using well known formula for a determinant of a block-matrix

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \cdot \det(D - CA^{-1}B), \tag{4.16}$$

one can reduce $\det(U\Omega)$ to the determinants of $N \times N$ matrices. However, in order to simplify the calculations, we shall fix the parameters p_j and q_j. Let, for example, $p_j = \lambda_j - ic/2$ and $q_j = \lambda_j + ic/2$. Then we have for two left blocks:

$$(PM)_{jk} = -e^\alpha \prod_{\substack{a=1\\a\neq k}}^{N} (\mu_a - \lambda_j) \prod_{a=1}^{N} (\lambda_{aj} - ic), \tag{4.17}$$

$$(QM)_{jk} = \prod_{\substack{a=1\\a\neq k}}^{N}(\mu_a - \lambda_j)\prod_{a=1}^{N}(\lambda_{aj} + ic), \tag{4.18}$$

Here $\lambda_{aj} = \lambda_a - \lambda_j$. Observe that

$$(QM)_{jk} = -u_j(PM)_{jk}, \qquad \text{where} \qquad u_j = e^{-\alpha}\prod_{a=1}^{N}\left(\frac{\lambda_{aj} + ic}{\lambda_{aj} - ic}\right). \tag{4.19}$$

Hence, due to (4.16) we have

$$\det(U\Omega) = \det(PM)_{jk}\cdot\det\Big((Q\Lambda)_{jk} + u_j(P\Lambda)_{jk}\Big). \tag{4.20}$$

After simple algebra we arrive at

$$\det(U\Omega) = W\det G \tag{4.21}$$

where

$$W = \prod_{a,b=1}^{N}[(\mu_a - \lambda_b)(\lambda_{ab} - ic)(\lambda_{ab} + ic)]\prod_{a>b}^{N}[\mu_{ab}\lambda_{ba}], \tag{4.22}$$

and

$$G_{jk} = \delta_{jk}\frac{\prod\limits_{\substack{a=1\\a\neq j}}^{N}\lambda_{aj}}{\prod\limits_{a=1}^{N}(\mu_a - \lambda_j)}\left\{e^{\alpha}\prod_{a=1}^{N}\left(\frac{\mu_a - \lambda_j + ic}{\lambda_{aj} + ic}\right) - \prod_{a=1}^{N}\left(\frac{\mu_a - \lambda_j - ic}{\lambda_{aj} - ic}\right)\right\}$$
$$+\frac{e^{\alpha}}{\lambda_{jk} - ic} - \frac{1}{\lambda_{jk} + ic}. \tag{4.23}$$

In order to reduce $\det G$ to the determinant (4.9), one need to make one more step:

$$\det G = \frac{\det(G\Gamma)}{\det\Gamma}, \tag{4.24}$$

where

$$\Gamma_{jk} = \frac{1}{\lambda_j - \mu_k}\frac{\prod\limits_{a=1}^{N}(\lambda_j - \mu_a)}{\prod\limits_{\substack{a=1\\a\neq j}}^{N}\lambda_{ja}}, \quad \text{and} \quad \det\Gamma = \prod_{a>b}^{N}\frac{\mu_{ba}}{\lambda_{ba}}. \tag{4.25}$$

The product $G\Gamma$ can be computed explicitly via the method, describing in the Appendix 1. The resulting representation for the form factor F_{Q_1} exactly coincides with (4.8) (4.9).

One can use the same idea in order to reduce the determinant of $(2N+1)\times(2N+1)$ matrix (4.5) to the determinant of the $N\times N$ matrix (3.6). A small complication is caused by the fact that in this case the blocks in the matrix (4.14) have different dimension, for example, the block PM is $N\times N$ matrix, while the block QM is $(N+1)\times N$ matrix etc. This leads to the appearance of the term $S_{N+1,k}$ in (3.19).

Thus we have demonstrated different representations for form factors and proved their equivalency. The representations in terms of elementary symmetric polynomials are quite characteristic for the form factors of the relativistic models (see, for instance, [10]). At the same time the representations (3.19), (4.8) look more natural, since they depend only on the functions, entering the R-matrix of the model. These representations also are suitable for the computation of the correlation functions.

