SPECTRUM AND SCATTERING OF EXCITATIONS IN THE ONE-DIMENSIONAL ISOTROPIC HEISENBERG MODEL

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The work gives a consistent and uniform exposition of all known results related to Heisenberg model. The classification of excitations is presented and their scattering is described both in ferromagnetic and the antiferromagnetic cases. It is shown that in the antiferromagnetic case there exists only one excitation with spin 1/2 which is a kink in the following sense: in physical states there is only an even number of kinks—spin waves, therefore they always have an integer spin. Thus, it is shown that the conventional picture of excitations is wrong in the antiferromagnetic case and the spin wave has spin 1/2, matrix is calculated.

INTRODUCTION

The one-dimensional isotropic Heisenberg model describes a system of N interacting particles with spin 1/2 on a one-dimensional lattice. The state space $\mathfrak{H}_N$ and the energy operator H_N are as follows:

$$\mathfrak{H}_N = \prod_{n=1}^{\infty} \otimes \eta_n, \quad \eta_n \approx \mathbb{C}^2, \tag{1}$$

$$H_N = \frac{J}{4} \sum_{n=1}^{N} (\sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \sigma_n^3 \sigma_{n+1}^3 - I_N). \tag{2}$$

Here I_N is the identity operator in the space $\mathfrak{H}_N$; the operators σ_n^a have the following form:

$$\sigma_n^a = I \otimes \ldots I \otimes \underbrace{\sigma^a}_{n} \otimes I \ldots \otimes I, \quad a = 1,2,3 \tag{3}$$

and they act nontrivially only in η_n from the product (1); the σ^a are Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

I is the identity matrix in $\mathbb{C}^2$. We assume in the sum (2) the periodic boundary conditions

$$\sigma_{N+1}^a = \sigma_1^a, \quad a = 1,2,3. \tag{5}$$

Depending on the sign of J we distinguish the ferromagnetic case $J<0$ and the antiferromagnetic case $J>0$. The problem of the most interest is to find the eigenvectors and eigenvalues of the operator H_N and to investigate their asymptotic behavior as $N \to \infty$.

The model under examination has been introduced by Heisenberg [1] in 1928 and has a long history. At first Bethe [2] proposed the procedure for finding eigenvectors and eigenvalues; this method is called now the method of Bethe substitution—Bethe Ansatz. Then Hulthén [3], des Cloiseaux and Pearson [4], Orbach [5], Yang and Yang [6-7], Baxter [8], Gaudin [9], Takahashi [10], Ovchinnikov [11], Kulish and Reshetikhin [12] et al. obtained, using Bethe's method, important results.

Nevertheless, in spite of the long history and extensive literature devoted to the Heisenberg model the series of generally accepted results concerning this model is wrong. This is especially true for the classification of excitations in the antiferromagnetic case. Starting with the work [4] it has been accepted to consider that the simplest among these

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excitations — the spin wave — has spin 1. Moreover, in [11] it has been asserted that there exists, in addition, a whole series of one-particle excitations — bound states of spin waves, which are singlets excitations with spin 0. We show here that in the antiferromagnetic case there exists only one excitation with spin 1/2 which is a kink in the following sense: in the physical states there is always an even number of kinks-spin waves, so these states always have integer spin. However, the state of a kink can be localized, hence, we can talk about their scattering.

In connection with this we have decided to reconsider the Heisenberg model. The present paper has the systematic presentation of all known results from a uniform point of view. An important methodological role in this is played by the recently created quantum method of the inverse problem [13-14].

We have already considered in [15] the most general anisotropic Heisenberg magnetic (XYZ-model) using the quantum method of the inverse problem. The difference between the present paper and that work is that here in a simpler situation we obtain more detailed results. Our hightened interest in the isotropic model is also related to the fact that this model is a component in the hierarchy of Bethe's substitutions used for solving quantum field theory models with color degrees of freedom (see [16-18, 12]). At first, this role of the model appeared clearly in the works of Gaudin [19] and Yang [20], devoted to the problem of two-component Fermi-gas and Bose-gas, respectively, and in the work of Lieb and Wu [21] devoted to Hubbard's model [22]. At the present time it becomes clear that Hubbard's model and its natural generalization for the cases of many colors and Bose-statistics is an interesting model of quantum field theory. It is completely integrable in the limit as $\Delta \to 0$, where Δ is the lattice size, and has both nonrelativistic and relativistic continuous limits. In the latter case we obtain a quantum field theory model with the asymptotic freedom condition. We devote the next paper to the detailed investigation of the relativistic limit in Hubbard's model.

Now a few words about the contents of the paper. In Sec. 1 we recall the main elements of the quantum method of the inverse problem for the model under consideration and present the algebraic form of Bethe's Ansatz. In Sec. 3 we consider the ferromagnetic case, give the classification of ground state excitations and describe their scattering. In Sec. 4 we give a more complete classification of all eigenvectors of the operator H_N for a finite N and use it to construct the ground state and excitations in the antiferromagnetic case. We also discuss there the spin of spin waves.

In the process of work we consult repeatedly with our colegues: A. G. Izergin, V. E. Korepin, P, P. Kulish, N. Yu. Reshetikhin, and E. K. Sklyanin. We would like to express our deep gratitude to them.

1. Algebraic Form of Bethe's Ansatz

The main components of the quantum method of the inverse problem for the considered model were introduced explicitly in [23, 15]. A great role in their formulation as well as in the general formulation of the quantum method of the inverse problem has been played by the deep work of Baxter [8]. At the present time there exists a sufficiently detailed description of this method (see [13-14]); therefore, we shall not describe it here once again, rather, we imply write out necessary formulas. The derivation of these formulas in a more general situation of the XYZ model can be found in [15].

We consider the local transition matrix — an operator-valued matrix of order 2 × 2

$$L_n(\lambda) = \begin{pmatrix} \lambda I + \frac{i}{2}\sigma_n^3 & \frac{i}{2}\sigma_n^- \\ \frac{i}{2}\sigma_n^+ & \lambda I - \frac{i}{2}\sigma_n^3 \end{pmatrix}, \tag{1.1}$$

where

$$\sigma_n^+ = \sigma_n^1 + i\sigma_n^2, \quad \sigma_n^- = \sigma_n^1 - i\sigma_n^2. \tag{1.2}$$

The space $\mathbb{C}^2$, where the matrix $L_n(\lambda)$ acts, is called auxiliary to distinguish it from the quantum space $\mathfrak{H}_N$ where the matrix elements act. The matrix $L_n(\lambda)$ can also be presented in the form

$$L_n(\lambda) = \lambda I \otimes I + \frac{i}{2} \sum_{a=1}^{3} \sigma^a \otimes \sigma_n^a , \tag{1.3}$$

where the identity matrix I on the left and Pauli matrices σ^a act in the auxiliary space.

The main relation

$$R(\lambda-\mu)(L_n(\lambda) \otimes L_n(\mu)) = (L_n(\mu) \otimes L_n(\lambda))\, R(\lambda-\mu) \tag{1.4}$$

holds, where the tensor product is taken in the auxiliary space. The corresponding 4 × 4-matrix $R(\lambda)$ has the form

$$R(\lambda) = \frac{1}{\lambda+i} \left(\left(\frac{\lambda}{2} + i\right) I \otimes I + \frac{\lambda}{2} \sum_{a=1}^{3} \sigma^a \otimes \sigma^a \right). \tag{1.5}$$

In the natural basis of the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ it has the following form:

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.6}$$

where

$$b(\lambda) = \frac{i}{\lambda+i}, \qquad c(\lambda) = \frac{\lambda}{\lambda+i}. \tag{1.7}$$

The monodromy matrix $\mathcal{T}_N(\lambda)$ is defined by

$$\mathcal{T}_N(\lambda) = L_N(\lambda) \ldots L_1(\lambda) = \overset{\curvearrowleft}{\prod_{n=1}^{N}} L_n(\lambda). \tag{1.8}$$

Similar to the local transition matrix $L_n(\lambda)$ it satisfies the relation

$$R(\lambda-\mu)(\mathcal{T}_N(\lambda) \otimes \mathcal{T}_N(\mu)) = (\mathcal{T}_N(\mu) \otimes \mathcal{T}_N(\lambda))\, R(\lambda-\mu). \tag{1.9}$$

We introduce matrix elements of the monodromy matrix as matrices in the auxiliary space

$$\mathcal{T}_N(\lambda) = \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & \mathcal{D}_N(\lambda) \end{pmatrix}. \tag{1.10}$$

The operators $A_N(\lambda)$, $B_N(\lambda)$, $C_N(\lambda)$, and $D_N(\lambda)$ act in the space $\mathfrak{H}_N$. We set

$$T_N(\lambda) = A_N(\lambda) + \mathcal{D}_N(\lambda). \tag{1.11}$$

From (1.9), in particular, we obtain commutation relations

$$[T_N(\lambda), T_N(\mu)] = 0 , \tag{1.12}$$

$$[B_N(\lambda), B_N(\mu)] = 0 , \tag{1.13}$$

$$A_N(\lambda) B_N(\mu) = \frac{1}{c(\mu-\lambda)} B_N(\mu) A_N(\lambda) - \frac{b(\mu-\lambda)}{c(\mu-\lambda)} B_N(\lambda) A_N(\mu), \tag{1.14}$$

$$\mathcal{D}_N(\lambda) B_N(\mu) = \frac{1}{c(\lambda-\mu)} B_N(\mu) \mathcal{D}_N(\lambda) - \frac{b(\lambda-\mu)}{c(\lambda-\mu)} B_N(\lambda) \mathcal{D}_N(\mu). \tag{1.15}$$

The family of commuting operators $T_N(\lambda)$ contains the momentum operator P_N and the energy operator H_N

$$P_N = \frac{1}{i} \log i^{-N} T_N\left(\frac{i}{2}\right), \tag{1.16}$$

$$H_N = \frac{iJ}{2}\,\frac{d}{d\lambda}\log T_N(\lambda)\Big|_{\lambda=\frac{i}{2}} - \frac{NJ}{2} I_N . \tag{1.17}$$

First, we explain the formula (1.16). Putting $\lambda = i/2$ in (1.3) and using the formula (1.8) we obtain that the operator $i^{-N}T_N(i/2)$ is unitary and coincides with the cyclic shift operator in the space

$$e^{-iP_N}\sigma_n^a e^{iP_N} = \sigma_{n+1}^a ,\ a=1,2,3 ; \quad n=1,\dots,N . \tag{1.18}$$

Its eigenvalues are $e^{i\pi j}$, $0 \leqslant P_j < 2\pi$, $j = 1,\dots,2^N$, it is natural to call P_j the momentum of the corresponding state. Formula (1.17) is obtained by expanding the product (1.8) in the neighborhood of the point $\lambda = i/2$ and taking into account (1.18).