5 Form factors of Heisenberg chains

As we have seen already, the calculation of the form factors is based on the explicit determinant representation (3.5), (3.6) for the scalar products of the special type (3.1). However the calculation it self of the scalar products in the framework of the Algebraic Bethe Ansatz deals with rather complicated combinatorial problems. The method based on the explicit representations for the functions Σ^α in terms of elementary symmetric polynomials also makes use of solving of complicated recurrence. In the paper [24] a new approach was proposed, which allows to simplify significantly the calculation of scalar products and form factors. There a factorizing Drinfel'd twist F was constructed. The new basis generated by the twist F is extremely suitable to study the structure of BA-vectors. and for representation of local operators in terms of the entries of the monodromy matrix. In the paper [25] this method was used for calculation of the form factors in Heisenberg chains. We would like to present here the main results of the paper [25].

Consider inhomogeneous XXZ (XXX) model (2.14):

$$T(\lambda) = L_M(\lambda - \xi_M) \cdots L_1(\lambda - \xi_1). \tag{5.1}$$

Then the local spin operators $\sigma_a^{(m)}$ can be presented in terms of the monodromy matrix entries as

$$\begin{aligned}
\sigma_-^{(m)} &= \left(\prod_{a=1}^{m-1} \tau(\xi_a)\right) B(\xi_m) \left(\prod_{a=m+1}^{M} \tau(\xi_a)\right), \\
\sigma_+^{(m)} &= \left(\prod_{a=1}^{m-1} \tau(\xi_a)\right) C(\xi_m) \left(\prod_{a=m+1}^{M} \tau(\xi_a)\right), \\
\sigma_z^{(m)} &= \left(\prod_{a=1}^{m-1} \tau(\xi_a)\right) (A(\xi_m) - D(\xi_m)) \left(\prod_{a=m+1}^{M} \tau(\xi_a)\right).
\end{aligned} \tag{5.2}$$

Here $\tau(\lambda) = A(\lambda) + D(\lambda)$ is the transfermatrix.

The representation (5.2) allows to find the form factors of local spin operators of XXZ (XXX) chains. Consider, for example, form factor $F_-^{(m)}$ of the operator $\sigma_-^{(m)}$:

$$F_-^{(m)} = \langle 0| \left(\prod_{j=1}^{N+1} C(\mu_j)\right) \sigma_-^{(m)} \left(\prod_{j=1}^{N} B(\lambda_j)\right) |0\rangle. \tag{5.3}$$

Both states in (5.3) are eigenstates of the transfermatrix, hence substituting here (5.2) we obtain

$$F_{-}^{(m)} = \prod_{a=1}^{m-1} \theta(\xi_a|\{\mu\}) \prod_{a=m+1}^{M} \theta(\xi_a|\{\lambda\}) \cdot \langle 0| \left(\prod_{j=1}^{N+1} C(\mu_j) \right) B(\xi_m) \left(\prod_{j=1}^{N} B(\lambda_j) \right) |0\rangle. \tag{5.4}$$

Here θ is eigenvalue of the transfermatrix (2.11). Thus we again have reduced the form factor to the scalar product of eigenstate by BA-vector. It is enough now to use the representation (3.5), which gives us the representation for the form factor $F_{\sigma_-^{(m)}}$ in terms of a determinant of $(N+1)\times(N+1)$ matrix.