Consider the vectors

$$\omega_n = \binom{1}{0} ,\quad \omega_n \in \mathfrak{h}_n ,\quad n=1,\dots,N ;$$

$$\Omega_N = \prod_{n=1}^{N} \otimes\, \omega_n ,\quad \Omega_N \in \mathfrak{H}_N .$$

The following relations hold:

$$\begin{aligned} A_N(\lambda)\Omega_N &= (\lambda+\tfrac{i}{2})^N \Omega_N , \\ D_N(\lambda)\Omega_N &= (\lambda-\tfrac{i}{2})^N \Omega_N , \end{aligned} \qquad C_N(\lambda)\Omega_N = 0 . \tag{1.19}$$

It follows from the formulas (1.13)-(1.15) and (1.19) that the vector

$$\Psi_N(\lambda_1,\dots,\lambda_l) = B_N(\lambda_1)\dots B_N(\lambda_l)\Omega_N \tag{1.20}$$

is an eigenvector of the family of operators $T_N(\lambda)$ if the numbers $\lambda_1,\dots,\lambda_l$ satisfy the system of equations

$$\left(\frac{\lambda_j - \frac{i}{2}}{\lambda_j + \frac{i}{2}}\right) = \prod_{\substack{k=1\\ k\neq j}}^{l} \frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k + i} ,\quad j=1,\dots,l . \tag{1.21}$$

The corresponding eigenvalue $\Lambda(\lambda;\ \lambda_1,\dots,\lambda_l)$ has the form

$$\Lambda(\lambda;\lambda_1,\dots,\lambda_l) = (\lambda+\tfrac{i}{2})^N \prod_{j=1}^{l} \frac{\lambda-\lambda_j-i}{\lambda-\lambda_j} + (\lambda-\tfrac{i}{2})^N \prod_{j=1}^{l} \frac{\lambda-\lambda_j+i}{\lambda-\lambda_j} . \tag{1.22}$$

We note that since the operators $B_N(\lambda)$ commute, both the vector $\Psi_N(\lambda_1,\dots,\lambda_l)$ and the eigenvalue $\Lambda(\lambda;\ \lambda_1,\dots,\lambda_l)$ are symmetric functions of $\lambda_1,\dots,\lambda_l$.

Comparing the formulas (1.16), (1.17) with (1.22) we see that the eigenvalues of the operators P_N and H_N have the form

$$p(\lambda_1,\dots,\lambda_l) = \frac{1}{i}\sum_{j=1}^{l} \log\frac{\lambda_j+\frac{i}{2}}{\lambda_j-\frac{i}{2}} \pmod{2\pi} \tag{1.23}$$

$$h(\lambda_1,\dots,\lambda_l) = -\frac{J}{2}\sum_{j=1}^{l} \frac{1}{\lambda_j^2+\frac{1}{4}} . \tag{1.24}$$

In addition to λ, it is convenient to introduce the variable $P(\lambda)$

$$e^{ip} = \frac{\lambda+\frac{i}{2}}{\lambda-\frac{i}{2}} ,\qquad p(\lambda) = -2\,\mathrm{arctg}\, 2\lambda + \pi \pmod{2\pi} . \tag{1.25}$$

In the variables p_j the momentum p and the energy h have the form

$$p(\lambda_1,\dots,\lambda_\ell) = \sum_{j=1}^{\ell} p_j \,(\mathrm{mod}\, 2\pi), \tag{1.26}$$

$$h(\lambda_1,\dots,\lambda_\ell) = -J \sum_{j=1}^{\ell} (1-\cos p_j). \tag{1.27}$$

The formulas presented above show that the vector Ω_N plays the role of a vacuum, and the operator $B_N(\lambda)$ has the meaning of the particle birth operator with the momentum $p(\lambda)$ and the energy

$$h(\lambda) = \frac{J}{2}\,\frac{dp(\lambda)}{d\lambda} = -J(1-\cos p(\lambda)). \tag{1.28}$$

Below, the eigenvectors of the form (1.20) are called Bethe's vectors.

We have not yet discussed the question of conditions which guarantee that vectors of that kind do not vanish. We shall consider this matter below. We shall show that a necessary condition for Bethe's vector not to vanish is $l \leqslant N/2$.

We can calculate the normalization of the vector (1.20) using the commutation relation

$$[C_N(\lambda), B_N(\mu)] = \frac{b(\lambda-\mu)}{c(\lambda-\mu)}\left(A_N(\mu)\mathcal{D}_N(\lambda) - A_N(\lambda)\mathcal{D}_N(\mu)\right), \tag{1.29}$$

which follows from (1.9) and the relations (1.19). Nevertheless, the corresponding combinatorial problem has not been solved yet.

With the help of (1.29) we can also prove the simpler assertion that Bethe's vectors with different collections $(\lambda_1,\dots,\lambda_l)$, $(\lambda'_1,\dots,\lambda'_{l'})$ are orthogonal. It would be also desirable to show that Bethe's vectors vanish if among the numbers $\lambda_1,\dots,\lambda_l$ are those that coincide. For the coordinate form of Bethe's Ansatz this fact is well known and we shall use it below. The question about the normalization of Bethe's vectors is also discussed in the work of Gaudin [9] and in the recent paper of Wu, Gaudin, and McCoy [24].

2. Spin of Bethe's Vectors

In addition to P and H, among the observable values of our system, is also the spin vector

$$S_a = \frac{1}{2}\sum_{n=1}^{N} \sigma_n^a, \quad a = 1,2,3. \tag{2.1}$$

Here and further in this paper we omit the index N in the cases when this can not cause misunderstanding. It is clear that operators P and H commute with S_a. In other words, a representation of the group SU(2) acts in the space $\mathfrak{H}_N$ and the corresponding eigenvectors are classified by irreducible representations of this group. We show here that Bethe's vectors are the leading vectors with respect to this action, i.e., the following relation holds:

$$S_+\Psi = 0, \tag{2.2}$$

where we use standard denotations

$$S_+ = \frac{1}{2}\sum_{n=1}^{N}\sigma_n^+, \quad S_- = \frac{1}{2}\sum_{n=1}^{N}\sigma_n^-. \tag{2.3}$$

It follows from (2.2) that in the isotropic case Bethe's Ansatz does not determine all eigenvectors of the energy operator. Together with Bethe's vectors, Ψ vectors of the form $S_-^m\Psi$ are also eigenvectors, where $1 \leqslant m \leqslant 2L$ and L is the spin of the representation to which Ψ belongs.

Moreover, the equality

$$S_3\Psi = \left(\frac{N}{2}-\ell\right)\Psi \tag{2.4}$$

holds, which we shall derive below together with (2.2). From the formula for the square of the spin S^2 with eigenvalues $L(L+1)$, $L \geqslant 0$

$$S^2 = \sum_{a=1}^{3} S_a^2 = S_- S_+ + S_3(S_3+1) \tag{2.5}$$

it follows that for Bethe's vectors

$$L = \frac{N}{2} - \ell . \tag{2.6}$$

From (2.6) the important inequality follows:

$$\ell \leqslant \frac{N}{2}, \tag{2.7}$$

which was already mentioned in Sec. 1. It would be interesting to show that if $l > N/2$, then the system of equations (1.21) does not have solutions in finite $\lambda_1, \ldots, \lambda_l$.

Now we are going to prove formulas (2.2) and (2.4). To accomplish this we obtain permutation relations between operators S_a and $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$. Following the ideas of the quantum method of the inverse problem we start with local relations and calculate the commutator of S_a with matrix elements of the matrix $L_n(\lambda)$. We have

$$[L_n(\lambda), S_a] = \frac{1}{2}[L_n(\lambda), \sigma_n^a]. \tag{2.8}$$

Here [,] means the commutator in the space $\mathfrak{H}_N$; we have also used the fact that among all spin operators only σ_n^a are contained in the matrix $L_n(\lambda)$. Further, from the representation (1.3) we obtain

$$\frac{1}{2}[L_n(\lambda), \sigma_n^a] = \frac{i}{4}\sum_{b=1}^{3}[\sigma^b \otimes \sigma_n^b, \sigma_n^a] = -\frac{1}{4}\sum_{b,c=1}^{3} \varepsilon^{bac}\sigma^b \otimes \sigma_n^c . \tag{2.9}$$

We rewrite the factor $\varepsilon^{bac}\sigma^b$ in the form

$$\varepsilon^{bac}\sigma^b = i[\sigma^c, \sigma^a]. \tag{2.10}$$

As a result we obtain

$$[L_n(\lambda), S_a]_{\text{кв.}} = -\frac{1}{2}[L_n(\lambda), \sigma^a]_{\text{бсп.}}, \tag{2.11}$$

where we emphasize that the commutator on the left-hand side is taken in the quantum space, and the commutator on the right-hand side in the auxiliary space. Note that we managed to replace the quantum commutator (2.8) by the numerical one by using essentially isotropic properties of the model under consideration. Now we consider the monodromy matrix. Using the "differentiation" property of the commutator we obtain

$$[\mathcal{T}(\lambda), S_a]_{\text{кв.}} = \sum_{n=1}^{N} L_n(\lambda)\ldots[L_n(\lambda), S_a]_{\text{кв.}}\ldots L_1(\lambda)$$

$$= -\frac{1}{2}\sum L_N(\lambda)\ldots[L_n(\lambda), \sigma^a]_{\text{бсп.}}\ldots L_1(\lambda) = -\frac{1}{2}[\mathcal{T}(\lambda), \sigma^a], \tag{2.12}$$

i.e., we obtain a formula absolutely analogous to (2.11). The formula (2.12) can be rewritten clearly in the form

$$[\mathcal{T}_{\alpha\beta}(\lambda), S_a] = \frac{1}{2}(\sigma^a_{\alpha\gamma}\mathcal{T}_{\gamma\beta}(\lambda) - \mathcal{T}_{\alpha\gamma}(\lambda)\sigma^a_{\gamma\beta}), \tag{2.13}$$

where $\alpha, \beta = 1, 2$ and $\sigma^a_{\alpha\beta}$ are matrix elements of the Pauli matrix. In particular, formulas (2.13) contain the relations

$$[S_a, T(\lambda)] = 0, \quad a = 1, 2, 3; \tag{2.14}$$

$$[S_3, B(\lambda)] = -B(\lambda), \tag{2.15}$$

$$[S_+,\ B(\lambda)] = A(\lambda) - \mathcal{D}(\lambda). \tag{2.16}$$

Now we are ready to prove formulas (2.2) and (2.4). We note first that by construction the vector Ω is also an eigenvector for the operators S_+, S_3, moreover

$$S_+\Omega = 0\ ,\ \ S_3\Omega = \frac{N}{2}\Omega . \tag{2.17}$$

The relation (2.4) follows immediately from (2.15) and from the second inequality in (2.17) Now we consider (2.2). We have, after carrying S_+ through all operators $B(\lambda_j)$ to the vector Ω,

$$S_+\Psi = \sum_{j=1}^{\ell} B(\lambda_1)\ldots B(\lambda_{j-1})(A(\lambda_j) - \mathcal{D}(\lambda_j))B(\lambda_{j+1})\ldots B(\lambda_\ell)\Omega . \tag{2.18}$$

Using the permutation relations (1.14) and (1.15) we carry the operators $A(\lambda_j)$ and $D(\lambda_j)$ through the $B(\lambda_k)$ to the vector Ω. We obtain the following sum

$$S_+\Psi = \sum_{j=1}^{\ell} M_j(\lambda_1,\ldots,\lambda_\ell)B(\lambda_1)\ldots B(\lambda_{j-1})B(\lambda_{j+1})\ldots B(\lambda_\ell)\Omega . \tag{2.19}$$

To find the coefficients $M_j(\lambda_1,\ldots,\lambda_l)$ we apply the method used in the algebric derivation of equations of the type (1.21) (see [13-15]). Note that to obtain $M_1(\lambda_1,\ldots,\lambda_l)$ we have to carry $A(\lambda_1)-D(\lambda_1)$ through the chain $B(\lambda_2)\ \ldots\ B(\lambda_l)$ to the vector Ω using only first summands on the right-hand sides of the relations (1.14) and (1.15). Therefore,

$$M_1(\lambda_1,\ldots,\lambda_\ell) = (\lambda_1 + \tfrac{i}{2})^N \prod_{j=2}^{\ell} \frac{\lambda_1-\lambda_j-i}{\lambda_1-\lambda_j} - (\lambda_1 - \tfrac{i}{2})^N \prod_{j=2}^{\ell} \frac{\lambda_1-\lambda_j+i}{\lambda_1-\lambda_j} . \tag{2.20}$$

The remaining coefficients $M_j(\lambda_1,\ldots,\lambda_l)$ are obtained from $M_1(\lambda_1,\ldots,\lambda_l)$ by the corresponding permutation of the numbers $\lambda_1,\ldots,\lambda_l$; they have the following form:

$$M_j(\lambda_1,\ldots,\lambda_\ell) = (\lambda_j + \tfrac{i}{2})^N \prod_{\substack{k=1\\k\neq j}}^{\ell} \frac{\lambda_j-\lambda_k-i}{\lambda_j-\lambda_k} - (\lambda_j - \tfrac{i}{2})^N \prod_{\substack{k=1\\k\neq j}}^{\ell} \frac{\lambda_j-\lambda_k+i}{\lambda_j-\lambda_k}\ ,\quad j = 1,\ldots,\ell . \tag{2.21}$$

We now note that the system of equations (1.21) means exactly that $M_j(\lambda_1,\ldots,\lambda_l) = 0$; $j = 1,\ldots,l$. Thus, we have proved the formula (2.2).

At the end of this section we note that the lowering operator S_- participates in the asymptotic formula for the operator $B(\lambda)$ as $\lambda \to \infty$. Indeed, when $\lambda \to \infty$

$$\mathcal{T}(\lambda) = \lambda^N \begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix} + \lambda^{N-1} \sum_{n=1}^{N} \left(L_n(\lambda) - \lambda \begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix} \right) , \tag{2.22}$$

from this we obtain

$$B(\lambda) \sim \frac{i\lambda^{N-1}}{2} \sum_{n=1}^{N} \sigma_n^- = i\lambda^{N-1} S_- . \tag{2.23}$$

Hence the vector of the form

$$S_-^m\Psi = S_-^m B(\lambda_1)\ldots B(\lambda_\ell)\Omega , \tag{2.24}$$

which satisfies the conditions

$$S_3 S_-^m\Psi = \left(\frac{N}{2} - \ell - m\right) S_-^m\Psi \tag{2.25}$$

and

$$S^2 S_-^m\Psi = \left(\frac{N}{2} - \ell\right)\left(\frac{N}{2} - \ell + 1\right) S_-^m\Psi \tag{2.26}$$

can be interpreted as Bethe's vector for which m values of λ in the chain $\lambda_1,\ldots,\lambda_{l+m}$ equal infinity. Note that formally the system of equations (1.21) also allows us to add to any solution with finite λ's the values $\lambda = \infty$. We note that in the variable p values $p \equiv 0 \pmod{2\pi}$ correspond to these λ.

The formulas (2.2) and (2.4) are essentially in Ovchinnikov's paper [11]. Gaudin in [9] derived them using the explicit coordinate form of Bethe's Ansatz.

3. Ground State and Excitations

The Case $J<0$. When $J<0$ the energy operator H is positive. Indeed, from the easily proved relation

$$\left(I\otimes I-\sum_{a=1}^{3}\sigma^a\otimes\sigma^a\right)^2=4\left(I\otimes I-\sum_{a=1}^{3}\sigma^a\otimes\sigma^a\right) \tag{3.1}$$

it follows that H can be represented in the form

$$H=-J\sum_{n=1}^{N}\left(\sigma_n^1\otimes\sigma_{n+1}^1+\sigma_n^2\otimes\sigma_{n+1}^2+\sigma_n^3\otimes\sigma_{n+1}^3-I_N\right)^2. \tag{3.2}$$

The operator H annihilates the vector Ω

$$H\Omega=0; \tag{3.3}$$

therefore, we can consider this vector as the ground state — the ferromagnetic vacuum. The spin of this state is maximal

$$L_\Omega=\frac{N}{2}. \tag{3.4}$$

Together with it we have N more eigenvectors

$$\Omega_l=S_-^l\,\Omega\,,\quad l=1,\ldots,N\,, \tag{3.5}$$

which annihilate H. We show that the other eigenvectors of the operator H have positive eigenvalues.

For the "one-particle" Bethe vector

$$\Psi(\lambda)=B(\lambda)\Omega \tag{3.6}$$

Eq. (1.21) gives the following relation between admissible values λ

$$\left(\frac{\lambda-\frac{i}{2}}{\lambda+\frac{i}{2}}\right)^N=1\,,\quad e^{ipN}=1\,. \tag{3.7}$$

In the limit as $N\to\infty$ admissible p fill the whole interval $[0, 2\pi)$; the corresponding λ run over the whole real axis. The excitations of this type are commonly called magnons.

For the "two-particle" Bethe vector

$$\Psi(\lambda_1,\lambda_2)=B(\lambda_1)B(\lambda_2)\Omega\,. \tag{3.8}$$

The system of equations (1.21) has the following form:

$$\begin{aligned}\left(\frac{\lambda_1-\frac{i}{2}}{\lambda_1+\frac{i}{2}}\right)^N&=\frac{\lambda_1-\lambda_2-i}{\lambda_1-\lambda_2+i}\,,\\ \left(\frac{\lambda_2-\frac{i}{2}}{\lambda_2+\frac{i}{2}}\right)^N&=\frac{\lambda_2-\lambda_1-i}{\lambda_2-\lambda_1+i}\,.\end{aligned} \tag{3.9}$$

In the limit as $N\to\infty$ its real solutions independently run over the whole real axis. Indeed, extracting the N-th root of both sides of the equations (3.9) we obtain that

$$\frac{\lambda_1 - \frac{i}{2}}{\lambda_1 + \frac{i}{2}} = \zeta_1 e^{\frac{i}{N}\varphi(\lambda_1-\lambda_2)} ,$$
$$\frac{\lambda_2 - \frac{i}{2}}{\lambda_2 + \frac{i}{2}} = \zeta_2 e^{-\frac{i}{N}\varphi(\lambda_1-\lambda_2)} , \tag{3.10}$$

where ζ_1 and ζ_2 are two arbitrary N-th roots of 1, and $\varphi(\lambda)$ is the principal value of the argument of the function $(\lambda - i)/(\lambda + i)$, $0 \leqslant \varphi(\lambda) < 2\pi$. In the limit as $N \to \infty$ the arguments of ζ_1 and ζ_2 independently run over the whole segment $[0, 2\pi)$ and $\exp\{\frac{i}{N}\varphi\}$ tends to 1, so Eqs. (3.10) become uncoupled and can be solved trivially. The corresponding Bethe's vector describes the state of scattering of two magnons.

Equations (2.9), however, also have complex solutions. We set

$$\lambda_1 = x_1 + iy, \quad \lambda_2 = x_2 + iy_2. \tag{3.11}$$

Taking the moduli of both sides of the first equation (3.9) we obtain

$$\left(\frac{x_1^2 + (y_1 - \frac{1}{2})^2}{x_1^2 + (y_1 + \frac{1}{2})^2}\right)^N = \frac{(x_1 - x_2)^2 + (y_1 - y_2 - 1)^2}{(x_1 - x_2)^2 + (y_1 - y_2 + 1)^2}. \tag{3.12}$$

Assuming $y_1 > 0$ we obtain that the left-hand side of (3.12) decreases exponentially as $N \to \infty$. Thus, with the exponential accuracy the relations

$$x_1 = x_2, \quad y_1 - y_2 = 1 \tag{3.13}$$

hold. Considering the absolute value of the second equation in (3.9) we see that $y_2 < 0$, so, we do not lose generality assuming that y_1 is positive. Further, multiplying the first equation in (3.9) by the second we obtain

$$\left(\frac{x_1 + i(y_1 - \frac{1}{2})}{x_1 + i(y_1 + \frac{1}{2})} \cdot \frac{x_2 + i(y_2 - \frac{1}{2})}{x_2 + i(y_2 + \frac{1}{2})}\right)^N = 1. \tag{3.14}$$

Substituting here (3.13) we get the relation

$$\left(\frac{x_1 + i(y_1 - \frac{3}{2})}{x_1 + i(y_1 + \frac{1}{2})}\right)^N = 1, \tag{3.15}$$

from this we conclude that $y_1 = 1/2$.

Therefore, asymptotically as $N \to \infty$ the solution of (3.11) is characterized by one real number x and has the form

$$\lambda_1 = x + \frac{i}{2}, \quad \lambda_2 = x - \frac{i}{2}, \tag{3.16}$$

where x parametrizes the total momentum of the state

$$p(\lambda_1, \lambda_2) = p(\lambda_1) + p(\lambda_2) = \frac{1}{i}\log\frac{x+i}{x-i} = p\left(\frac{x}{2}\right) = p\left(\frac{\lambda_1+\lambda_2}{4}\right). \tag{3.17}$$

For the corresponding energy we obtain

$$h(\lambda_1, \lambda_2) = h(\lambda_1) + h(\lambda_2) = \frac{J}{2}\left(\frac{dp(\lambda_1)}{d\lambda_1} + \frac{dp(\lambda_2)}{d\lambda_2}\right) = \frac{J}{2}\left(\frac{\partial}{\partial\lambda_1} + \frac{\partial}{\partial\lambda_2}\right)(p(\lambda_1) + p(\lambda_2))$$
$$= \frac{J}{2}\left(\frac{\partial}{\partial\lambda_1} + \frac{\partial}{\partial\lambda_2}\right)p\left(\frac{\lambda_1+\lambda_2}{4}\right) = \frac{J}{4}\left.\frac{dp}{d\lambda}\right|_{\lambda = \frac{\lambda_1+\lambda_2}{4}} , \tag{3.18}$$

or

$$h(\lambda_1,\lambda_2) = -\frac{J}{2}(1-\cos p(\lambda_1,\lambda_2)). \tag{3.19}$$

To shorten the calculations presented above we have taken some liberties in treating λ_1 and λ_2 as independent variables. The energy $h(\lambda_1, \lambda_2)$ of the constructed state is positive and less than the energy of two magnons with the momenta p_1 and p_2 which are bound by the relation $p_1 + p_2 \equiv p(\lambda_1, \lambda_2) \pmod{2\pi}$. Therefore, it is natural to call this state the bound state of two magnons.

Bethe's vector

$$\Psi(\lambda_1,\ldots,\lambda_l) = B(\lambda_1)\ldots B(\lambda_l)\Omega \tag{3.20}$$

for any arbitrary l can be investigated similarly. To the real λ's there corresponds a state of l independent magnons. The complex λ's gather into "strings" of length $2M + 1$ — collections of $2M + 1$ numbers of the form

$$\lambda_m = x + im, \quad m = -M, -M+1, \ldots, M-1, M; \tag{3.21}$$

where M is a positive integer or half-integer number. Magnons themselves can be considered as strings of length 1.

The proof of the fact that complex λ's have the form (3.21) when $N \to \infty$ is similar to the case of the string of length 2 which was considered above. Namely, together with the number $\lambda = x + iy$ with $y > 0$, the solution $\lambda_1,\ldots,\lambda_l$ of the system (1.21) also contains $\lambda = x + i(y - 1)$ [cf. (3.13)]. Further, the fact that the total momentum is real [cf. (3.15)] shows that such λ's group in chains symmetric with respect to the real axis — the strings of the form (3.21).

Thus, in the general Bethe vector (3.20) l numbers $\lambda_1,\ldots,\lambda_l$ gather into strings of different length. We denote by ν_M the number of strings of length $2M + 1$, $M = 0, 1/2,\ldots$ and by $\lambda_{j,M}$, $j = 1,\ldots,\nu_M$ the real parts of the parameters λ which belong to the j-th string. We denote the total number of strings by q. We have

$$q = \sum_M \nu_M, \qquad l = \sum (2M+1)\nu_M. \tag{3.22}$$

The collection of integers $(l, q, \{\nu_M\})$ constrained by the relation (3.22) characterizes Bethe's vector (3.20) up to the determination of the q numbers $\lambda_{j,M}$. We call this collection the configuration. The energy and the momentum of Bethe's vector, which corresponds to the given configuration within exponential accuracy as $N \to \infty$ consist of q summands which represent the energy and the momentum of separate strings. The energy and the momentum of the string of length $2M + 1$ will be calculated later.