Let us present the final result in the form given in [25]. Introduce $(N+1)\times(N+1)$ matrix

$$T_{jk}(\{\mu\}_{N+1}|\{\nu\}_{N+1}) = \frac{\partial}{\partial \nu_k}\theta(\mu_j|\{\nu\}_{N+1}). \tag{5.5}$$

Then the form factor $F_{-}^{(m)}$ is proportional to the ratio of two determinants

$$F_{-}^{(m)} = \frac{\prod_{a=1}^{N+1}\prod_{b=1}^{m-1} f(\mu_a,\xi_b)}{\prod_{a=1}^{N}\prod_{b=1}^{m} f(\lambda_a,\xi_b)} \cdot \frac{\det T_{jk}(\{\mu\}|\{\xi_m,\lambda_1,\dots,\lambda_N\})}{\det K_{jk}(\{\mu\}|\{\xi_m,\lambda_1,\dots,\lambda_N\})}. \tag{5.6}$$

Here K_{jk} is the Cauchy matrix

$$K_{jk}(\{\mu\}|\{\nu\}) = \varphi^{-1}(\mu_j-\nu_k), \tag{5.7}$$

where $\varphi(\lambda)=\lambda$ for XXX chain and $\varphi(\lambda)=\sinh\lambda$ for XXZ chain. Actually the function $\varphi^{-1}(\lambda)$ coincides with the function $g(\lambda)$ up to constant factor. The representation similar to (5.6) also exists for the scalar products (3.5).

6 Conclusion

We considered form factors of exactly solvable (completely integrable) models in the finite volume. We demonstrated that the form factors can be represented as determinants of matrices. The dimension of the matrix is related to the number of particles in the corresponding state. In our earlier publications we showed that this representation is useful for the theory of correlation functions. The contributions of all form factors can be taken into account and the correlation function also can be represented as a determinant.

A Zero eigenvector

Consider for simplicity the case of the rational R-matrix. Let $N\times N$ matrix (3.7) is given

$$\tilde{M}_{jk} = t(\mu_k,\lambda_j) + V_j t(\lambda_j,\mu_k). \tag{A.1}$$

Here

$$V_j = \prod_{m=1}^{N} \frac{(\lambda_j - \mu_m + ic)(\lambda_m - \lambda_j + ic)}{(\mu_m - \lambda_j + ic)(\lambda_j - \lambda_m + ic)}. \tag{A.2}$$

Recall also that $t(\lambda, \mu) = (ic)^2/(\lambda - \mu)(\lambda - \mu + ic)$. Let us prove that

$$\sum_{k=1}^{N} \tilde{M}_{jk}\xi_k = 0, \qquad \text{where} \qquad \xi_k = \frac{\prod\limits_{m=1}^{N}(\mu_k - \lambda_m)}{\prod\limits_{\substack{m=1\\m\neq k}}^{N}(\mu_k - \mu_m)}. \tag{A.3}$$

We have

$$\sum_{k=1}^{N} \tilde{M}_{jk}\xi_k = (ic)^2\Big(G_+ + V_j G_-\Big), \tag{A.4}$$

where

$$G_\pm = \sum_{k=1}^{N} \frac{1}{(\mu_k - \lambda_j)(\mu_k - \lambda_j \pm ic)} \cdot \frac{\prod\limits_{m=1}^{N}(\mu_k - \lambda_m)}{\prod\limits_{\substack{m=1\\m\neq k}}^{N}(\mu_k - \mu_m)}, \tag{A.5}$$

In order to find $G_\pm$ consider auxiliary integral

$$I_\pm = \frac{1}{2\pi i} \int\limits_{|z|=R\to\infty} \frac{dz}{(z - \lambda_j)(z - \lambda_j \pm ic)} \prod_{m=1}^{N} \frac{(z - \lambda_m)}{(z - \mu_m)}. \tag{A.6}$$

The integral is taken with respect to the closed contour around infinity (in other words all the poles of the integrand lie within the contour). Obviously the integral $I_\pm$ is equal to the residue at infinity, which in turn is equal to zero, hence $I_\pm = 0$. On the other hand the value of the integral $I_\pm$ is equal to the sum of the residues in the poles $z = \mu_k, \quad k = 1, \ldots, N$ and $z = \lambda_j - ic$. Thus, we obtain

$$I_\pm = G_\pm \mp \frac{1}{ic} \prod_{m=1}^{N} \left(\frac{\lambda_m - \lambda_j \pm ic}{\mu_m - \lambda_j \pm ic}\right). \tag{A.7}$$