For the parameters $\lambda_{j,M}$ of the given configuration the system of equations holds which is obtained from (1.21) in the following way. For a chosen string of length $2M + 1$ we find the product of those equations from (1.21) which contain the parameters λ_j belonging to this string. On the right-hand side we find the product over k of the factors $(\lambda_j - \lambda_k - i)/(\lambda_j - \lambda_k + i)$ according to the splitting of the variables λ into strings in the given configuration. To present the equations obtained it is convenient to introduce the denotation

$$V_0(\lambda) = \frac{\lambda - i}{\lambda + i}. \tag{3.23}$$

We have the equalities

$$\prod_{m=-M}^{M} V_0(2(\lambda + im)) = V_0\left(\frac{\lambda}{M+\frac{1}{2}}\right), \tag{3.24}$$

$$\prod_{m=-M}^{M} V_0(\lambda + im) = V_0\left(\frac{\lambda}{M}\right) V_0\left(\frac{\lambda}{M+1}\right) = V_M(\lambda), \tag{3.25}$$

$$\prod_{m_1=-M_1}^{M_1}\prod_{m_2=-M_2}^{M_2} V_0(\lambda+i(m_1+m_2))=\prod_{L=|M_1-M_2|}^{M_1+M_2} V_L(\lambda)=V_{M_1,M_2}(\lambda). \tag{3.26}$$

The required system of equations has the form

$$V_0^N\left(\frac{\lambda_{j,M_1}}{M+\frac{1}{2}}\right)=\prod_{M_2}\prod_{\substack{\kappa=1\\(\kappa,M_2)\neq(j,M_1)}}^{\nu_{M_2}} V_{M_1,M_2}(\lambda_{j,M_1}-\lambda_{\kappa,M_2}). \tag{3.27}$$

These equations were introduced and used by Takahashi [10] and Gaudin [9]. Their distinctive feature is that they contain only real parameters $\lambda_{j,M}$. The unused equations from the system (1.21) for the complex λ_j are satisfied with exponential accuracy as $N \to \infty$. Here we have only outlined the proof of this assertion. The detailed proof can be found in [9].

We calculate the momentum and the energy of the separate string of length $2M + 1$. For this we use the formula (3.24) which implies that

$$p(\lambda_{-M},\ldots,\lambda_M)=\frac{1}{i}\log\frac{x+i(M+\frac{1}{2})}{x-i(M+\frac{1}{2})}=p\left(\frac{x}{2M+1}\right)=-2\operatorname{arctg}\frac{x}{M+\frac{1}{2}}+\pi \pmod{2\pi} \tag{3.28}$$

and

$$h(\lambda_{-M},\ldots,\lambda_M)=-J\,\frac{M+\frac{1}{2}}{x^2+(M+\frac{1}{2})^2}=-\frac{J}{2M+1}(1-\cos p(\lambda_{-M},\ldots,\lambda_M)). \tag{3.29}$$

Thus, as $N \to \infty$ Bethe's vectors can be interpreted in terms of many-particle states of scattering of magnons and also in terms of their bound states — the strings of length greater than 1. The one-particle state Ψ_m is characterized by the momentum p and the energy

$$h_M(p)=-\frac{J}{2M+1}(1-\cos p),\quad M=0,\frac{1}{2},\cdots. \tag{3.30}$$

With N growing these states become more and more degenerate; the degree of their degeneration equals $N - 4M - 1$. We can omit this degeneration if in the passage to the limit as $N \to \infty$ we consider states which belong to the incomplete tensor product, in the sense of von Neumann [25], of the spaces η_n which adjoins the state Ω. This limit space $\mathfrak{H}_F$ is isomorphic to the incomplete tensor product of the Fock spaces for the infinite number of excitations Ψ_M. Among all actions of the group SU(2) in the space $\mathfrak{H}_N$ only the action of the operator $S_3 - (N/2)I_N$ survives in the space $\mathfrak{H}_F$.

This limit situation can be naturally described in the formalism of the quantum method of the inverse problem if we consider the limit of the monodromy matrix $\mathcal{T}_N(\lambda)$ as $N \to \infty$ regularized with respect to state Ω. The corresponding procedure was explicitly presented by Kulish and Sklyanin in [23] and is analogous to the procedure described in [13, 26] for the case of bosons with δ-type interaction (the quantum nonlinear Schrödinger equation). It is natural, for this purpose to renumber the lattice nodes assuming that n changes asymptotically with respect to 0; e.g., for N odd $-(N-1)/2 \leqslant n \leqslant (N-1)/2$. In the space $\mathfrak{H}_F$ there exists the limit

$$\mathcal{T}(\lambda)=\lim_{N\to\infty}\begin{pmatrix}\lambda+\frac{i}{2} & 0\\ 0 & \lambda-\frac{i}{2}\end{pmatrix}^{-\frac{N-1}{2}}\mathcal{T}_N(\lambda)\begin{pmatrix}\lambda+\frac{i}{2} & 0\\ 0 & \lambda-\frac{i}{2}\end{pmatrix}^{-\frac{N-1}{2}}=\begin{pmatrix}A(\lambda) & B(\lambda)\\ C(\lambda) & D(\lambda)\end{pmatrix}. \tag{3.31}$$

For limit operators the following relations hold:

$$[B(\lambda),B(\mu)]=0,\quad A(\lambda)\Omega=\Omega, \tag{3.32}$$

$$A(\lambda)B(\mu)=\frac{1}{c(\mu-\lambda)}B(\mu)A(\lambda).$$

The relation (3.32) can be taken as the base for the definition of the excitations spectrum described above in the same way as it was done in [27].

The formula (3.31) contains the diagonal matrix which is obtained from the operator $L_n(\lambda)$ after formal substitution

$$\sigma_n^1 = 0, \quad \sigma_n^2 = 0, \quad \sigma_n^3 = I. \tag{3.33}$$

This substitution symbolizes the boundary conditions imposed on the spin operators in the space

$$\sigma_n^1 \longrightarrow 0, \quad \sigma_n^2 \longrightarrow 0, \quad \sigma_n^3 \longrightarrow I \quad \text{for } |n| \longrightarrow \infty. \tag{3.34}$$

The state space $\mathfrak{H}_F$ in the coordinate realization is the closure of the linear span of vectors of the form

$$\sum_{|n_1|\leq m_1} \cdots \sum_{|n_k|\leq m_k} f_{n_1,\ldots,n_k} \sigma_{n_1}^- \cdots \sigma_{n_k}^- \Omega, \tag{3.35}$$

where $k, m_1, \ldots, m_k < \infty$. The operator σ_n^+ acts in this space and the vectors (3.35) annihilate this operator as $|n| \to \infty$. On these vectors in the limit for large $|n|$ the operator σ_n^3 becomes an identity operator. As to the operator σ_n^-, for $|n| \to \infty$, its image lies outside of the space $\mathfrak{H}_F$ and so it vanishes in the weak sense. The rigorous investigation of the operator H in the Hilbert space $\mathfrak{H}_F$ is presented in a number of works by Thomas and Babbitt [28-30]. In [28-29] a proof for the expansion theorem is given in terms of the eigenvectors of the operator H in the space $\mathfrak{H}_F$; moreover, it is shown that the limits of Bethe's vectors as $N \to \infty$ form a complete system. In [30] the scattering theory is constructed for these states and the eigenvalues of the S-matrix are calculated. In particular, it is shown in [30] that the configuration does not change during the scattering and the whole effect of the scattering is reduced to the multiplication by a phase factor.

In the framework of the quantum method of the inverse problem the calculation of the S-matrix appears to be the simplest. By analogy with the case of the quantum nonlinear Schrödinger equation considered in [13] we introduce Zamolodchikov's operators

$$Z_M(\lambda) = \prod_{m=-M}^{M} B(\lambda + im) A^{-1}(\lambda). \tag{3.36}$$

As follows from (3.32) these operators satisfy the permutation relations

$$Z_M(\lambda)\, Z_{M'}(\mu) = Z_{M'}(\mu)\, Z_M(\lambda)\, S_{MM'}(\mu - \lambda), \tag{3.37}$$

where

$$S_{M,M'}(\lambda) = V_{M,M'}(\lambda) \tag{3.38}$$

[see (3.26)].

For the given configuration in and out — states are described by the general formula (see [31])

$$\prod_M \prod_{j=1}^{\nu_M} Z_M(\lambda_{j,M})\, \Omega, \tag{3.39}$$

where for the in-state all parameters $\lambda_{j,M}$ are ordered from right to left in increasing order and for the out-state they are ordered in decreasing order. The process of the multiparticle scattering is reduced to a sequence of two-particle ones and the complete phase factor is the product of $q(q-1)/2$ two-particle factors $S_{M,M'}(|\lambda_{j,M} - \lambda_{j',M'}|)$. The structure of the poles of the eigenvalues of the S-matrix agrees with the interpretation of the strings of length $2M + 1$, $M > 0$ as the bound states of $2M + 1$ elementary magnons.

4. Ground State and Excitations

The Case $\mathfrak{J}>0$. When $\mathfrak{J}>0$ the energy operator is negative and the eigenvector with the eigenvalue of minimal modulus is the ground state — the antiferromagnetic vacuum. It is clear from the picture of excitations described in Sec. 3 that for a given N this vector corresponds to the largest possible number of strings of length 1; $l = [N/2]$. In this section we assume that N is even; in this case the ground state is not degenerate, its spin equals zero and the spin of all states is even. To characterize the ground state excitations we have to carry out more detailed investigations of the system of equations (3.27). Therefore, at the beginning of this section we shall consider the question concerning the parametrization of all its solutions for large but finite N. For this following Bethe and all subsequent authors, we pass to the logarithms in this system. We note that (3.27) contains only the factors of the form

$$V_0\left(\frac{\lambda}{a}\right) = \frac{\lambda - ia}{\lambda + ia}, \tag{4.1}$$

where λ and a are real. We define the branch of $\log V_0(\lambda)$ making cuts in the complex plane along the imaginary axis from i to $i\infty$ and from $-i\infty$ to i. In this case

$$\frac{1}{i}\log V_0(\lambda) = 2\,\mathrm{arctg}\,\lambda + \pi, \tag{4.2}$$

where $\arctan\lambda$ is the main branch of the arctangent, $-\pi/2 < \arctan\lambda < \pi/2$ for λ real.

Taking the logarithm of both sides of the equations (3.27) we obtain the system of equations

$$2N\,\mathrm{arctg}\,\frac{\lambda_{j,M_1}}{M_1+\frac{1}{2}} = 2\pi Q_{j,M_1} - \sum_{M_2}\sum_{k=1}^{\nu_{M_2}} \Phi_{M_1,M_2}(\lambda_{j,M_1}-\lambda_{k,M_2}), \tag{4.3}$$

where

$$\Phi_{M_1,M_2}(\lambda) = 2\sum_{L=|M_1-M_2|}^{M_1+M_2\ \prime}\left(\mathrm{arctg}\,\frac{\lambda}{L} + \mathrm{arctg}\,\frac{\lambda}{L+1}\right) \tag{4.4}$$

the prime here means that in the case $M_1 = M_2$ we omit the summand $\arctan\lambda/L$ with $L = 0$. Each of the numbers $Q_{j,M}$ is a sum of some integer, which characterizes the full increment of the argument of the right-hand side of (3.27), of the number N/2, which arises from the replacement of $(1/i)\log V_0(\lambda)$ by $2\arctan\lambda$, and of the analogous summand in the right-hand side of (4.3).

Integer and half-integer numbers $Q_{j,M}$ parametrize the possible solutions of the system of equations (4.3). While investigating this system we assume that for M given the numbers $Q_{j,M}$ do not coincide. According to the remark at the end of Sec. 1 this leads to the vanishing of the corresponding Bethe's vector. We also note that from the system (4.3) it follows easily that if for a given collection of the numbers $Q_{j,M}$, which are different for every M, there exists a solution $\{\lambda_{j,M}\}$, then for every M the numbers $\lambda_{j,M}$ do not coincide. Moreover, the corresponding Bethe's vector does not vanish.

We define the possible values of the numbers $Q_{j,M}$ for a given configuration $(l, q, \{\nu_M\})$. Since $\arctan\lambda$ is odd admissible values of $Q_{j,M}$ are located symmetrically with respect to 0

$$-Q_M^{max} \leq Q_{1,M} < Q_{2,M} < \ldots < Q_{\nu_M,M} \leq Q_M^{max} \tag{4.5}$$

and are integers of half-integers depending on Q_M^{max}. We assume that for every M the numbers $\lambda_{j,M}$ are ordered in such a way that they increase from 1 to ν_M when j is growing. The largest admissible value Q_M^{max} is determined from the following principle: for $Q_{j,M} = Q_M^{max} + 2M + 1$ the corresponding solution $\lambda_{j,M}$ equals ∞. We recall that $\lambda_{j,M}$ parametrizes a string of length $2M + 1$, so to "push" this string to ∞ it is natural to assume that the maximal admissible value Q_M^{max} must be exceeded by $2M + 1$. We note now that

$$\Phi_{M_1,M_2}(\infty) = -\Phi_{M_1,M_2}(-\infty) = 2\pi\mathfrak{J}(M_1,M_2), \tag{4.6}$$

where

$$\mathfrak{J}(M_1,M_2)=\begin{cases} 2\min(M_1,M_2)+1 & \text{for } M_1\neq M_2, \\ 2M_1+\frac{1}{2} & \text{for } M_1=M_2. \end{cases} \tag{4.7}$$

From Eqs. (4.3) and (4.6) we obtain the following value for Q_M^{max}

$$Q_M^{max} = \frac{N}{2} - \sum_{M'} \mathfrak{J}(M,M')\nu_{M'} - \frac{1}{2}. \tag{4.8}$$

The restrictions for integer and half-integer numbers, which participate in equations of the type (4.3) were first obtained by Bethe himself in [2]. He used the different branch of the function $\log V_0(\lambda)$; however, we can recalculate his restriction using our terms and obtain the formula (4.8). A different derivation of (4.8) is presented in Gaudin's work [8].

We call admissible values for the numbers $Q_{j,M}$ the vacancies. We denote number of vacancies for every M by P_M. We have

$$P_M = 2Q_M^{max}+1 = N-2\sum_{M'} \mathfrak{J}(M,M')\nu_{M'}. \tag{4.9}$$

Now we are going to describe the parametrization of solutions of the system (3.27) and together with them of all Bethe's vectors. We formulate the main hypothesis: to every admissible collection $Q_{j,M}$ there corresponds a unique solution of the system (4.3). This hypothesis explicitly or implicitly was accepted by all specialists beginning with Bethe. Apparently for its justification we should use an extension to arbitrary admissible configurations of the Yang and Yang variational principle [6], which they formulated for the vacuum configuration $l = q = \nu_0 = N/2$, $\nu_M = 0$ for $M > 0$. Thus, we assume that the numbers $Q_{j,M}$ play the role of the complete collection of parameters which classify Bethe's vectors.

We show that the hypothesis just formulated is consistent with the number of states in $\mathfrak{H}_N$, $\dim \mathfrak{H}_N = 2^N$. To a given configuration (l, q, $\{\nu_M\}$) there corresponds the number of states $Z(N|\{\nu_M\})$ determined in the following way:

$$Z(N|\{\nu_M\}) = \prod_M C_{P_M}^{\nu_M}. \tag{4.10}$$

Our nearest goal is to calculate the number of states for all configurations with fixed l

$$Z(N,l) = \sum_{\sum_M (2M+1)\nu_M = l} Z(N|\{\nu_M\}). \tag{4.11}$$

To calculate $Z(N, l)$, we, following Bethe [2], use an important property of the numbers P_M, which follows directly from (4.9),

$$P_M(N|\{\nu_M\}) = P_{M-\frac{1}{2}}(N-2q|\{\nu'_M\}), \quad M>0, \tag{4.12}$$

where $\nu'_M = \nu_{M+1/2}$ and, or course,

$$q = \sum_M \nu_M. \tag{4.13}$$

We introduce the partial number of states

$$Z(N,l,q) = \sum_{\substack{\sum (2M+1)\nu_M = l \\ \sum \nu_M = q}} Z(N|\{\nu_M\}). \tag{4.14}$$

Thus,

$$Z(N,l) = \sum_{q=0}^{l} Z(N,l,q). \tag{4.15}$$

We obtain from (4.12) that

$$Z(N|\{\nu_M\}) = C_{N-2q+\nu_0}^{\nu_0} Z(N-2q|\{\nu'_M\}); \tag{4.16}$$

from this we obtain the recurrence relation for $Z(N, l, q)$

$$Z(N,l,q) = \sum_{\nu=0}^{q-1} C_{N-2q+\nu}^{\nu} Z(N-2q, l-q, q-\nu) \tag{4.17}$$

with the initial condition

$$Z(N,1,1) = N-1. \tag{4.18}$$

The solution of this relation with the given initial condition was found by Bethe [2] and has the form

$$Z(N,l,q) = \frac{N-2l+1}{N-l+1} C_{N-l+1}^{q} C_{l-1}^{q-1}. \tag{4.19}$$

The summation in the formula (4.15), taking into account (4.19), can be performed trivially and leads to the result

$$Z(N,l) = \frac{N-2l+1}{N-l+1} C_N^l = C_N^l - C_N^{l-1}. \tag{4.20}$$

This is exactly the formula that was obtained by Bethe.

We now recall that every Bethe's vector with a fixed $l \leqslant N/2$ is the leading vector in the multiplet of dimension $N - 2l + 1$. Thus, the full number of states Z generated by the admissible numbers $Q_{j,M}$ equals

$$Z = \sum_{l=0}^{\frac{N}{2}} (N-2l+1)(C_N^l - C_N^{l-1}). \tag{4.21}$$

By integration by parts the sum (4.21) can easily be transformed to the form

$$Z = C_N^{\frac{N}{2}} + 2\sum_{l=0}^{\frac{N}{2}-1} C_N^l = 2^N. \tag{4.22}$$

Thus, we have shown that the main hypothesis about the parameterization of Bethe's vectors yields a complete system of the eigenvectors of the operator H. We can now consider the main problem of this section — the classification of states with energy close, to the energy of the ground state as $N \to \infty$.

The ground state, as was already remarked, corresponds to the configuration

$$l = q = \nu_0 = \frac{N}{2}; \quad \nu_M = 0, \ M>0. \tag{4.23}$$

In this configuration the number of vacancies for strings of length 1 also equals N/2. Thus, the corresponding numbers $Q_{j,0}$ fill in all vacancies and belong to the segment

$$-\frac{N}{4} + \frac{1}{2} \leqslant Q_{j,0} \leqslant \frac{N}{4} - \frac{1}{2}; \tag{4.24}$$

this proves once again that this state is not degenerate.

Consider now examples of the simplest excitations:

1. $$l = \frac{N}{2}, \ q = \frac{N}{2} - 1, \quad \nu_0 = \frac{N}{2} - 2, \quad \nu_{\frac{1}{2}} = 1; \nu_M = 0, M > \frac{1}{2}. \tag{4.25}$$

The spin of this state also equals 0. For the strings of length 1 the number of vacancies in this configuration equals N/2; for the string of length 2 there is one vacancy, the only admissible $Q_{j,1}$ equals 0. Thus the configuration described is determined by two parameters:

the positions of two unfilled vacancies — "holes" — of the strings of length 1 which vary independently in the interval (4.24).

2.
$$l = q = \nu_0 = \frac{N}{2} - 1; \quad \nu_M = 0,\ M > 0. \tag{4.26}$$

The spin of this state equals 1, so it is the leading vector of the triplet. The number of vacancies for the strings of length 1 equals N/2 + 1. Thus, this configuration also corresponds to the two-parameter family of Bethe's vectors and is paramerized by two holes.

Examples 1 and 2 belong to the configuration class $\mathfrak{M}_{AF}$, which can be characterized in the following way: the number of strings of length 1 in each configuration from $\mathfrak{M}_{AF}$ differs by a finite quantity from N/2; therefore, the number of strings of length greater than 1 is finite. If $\nu_0 = N/2 - k_0$, where k_0 is positive and finite, then (4.9) implies

$$P_0 = \frac{N}{2} + k_0 - 2\sum_{M>0} \nu_M, \tag{4.27}$$

$$P_M = 2k_0 - 2\sum_{M'>0} J(M,M')\nu_{M'},\ M > 0; \tag{4.28}$$

hence, we have

$$P_0 \geq \frac{N}{2}, \quad P_M < 2k_0,\ M > 0. \tag{4.29}$$

Moreover, it follows from (4.27) that the number of holes for the strings of length 1 is always even and equals 2 only for Examples 1 and 2. It is useful to imagine the class $\mathfrak{M}_{AF}$ as "the sea" of strings of length 1 with a finite number of strings of length greater than 1 immersed into it. We shall prove below that the class $\mathfrak{M}_{AF}$ can be characterized as the class of such configurations for which the corresponding states have, when $N \to \infty$, finite ener energy and momentum with respect to the antiferromagnetic vacuum.

Following Hulthén and des Cloiseaux—Pearson we can give the complete characterization of the states described above for $N \to \infty$. The fact is that for $N \to \infty$ the numbers $\lambda_{j,0}$ are distributed uniformly over the whole real axis with some density. This density satisfies a linear integral equation which easily can be solved explicitly. With the help of this density we calculate the main observables for a given state.

We consider this situation in greater detail. We begin with the case of the ground state. Equation (4.3) has the form

$$\operatorname{arctg} 2\lambda_j = \frac{\pi Q_j}{N} + \frac{1}{N}\sum_{k=1}^{N/2} \operatorname{arctg}(\lambda_j - \lambda_k). \tag{4.30}$$

Here and below we omit the subscript 0 of the numbers $\lambda_{j,0}$ and $Q_{j,0}$; the numbers Q_j increase monotonically with j in the segment [—N/4 + 1/2, N/4 — 1/2]. For $N \to \infty$ we have

$$\frac{Q_j}{N} \to x, \quad -\frac{1}{4} \leq x \leq \frac{1}{4}, \quad \lambda_j \to \lambda(x), \tag{4.31}$$

where $\lambda(x)$ is a monotone function, moreover, $\lambda(-1/4) = -\infty$ and $\lambda(1/4) = \infty$. A rigorous proof of this can be found in the Yang and Yang work [6]. Replacing the sum by the integral in (4.30) we obtain a limit equation

$$\operatorname{arctg} \lambda(x) = \pi x + \int_{-1/4}^{1/4} \operatorname{arctg}(\lambda(x) - \lambda(y))\, dy. \tag{4.32}$$

More detailed investigation with the application of Sonin's formula for the replacement of sums by integrals shows that Eq. (4.32) approximates (4.30) with an error which is $O(1/N^2)$. We introduce the function $x(\lambda)$ inverse to $\lambda(x)$. The function

$$\rho(\lambda) = \frac{1}{\left.\frac{d\lambda(x)}{dx}\right|_{x=x(\lambda)}} \tag{4.33}$$

plays the role of the density of the numbers λ_j in the interval $d\lambda$. Differentiating (4.32) we obtain for $\rho(\lambda)$ Hulthén's integral equation

$$\pi\rho(\lambda) + \int_{-\infty}^{\infty} \frac{\rho(\mu)}{1+(\lambda-\mu)^2}\,d\mu = \frac{2}{1+4\lambda^2}, \tag{4.34}$$

which can be solved by the Fourier transform. Note that

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda\xi}}{\lambda^2+1}\,d\lambda = \pi e^{-|\xi|}; \tag{4.35}$$

hence,

$$\rho(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}|\xi|}}{1+e^{-|\xi|}}\, e^{-i\lambda\xi}\,d\xi = \frac{1}{2\operatorname{ch}\pi\lambda}. \tag{4.36}$$

The corresponding energy and momentum have the form

$$E = \sum_{j=1}^{\frac{N}{2}} h(\lambda_j) = N\int_{-\infty}^{\infty} h(\lambda)\rho(\lambda)\,d\lambda = -\frac{JN}{2}\int_{-\infty}^{\infty} \frac{e^{-|\xi|}}{1+e^{-|\xi|}}\,d\xi = -JN\log 2, \tag{4.37}$$

$$P = \sum_{j=1}^{\frac{N}{2}} p(\lambda_j) = N\int_{-\infty}^{\infty} p(\lambda)\rho(\lambda)\,d\lambda = \frac{N\pi}{2} (\mathrm{mod}\ 2\pi). \tag{4.38}$$

Moreover,

$$S = \frac{N}{2} - \sum_{j=1}^{\frac{N}{2}} 1 = \frac{N}{2} - N\int_{-\infty}^{\infty} \rho(\lambda)\,d\lambda = 0, \tag{4.39}$$

which, of course, was obvious beforehand.