The first term in the r.h.s. of (A.7) corresponds to the residues at the points $z = \mu_k, \quad k = 1, \ldots, N$; the second term corresponds to the residue at the point $z = \lambda_j - ic$. Hence

$$G_\pm = \pm\frac{1}{ic} \prod_{m=1}^{N} \left(\frac{\lambda_m - \lambda_j \pm ic}{\mu_m - \lambda_j \pm ic}\right). \tag{A.8}$$

Substituting $G_\pm$ into (A.4) and using explicit expression (A.2) for V_j, we arrive at

$$\sum_{k=1}^{N} \tilde{M}_{jk}\xi_k = 0. \tag{A.9}$$

In the case of trigonometric R-matrix, after the replacement $\exp\{2\lambda\} \to \lambda$ and $\exp\{2\mu\} \to \mu$ the proof reduces to the rational case.

References

[1] L. D. Faddeev and E. K. Sklyanin, Dokl. Acad. Nauk SSSR **243** (1978) 1430.

[2] E. K. Sklyanin, Dokl. Acad. Nauk SSSR **244** (1978) 1337.

[3] L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan, Theor. Math. Phys. **40** (1979) 194.

[4] F. A. Smirnov, Form factors in completely integrable models of quantum field theory (World Scientific, Singapore, 1992).

[5] A. B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. 120 (1979) 253; A. B. Zamolodchikov, Advanced Studies in Pure Math. 19 (1989) 641; Int. J. Mod. Phys. A 3 (1988) 743.

[6] A. E. Arinshtein, V. A. Fatteev, and A. B. Zamolodchikov, Phys. Lett. B 87 (1979) 3389.

[7] B. Berg, M. Karowski and P. Weisz, Phys. Rev. D 19 (1979) 2477; M. Karowski and P. Weisz, Nucl. Phys. B 139 (1978) 445; M. Karowski, Phys. Rep. 49 (1979) 229.

[8] Al. B. Zamolodchikov, Nucl. Phys. B 348 (1991) 619.

[9] J. L. Cardy, G. Mussardo, Nucl. Phys. B 340 (1990) 387.

[10] A. Koubek and G. Mussardo, Phys. Lett. B 311 (1993) 193.

[11] A. Fring, G. Mussardo and P. Simonetti, Nucl. Phys. B 393 (1990) 413.

[12] C. Ahn, G. Delfino and G. Mussardo, Phys. Lett. B 317 (1993) 573.

[13] S. Lukyanov, Phys. Lett. B 408 (1997) 192; V. Brazhnikov and S. Lukyanov, preprints RU-97-58, CLNS 97/1488, hep-th/9707091.

[14] J. Balog, T. Hauer and M. R. Niedermaier, Phys. Lett. B 386 (1996) 224; Nucl. Phys. B 440 (1995) 603.

[15] A. G. Izergin, V. E. Korepin and N. Yu. Reshetikin, J. Phys. **A20** (1987) 4799.

[16] V. E. Korepin, Commun. Math. Phys. **86** (1982) 391.

[17] A. G. Izergin and V. E. Korepin, Commun. Math. Phys. **99** (1985) 271.

[18] V. E. Korepin, Commun. Math. Phys. **113** (1987) 177.

[19] N. A. Slavnov, Theor. Math. Phys. **79** (1989) 502.

[20] N. A. Slavnov, Zap. Nauchn. Semin. POMI **245** (1997) 270.

[21] T. Kojima, V. E. Korepin and N. A. Slavnov, Commun. Math. Phys. **188** (1997) 657.

[22] A. N. Kirillov and F. A. Smirnov, Int. Journ. Mod. Phys. A **3** (1988) 731

[23] A. N. Kirillov and F. A. Smirnov, J. Sov. Math. **47** (1989) 2413.

[24] J. M. Maillet and J. Sanchez de Santos, Drinfel'd twists and Algebraic Bethe Ansatz, q-alg/9612012.

[25] N. A. Kitanin, J. M. Maillet and V. Terras, Form factors of the XXZ Heisenberg spin-1/2 finite chain, math-ph/9807020.