We consider now the excitations from Examples 1 and 2. We start with the simpler Example 2. Equation (4.3) has the form similar to (4.30)

$$\operatorname{arctg} 2\lambda_j = \frac{\pi Q_j}{N} + \frac{1}{N}\sum_{\kappa=1}^{\frac{N}{2}-1} \operatorname{arctg}(\lambda_j - \lambda_\kappa), \tag{4.40}$$

but now the numbers Q_j lie in the segment $[-N/4, N/4]$ and have two blanks — holes. We denote the blanks by $Q_1^{(h)}$ and $Q_2^{(h)}$, $Q_1^{(h)} < Q_2^{(h)}$. For

$$\frac{Q_1^{(h)}}{N} \to x_1, \quad \frac{Q_2^{(h)}}{N} \to x_2, \quad \frac{Q_j}{N} \to x + \frac{1}{N}(\theta(x-x_1)+\theta(x-x_2)), \tag{4.41}$$

where $\theta(x)$ is the Heaviside function, $\theta(x) = 0$ for $x \leqslant 0$, $\theta(x) = 1$ for $x > 0$. The limit equation

$$\operatorname{arctg} 2\lambda(x) = \pi x + \frac{\pi}{N}(\theta(x-x_1)+\theta(x-x_2)) + \int_{-\frac{1}{4}}^{\frac{1}{4}} \operatorname{arctg}(\lambda(x)-\lambda(y))\,dy \tag{4.42}$$

is again satisfied with an error which is $O(1/N^2)$. For the function $\rho_t(\lambda)$, determined by the formula similar to (4.33), we obtain the linear integral equation

$$\pi\rho_t(\lambda)+\int_{-\infty}^{\infty}\frac{\rho_t(\mu)}{1+(\lambda-\mu)^2}\,d\mu = \frac{2}{1+4\lambda^2}-\frac{\pi}{N}(\delta(\lambda-\lambda_1)+\delta(\lambda-\lambda_2)), \tag{4.43}$$

where λ_i are the parameters of holes, $\lambda_i = \lambda(x_i)$, $i = 1, 2$. It is obvious that $\rho_t(\lambda)$ has the form

$$\rho_t(\lambda)=\rho(\lambda)+\frac{1}{N}(\sigma(\lambda-\lambda_1)+\sigma(\lambda-\lambda_2)), \tag{4.44}$$

where the function $\sigma(\lambda)$ satisfies the des Cloizeaux—Pearson integral equation

$$\pi\sigma(\lambda)+\int_{-\infty}^{\infty}\frac{\sigma(\mu)}{1+(\lambda-\mu)}\,d\mu=-\pi\delta(\lambda). \tag{4.45}$$

The solution has the form

$$\sigma(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\sigma}(\xi)e^{-i\lambda\xi}\,d\xi, \quad \hat{\sigma}(\xi)=-\frac{1}{1+e^{|\xi|}}. \tag{4.46}$$

The energy and the momentum of this state measured from the ground state have the form

$$\varepsilon_t(\lambda_1,\lambda_2)=N\int_{-\infty}^{\infty}h(\lambda)(\rho_t(\lambda)-\rho(\lambda))\,d\lambda = \varepsilon(\lambda_1)+\varepsilon(\lambda_2), \tag{4.47}$$

$$\kappa_t(\lambda_1,\lambda_2)=N\int_{-\infty}^{\infty}p(\lambda)(\rho_t(\lambda)-\rho(\lambda))\,d\lambda = \kappa(\lambda_1)+\kappa(\lambda_2)(\mathrm{mod}\ 2\pi), \tag{4.48}$$

where

$$\varepsilon(\lambda)=\int_{-\infty}^{\infty}h(\mu)\sigma(\lambda-\mu)\,d\mu=J\frac{\pi}{2\,\mathrm{ch}\,\pi\lambda}, \tag{4.49}$$

$$\kappa(\lambda)=\int_{-\infty}^{\infty}p(\mu)\sigma(\lambda-\mu)\,d\mu=\mathrm{arctg}\,\mathrm{sh}\,\pi\lambda-\frac{\pi}{2},\ -\pi\le\kappa(\lambda)\le 0; \tag{4.50}$$

moreover,

$$\varepsilon=-\frac{J\pi}{2}\sin\kappa. \tag{4.51}$$

We see that the momentum $\kappa_t(\lambda_1, \lambda_2)$ varies over the interval $[0, 2\pi)$, i.e., over the full zone, when λ_1 and λ_2 independently run over the whole real axis. Moreover, the total spin of the state can be calculated by the formula

$$S=-\int_{-\infty}^{\infty}(\sigma(\lambda-\lambda_1)+\sigma(\lambda-\lambda_2))\,d\lambda=1. \tag{4.52}$$

We now consider Example 2. Denote by λ_s the only number among $\lambda_{j,1/2}$ which characterizes the string of length 2; we will denote by λ_j the numbers $\lambda_{j,0}$ for strings of length 1. Equations (4.3) are split up into two equations

$$\mathrm{arctg}\,2\lambda_j=\frac{\pi Q_j}{N}+\frac{1}{N}\varphi(\lambda_j-\lambda_s)+\frac{1}{N}\sum_{\kappa=1}^{\frac{N}{2}-2}\mathrm{arctg}\,(\lambda_j-\lambda_\kappa), \tag{4.53}$$

$$\mathrm{arctg}\,\lambda_s=\frac{1}{N}\sum_{j=1}^{\frac{N}{2}-2}\varphi(\lambda_s-\lambda_j), \tag{4.54}$$

where

$$\Phi(\lambda) = \operatorname{arctg} 2\lambda + \operatorname{arctg} \frac{2}{3}\lambda. \tag{4.55}$$

Moreover, $N/2 - 2$ numbers Q_j vary in the segment $-N/4 + 1/2 \leqslant Q_j \leqslant N/4 - 1/2$, so among them there are again two holes $Q_1^{(h)}$ and $Q_2^{(h)}$, $Q_1^{(h)} < Q_2^{(h)}$.

As above we can consider Eq. (4.53) assuming that λ_s is an arbitrary parameter. We determine it from Eq. (4.54). For the density $\rho_s(\lambda)$ we obtain the equation

$$\pi\rho_s(\lambda) + \int_{-\infty}^{\infty} \frac{\rho_s(\mu)}{1+(\lambda-\mu)^2}\, d\mu = \frac{2}{1+4\lambda^2} - \frac{1}{N}\Phi'(\lambda-\lambda_s) - \frac{\pi}{N}(\delta(\lambda-\lambda_1)+\delta(\lambda-\lambda_2)); \tag{4.56}$$

the solution has the form

$$\rho_s(\lambda) = \rho(\lambda) + \frac{1}{N}(\sigma(\lambda-\lambda_1)+\sigma(\lambda-\lambda_2)+\omega(\lambda-\lambda_s)), \tag{4.57}$$

where $\rho(\lambda)$ and $\sigma(\lambda)$ were introduced above, and $\omega(\lambda)$ satisfies the equation

$$\pi\omega(\lambda) + \int_{-\infty}^{\infty} \frac{\omega(\mu)}{1+(\lambda-\mu)^2}\, d\mu = -\Phi'(\lambda). \tag{4.58}$$

The solution of Eq. (4.58) has the form

$$\omega(\lambda) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}|\xi|} + e^{-\frac{3}{2}|\xi|}}{1+e^{-|\xi|}}\, e^{-i\lambda\xi}\, d\xi = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}|\xi| - i\lambda\xi}\, d\xi = -\frac{2}{\pi(1+4\lambda^2)}. \tag{4.59}$$

To determine λ_s we consider the passage to the limit as $N \to \infty$ in Eq. (4.54)

$$\operatorname{arctg} \lambda_s = \int_{-\infty}^{\infty} \Phi(\lambda_s-\lambda)\rho(\lambda)\, d\lambda + \frac{1}{N}\int_{-\infty}^{\infty} \Phi(\lambda_s-\lambda)(\sigma(\lambda-\lambda_1)+\sigma(\lambda-\lambda_2)+\omega(\lambda-\lambda_s))\, d\lambda. \tag{4.60}$$

The first integral on the right-hand side of (4.60) is equal to the left-hand side for every λ_s. Further, since $\Phi(\lambda)$ is odd and $\omega(\lambda)$ is even, the last summand in the second integral vanishes and we obtain the following condition to determine λ_s:

$$\int_{-\infty}^{\infty} \Phi(\lambda_s-\lambda)(\sigma(\lambda-\lambda_1)+\sigma(\lambda-\lambda_2))\, d\lambda = 0. \tag{4.61}$$

We have

$$\int_{-\infty}^{\infty} \Phi(\lambda-\mu)\sigma(\mu)\, d\mu = -\operatorname{arctg} 2\lambda. \tag{4.62}$$

Thus, the condition (4.61) has the form

$$\operatorname{arctg} 2(\lambda_s-\lambda_1) + \operatorname{arctg} 2(\lambda_s-\lambda_2) = 0, \tag{4.63}$$

which has the solution

$$\lambda_s = \frac{\lambda_1+\lambda_2}{2}. \tag{4.64}$$

Therefore, the string parameter λ_s is determined uniquely.

The remarkable fact is that our string does not have a contribution to the energy and the momentum of excitation. Indeed, it is easy to show that

$$\int_{-\infty}^{\infty} h(\lambda)\omega(\lambda-\lambda_s)d\lambda = \frac{J}{2}\int_{-\infty}^{\infty} e^{-|\xi|-i\lambda_s\xi}d\xi = \frac{J}{1+\lambda_s^2} = -h_{\frac{1}{2}}(\lambda_s) \tag{4.65}$$

and similarly for the momentum

$$\int_{-\infty}^{\infty} p(\lambda)\omega(\lambda-\lambda_s)d\lambda = -p_{\frac{1}{2}}(\lambda_s), \tag{4.66}$$

where $h_{1/2}(\lambda_s)$ and $p_{1/2}(\lambda_s)$ are the energy and the momentum of the string of length 2, given by the formulas (3.28) and (3.29). We have now the expressions for the energy and the momentum

$$\varepsilon_s(\lambda_1,\lambda_2) = h_{\frac{1}{2}}(\lambda_s)+N\int_{-\infty}^{\infty} h(\lambda)(\rho_s(\lambda)-\rho(\lambda))d\lambda = \varepsilon(\lambda_1)+\varepsilon(\lambda_2), \tag{4.67}$$

$$\kappa_s(\lambda_1,\lambda_2) = p_{\frac{1}{2}}(\lambda_s)+N\int_{-\infty}^{\infty} p(\lambda)(\rho_s(\lambda)-\rho(\lambda))d\lambda = \kappa(\lambda_1)+\kappa(\lambda_2), \tag{4.68}$$

where the functions $\varepsilon(\lambda)$ and $\kappa(\lambda)$ are introduced in (4.49) and (4.50). Thus, the energy and the momentum of the constructed state are the same as in Example 2. The only difference between these states is the value of the spin. The spin of the latter can be calculated with the help of the formla

$$S = -2 - \int_{-\infty}^{\infty} (2\sigma(\lambda)+\omega(\lambda))d\lambda = 0. \tag{4.69}$$

Formulas analogous to (4.52) and (4.69) were first introduced by Korepin in [32] for the calculation of the change in the massive Thirring model.

We also note that formulas (4.65) and (4.66) are particular cases of the general theorem which states that in configurations from the class $\mathfrak{M}_{AF}$ the contribution of the strings of length greater than 1 to the energy and the momentum vanishes. Indeed, we can prove, in the manner similar to that of Examples 1 and 2 above, the additivity of the contribution of the strings of length greater than 1 in the density $\rho(\lambda)$ for the strings of length 1. The function $\omega_M(\lambda)$, which corresponds to each string of length $2M+1>1$, satisfies the equation

$$\pi\omega_M(\lambda) + \int_{-\infty}^{\infty} \frac{\omega_M(\mu)}{1+(\lambda-\mu)^2}d\mu = -\varphi'_{M,0}(\lambda). \tag{4.70}$$

The solution has the form

$$\begin{aligned} \omega_M(\lambda) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\omega}_M(\xi)e^{-i\lambda\xi}d\xi = -\frac{M}{\pi(M^2+\lambda^2)}, \\ \hat{\omega}_M(\xi) &= -e^{-M|\xi|}. \end{aligned} \tag{4.71}$$

The corresponding contribution in the energy is

$$h_M(\lambda_s)+\int_{-\infty}^{\infty} h(\lambda)\omega_M(\lambda-\lambda_s)d\lambda = h_M(\lambda_s)+\frac{J}{2}\int_{-\infty}^{\infty} e^{-(M+\frac{1}{2})|\xi|-i\lambda\xi}d\xi = 0. \tag{4.72}$$

We consider the momentum in the same manner.

This fact is important for the classification of all states from the class $\mathfrak{M}_{AF}$. By virtue of the additivity mentioned above their energy and momentum consist of the energies and the momenta of an even number of holes in the sea of strings of length 1 and thus, are finite. The role of strings of length greater than 1 is reduced to the separation of this states with respect to the spin. The spin can be calculated by Korepin's formula

$$S = -\sum_{M>0} (2M+1)\nu_M - \int_{-\infty}^{\infty} \{2(\nu_0 - \sum_{M>0} \nu_M)\sigma(\lambda) + \sum_{M>0} \nu_M \omega_M(\lambda)\} d\lambda = -\sum_{M>0} (2M+1)\nu_M - 2\nu_0 \int_{-\infty}^{\infty} \sigma(\lambda) d\lambda. \quad (4.73)$$

Each of the states which corresponds to a configuration not included in the class $\mathfrak{M}_{AF}$ has an infinite relative energy for $N \to \infty$. We have not carried out an accurate investigation, however a simple consideration of all possible examples proves this convincingly. Indeed, the configurations $(l, q, \{\nu_M\})$ with finite q have an infinite relative energy. Further, the configurations with the sea of strings of length greater than 1 also have infinite energy with respect to the ground state, since the energy of the strings of length $2M + 1$, $M > 0$, is always greater than the energy of the collection of the strings of length 1 with the same momentum. Therefore, the condition that the relative energy is finite allows us to separate from the unseparable for $(N \to \infty)$ space $\mathfrak{H}_\infty$ the separable subspace $\mathfrak{H}_{AF}$ spanned by vectors which correspond to configurations from the class $\mathfrak{M}_{AF}$.

Consider now the interpretations of our results in terms of quantum field theory. We assume that the considerations presented above show that all antiferromagnetic vacuum excitations describe scattering states only of an even number of quasiparticles of one kind with momentum running over half of the Brillouin zone $-\pi \leqslant \kappa \leqslant 0$, and with the dispersion law $\varepsilon(\kappa) = -\frac{J\pi}{2} \sin\kappa$. Individual one-particle states do not exist. In this sense the quasiparticles are kinks and resemble the solutions in the quantum sine-Gordon model [32, 14]. The spin of a kink – a spin wave equals 1/2. Indeed, the states from Examples 1 and 2 have spin 0 and 1 and are the leading vectors composed of two kinks. In the first case the spins of kinks are antiparallel, in the second case – parallel.

More formally, we assume that the Hilbert space $\mathfrak{H}_{AF}$ is decomposed in the direct integral

$$\mathfrak{H}_{AF} = \sum_{n=0} \int_{\pi}^{2\pi} \ldots \int_{\pi}^{2\pi} d\kappa_1 \ldots d\kappa_{2n} \prod_{j=1}^{2n} \otimes \mathbb{C}^2 \quad (4.74)$$

and differs from the usual Fock space for bosons with spin 1/2 by the rule which forbids the states with an odd number of particles. Since their momenta change independently we may assume that a single kink can be localized and so, it makes sense to talk about their scattering. We calculate the corresponding S-matrix in the next section.

We note that the example considered above represents a phenomena which in quantum field theory is usually called decoloration. Indeed, the initial model is invariant with respect to the group SU(2) whereas physical states are classified by $SU(2)/Z_2 \approx SO(3)$, where Z_2 is the center of SU(2). Kinks (analogs of quarks) fly out but their number in physical states is necessarily even.

Our interpretation of excitations fundamentally differs from the generally accepted picture which can be traced back to des Cloizeaux–Pearson. Namely, in [4] for a finite N the set of states with spin 1 was presented, which, according to the authors, transfers to the one-particle triplet state when $N \to \infty$. The momentum of this state changes in the whole zone which in this case has the form $-\pi \leqslant \kappa \leqslant \pi$. The dispersion law has the following form

$$\varepsilon(\kappa) = \frac{J\pi}{2} |\sin\kappa|. \quad (4.75)$$

Further, in [11] it was stated that the des Cloizeaux–Pearson excitations are not the only one-particle excitations; in addition to them there exist bound singlet states with the dispersion law

$$\varepsilon(\kappa) = J\pi \left|\sin\frac{\kappa}{2}\right|, \quad -\frac{\pi}{2} \leqslant \kappa \leqslant \frac{\pi}{2}. \quad (4.76)$$

Finally in [33, 34], the results of [11] were criticized; however, the authors of these papers assumed as before that there exist many excitations, including singlet excitations, with the dispersion law (4.75). Thus, we have contradiction which requires resolution.

In one phrase this resolution consists of the following: the authors of all works mentioned above artificially separated a one-parameter family of vectors from the general two-parameter family of the general form by fixing one of the parameters.

We explain it in greater detail. The des Cloizeaux–Pearson excitations in our terms have the following form: the configuration of Example 2 is considered, where two nonfilled vacancies are chosen in a special way

a) $Q_1^{(h)} = -Q^{max}$, $Q_2^{(h)}$ is arbitrary, $2\pi Q_2/N = \kappa_2$, $-\pi/2 \leqslant \kappa_2 \leqslant \pi/2$;

b) $Q_1^{(h)}$ is arbitrary, $Q_2^{(h)} = Q^{max}$, $2\pi Q_1/N = \kappa_1$, $-\pi/2 \leqslant \kappa_1 \leqslant \pi/2$.

We assume that the set of vectors thus obtained is characterized by the parameter

$$\kappa = \begin{cases} \kappa_1 - \frac{\pi}{2}, & -\pi \leqslant \kappa \leqslant 0; \\ \kappa_2 + \frac{\pi}{2}, & 0 \leqslant \kappa \leqslant \pi. \end{cases} \tag{4.77}$$

From the calculations presented above it is clear that for such set of vectors the dispersion law indeed has the form (4.75). However, it is equally clear that this set is a particular case of the vectors from the two-parameters with the value of the momentum of one of the quasiparticles fixed.

We explain now in general terms what we have in mind distinguishing one-particle states from many-particle ones. The space of states $\mathfrak{H}$ can be decomposed in the direct integral

$$\mathfrak{H} = \int d\kappa\, \mathfrak{H}(\kappa) \tag{4.78}$$

over the eigensubspaces of the momentum operator. The one-particle state $\Psi(\kappa)$ with the momentum κ is a vector with the finite norm in $\mathfrak{H}(\kappa)$. At the same time, the two-particle state $\Psi(\kappa_1, \kappa_2)$ has in $\mathfrak{H}(\kappa)$, for $\kappa = \kappa_1 + \kappa_2$, an infinite norm. Therefore, by fixing one of the arguments in the two-particle state we can not obtain that it has a finite norm in $\mathfrak{H}(\kappa)$.

Of course, our considerations can become rigorous only after we prove the expansion (4.74). Nevertheless, we believe that our interpretation is much more natural than the construction of des Cloizeaux–Pearson, expecially after their construction was interpreted in terms of fillings of admissible vacancies.

We can similarly consider other false excitations. For example, the excitation from [11] with the dispersion law (4.76) are the sum of the two des Cloizeaux–Pearson excitations with coinciding momenta and, moreover, it requires the involvement of a string of length greater than 1 with a certain momentum. All this shows that even the computer calculations made in [4, 33] can lead to misunderstanding if interpreted incorrectly.

Apparently, technically, the simplest method of the resolution of the "misunderstanding" discussed above is the calculation of the norm of the state (3.20) in the space $\mathfrak{H}_N$. The states which correspond to configurations from Examples 1 and 2 should have a norm which differs from the ground state norm by a factor proportional to N^2; this would provide obvious evidence of their two-particle nature and justification of the expansion (4.74). In this connection the combinatorial problem, mentioned at the end of Sec. 1, becomes particularly appropriate.

With this we finish the classification of excitations in the antiferromagnetic case.

5. Scattering of Spin Waves

In this section we consider one on another scattering of two spin waves. As in the ferromagnetic case the full S-matrix can be factorized and all excitations are reduced to the two-particle ones. However, unlike the ferromagnetic case, for which the kind of particles (the length of the string) does not change during the scattering, in the antiferromagnetic case the only excitation — kink is characterized, in addition to momentum, by the spin. The general form of the two-particle S-matrix for quasiparticles with spin 1/2 is given by the formula

$$S_{ab;cd}(\lambda_1, \lambda_2) = S_1(\lambda_1, \lambda_2)\delta_{ac}\delta_{bd} + S_2(\lambda_1, \lambda_2)\delta_{ad}\delta_{bc}, \tag{5.1}$$

where a, b, c, d = 1, 2 and λ_1, λ_2 are parameters of colliding particles. The triplet and singlet phase factors have the form

$$S_t = S_1 + S_2\,, \quad S_s = S_1 - S_2\,. \tag{5.2}$$

At this writing only one method for the calculation of phase factors for scattering of holes in the sea has been discussed in the literature; this method was based on the momentum of one hole in the presence of another hole. The method was developed by Korepin [32] in quantum theory of solitons. In [12] Kulish and Reshetikhin calculated $S_t(\lambda_1, \lambda_2)$ using Korepin's method.

We present here a formal derivation of expressions for S_t and S_s by generalizing the procedure with Zamolodchikov's operators from Sec. 3. For this, it is sufficient to study eigenvalues of the S-matrix on Bethe's vectors, since they are the leading vectors, and in every nonreducible representation of SU(2) the S-matrix is proportional to the identity operator. We recall the structure of Bethe's vectors for the antiferromagnetic vacuum and triplet and singlet two-particle states

$$\Omega_{AF} = \prod_j B(\lambda_j)\Omega\,, \tag{5.3}$$

$$\Omega_t(\lambda_1,\lambda_2) = \prod_j B(\lambda_j^{(t)})\Omega\,, \tag{5.4}$$

$$\Omega_s(\lambda_1,\lambda_2) = B\left(\frac{\lambda_1+\lambda_2+i}{2}\right)B\left(\frac{\lambda_1+\lambda_2-i}{2}\right)\prod_j B(\lambda_j^{(s)})\Omega\,, \tag{5.5}$$

where for $N \to \infty$ the numbers λ_j, $\lambda_j^{(t)}$, and λ_2 are the hole parameters. We denote the operators of (λ), and $\rho_s(\lambda)$, respectively; and λ_1 are the hole parameters. We denote the operators of creation of the states $\Omega_t(\lambda_1, \lambda_2)$ and $\Omega_s(\lambda_1, \lambda_2)$ from the state Ω_{AF} by $B_t(\lambda_1, \lambda_2)$ and $B_s(\lambda_1, \lambda_2)$, respectively. Formally $B_t(\lambda_1, \lambda_2)$ is determined as follows:

$$\begin{aligned} B_t(\lambda_1,\lambda_2) &= \prod_j B(\lambda_j^{(t)})\cdot\prod_j B^{-1}(\lambda_j) = \exp\Big\{N\int_{-\infty}^{\infty}\log B(\lambda)\rho_t(\lambda)d\lambda\Big\}\exp\Big\{-N\int_{-\infty}^{\infty}\log B(\lambda)\rho(\lambda)d\lambda\Big\} \\ &= \exp\Big\{\int_{-\infty}^{\infty}\log B(\lambda)(\sigma(\lambda-\lambda_1)+\sigma(\lambda-\lambda_2))d\lambda\Big\}, \end{aligned} \tag{5.6}$$

where we used the fact that operators $B(\lambda)$ commute and the formula (4.44). We see that

$$B_t(\lambda_1,\lambda_2) = B_{\text{kink}}(\lambda_1)\,B_{\text{kink}}(\lambda_2)\,, \tag{5.7}$$

where the operator

$$B_{\text{kink}}(\lambda) = \exp\Big\{\int_{-\infty}^{\infty}\log B(\mu)\sigma(\lambda-\mu)d\mu\Big\} \tag{5.8}$$

can be interpreted as the kink creation operator. Of course, we know that this operator has its range outside the space $\mathfrak{H}_{AF}$, where only the product (5.7) has sense. Nevertheless, this formal object is convenient for the construction of the in and out-states for the leading vector in the triplet.

By analogy to the formula (3.36) we consider Zamolodchikov's operator

$$Z_{\text{kink}}(\lambda) = B_{\text{kink}}(\lambda)A_{\infty}^{-1}(\lambda)\,. \tag{5.9}$$

Here the operator $A_\infty(\lambda)$ is obtained from $A_N(\lambda)$ in the limit as $N \to \infty$ after separation of the diverging eigenvalue:

$$A_\infty(\lambda) = \lim_{N\to\infty} A_N(\lambda)\left(\lambda+\frac{i}{2}\right)^{-N}\exp\Big\{-N\int_{-\infty}^{\infty}\log\frac{\lambda-\mu-i}{\lambda-\mu}\,\rho(\mu)d\mu\Big\} \tag{5.10}$$

[see the formula (1.22)]. The commutation relation, similar to (3.32),

$$A_\infty(\lambda)\,B_{\text{kink}}(\mu) = B_{\text{kink}}(\mu)A_\infty(\lambda)\,\frac{1}{C_\infty(\mu-\lambda)}\,, \tag{5.11}$$

where

$$\frac{1}{C_\infty(\lambda)} = \exp\left\{\int_{-\infty}^{\infty} \log\frac{1}{c(\mu)}\sigma(\mu-\lambda)d\mu\right\}. \tag{5.12}$$

The relations (5.11) follow from (1.14) if we omit in it the second summand on the right-hand side. This can be justified after the correct passage to the limit as $N \to \infty$. It follows from (5.11) that Zamolodchikov's operators satisfy the following permutation relations:

$$Z_{\text{kink}}(\lambda)\, Z_{\text{kink}}(\mu) = Z_{\text{kink}}(\mu)\, Z_{\text{kink}}(\lambda) S_{\text{kink}}(\mu-\lambda), \tag{5.13}$$

where

$$\frac{1}{i}\log S_{\text{kink}}(\lambda) = \frac{1}{i}\int_{-\infty}^{\infty} \log V_0(\mu)\sigma(\mu-\lambda)d\mu. \tag{5.14}$$

We note that unlike (5.9) and (5.11) the products in (5.13) have sense in the space $\mathfrak{H}_{AF}$. To calculate the integral in (5.14) it is convenient to differentiate it with respect to λ. After that it is reduced to the integral

$$2\int_{-\infty}^{\infty} \frac{\sigma(\mu)}{1+(\lambda-\mu)^2}d\mu = -2\pi(\sigma(\lambda)+\sigma(\lambda)), \tag{5.15}$$

where we used Eq. (4.45). From (4.46) we have

$$\frac{1}{i}\frac{d}{d\lambda}\log S_{\text{kink}}(\lambda) = -\int_{-\infty}^{\infty} \frac{e^{-|\xi|}}{1+e^{-|\xi|}} e^{-i\lambda\xi} d\xi = \frac{1}{2}\left\{\Psi\left(\frac{1+i\lambda}{2}\right) + \Psi\left(\frac{1-i\lambda}{2}\right) - \Psi\left(1+\frac{i\lambda}{2}\right) - \Psi\left(1-\frac{i\lambda}{2}\right)\right\}, \tag{5.16}$$

where $\Psi(z)$ is the logarithmic derivative of the Γ-function (see [35]). Thus,

$$S_{\text{kink}}(\lambda) = \frac{1}{i}\,\frac{\Gamma\left(\frac{1+i\lambda}{2}\right)\Gamma\left(1-\frac{i\lambda}{2}\right)}{\Gamma\left(\frac{1-i\lambda}{2}\right)\Gamma\left(1+\frac{i\lambda}{2}\right)}. \tag{5.17}$$

Similarly to the considerations of Sec. 3 we construct the leading in and out-vector in the triplet state in the form

$$\Psi_{in}^{(t)} = Z_{\text{kink}}(\lambda_1)\; Z_{\text{kink}}(\lambda_2)\,\Omega_{AF}, \tag{5.18}$$

$$\lambda_1 > \lambda_2$$

$$\Psi_{out}^{(t)} = Z_{\text{kink}}(\lambda_2)\; Z_{\text{kink}}(\lambda_1)\,\Omega_{AF}. \tag{5.19}$$

Then it follows from (5.13) that the kink scattering S-matrix in the triplet state has the form

$$S_t(\lambda_1,\lambda_2) = S_{\text{kink}}(|\lambda_1-\lambda_2|); \tag{5.20}$$

this coincides with the result obtained in [12].

Unfortunately, the considerations presented above cannot be extended directly to the singlet state, since the corresponding operator of creation of the state $\Omega_S(\lambda_1, \lambda_2)$ from Ω_{AF} has the form

$$B_S(\lambda_1,\lambda_2) = B\left(\frac{\lambda_1+\lambda_2+i}{2}\right)B\left(\frac{\lambda_1+\lambda_2-i}{2}\right)\cdot\exp\left\{\int_{-\infty}^{\infty}\log B(\mu)\omega\left(\frac{\lambda_1+\lambda_2}{2}-\mu\right)d\mu\right\} B_{\text{kink}}(\lambda_1)\, B_{\text{kink}}(\lambda_2) \tag{5.21}$$

and cannot be factorized into two elementary creation operators.

However, we note that the in and out-states from (3.39) can be presented, up to the normalization factor, in the form

$$\Psi_{in} = A^{-1}(\lambda_1) B(\lambda_1) B(\lambda_2) A^{-1}(\lambda_2)\Omega \tag{5.22}$$

and

$$\Psi_{out} = A^{-1}(\lambda_2) B(\lambda_2) B(\lambda_1) A^{-1}(\lambda_1)\Omega, \tag{5.23}$$

where $\lambda_1 > \lambda_2$. In the same manner we can act in the case of the triplet (5.18), (5.19) already considered; we shall use this consideration for the singlet, setting for $\lambda_1 > \lambda_2$

$$\Psi_{in}^{(s)} = A_\infty^{-1}(\lambda_1) B_s(\lambda_1, \lambda_2) A_\infty^{-1}(\lambda_2)\Omega_{AF} \tag{5.24}$$

and

$$\Psi_{out}^{(s)} = A^{-1}(\lambda_2) B_s(\lambda_1, \lambda_2) A_\infty^{-1}(\lambda_1)\Omega_{AF}. \tag{5.25}$$

Then

$$\Psi_{out}^{(s)} = S_s(\lambda_1 - \lambda_2)\Psi_{in}^{(s)}, \tag{5.26}$$

where

$$S_s(\lambda) = S_{kink}(|\lambda|) S_0\left(\frac{|\lambda|}{2}\right). \tag{5.27}$$

Using (5.11), (5.12), and (5.21) for $(1/i)\log S_0(\lambda)$ we obtain the expression

$$\frac{1}{i}\log S_0(\lambda) = \frac{1}{i}\log V_{\frac{1}{2}}(\lambda) + \frac{1}{i}\int_{-\infty}^{\infty} \log V_0(\mu)\omega(\mu-\lambda)d\mu. \tag{5.28}$$

The logarithmic derivative of $S_0(\lambda)$ is simplified after applying the integral equation (4.58) and we have

$$\frac{1}{i}\frac{d}{d\lambda}\log S_0(\lambda) = -2\pi\omega(\lambda) = \frac{4}{1+4\lambda^2}; \tag{5.29}$$

hence,

$$S_0\left(\frac{\lambda}{2}\right) = \frac{\lambda - i}{\lambda + i}. \tag{5.30}$$

Thus, we have calculated S_t and S_s and we can write the S-matrix (5.1) in the form

$$\begin{gathered} S(\lambda_1, \lambda_2) = S(|\lambda_1 - \lambda_2|), \\ S(\lambda) = S_t(\lambda)\left(\frac{\lambda}{\lambda+i} I + \frac{i}{\lambda+i} P\right), \end{gathered} \tag{5.31}$$

where I and P are the identity operator and the permutation operator in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$, respectively. The formula (5.31) is consistent with the general formula of factorizable S-matrices of Zamolodchikov [31] and of Karovskii et al. [36]. It is clear that we would obtain the same result using Korepin's method [32]. In some sense this can be considered as justification of the formulas (5.18), (5.19) and (5.24), (5.25), since till now the only justification for them was the analog to the ferromagnetic case from Sec. 3.

We note that we did not construct the representation of the Zamolodchikov algebra, i.e., we did not find a set of operators $Z_a(\lambda)$ which satisfy the relations

$$Z_a(\lambda) Z_b(\mu) = Z_c(\mu) Z_d(\lambda) S_{ab;cd}(\mu - \lambda), \tag{5.32}$$

where we assume the summation from 1 to 2 over the repeating indices. However, to calculate $S_{ab,bc}(\lambda)$ it was sufficient to represent only the operators of creation of the leading vectors.

We note also that the S-matrix constructed above does not have poles in the strip $|\mathrm{Im}\times\lambda| < 1$. This domain plays the role of the physical sheet for our excitations. We can judge it from the expressions for additive observables, e.g., $\varepsilon(\lambda)$ and $\kappa(\lambda)$, which have as functions of λ the period $2i$ [this fact follows from (4.49) and (4.50)]. This proves once again that spin waves do not have bound states and represent the only elementary excitations in our model.

LITERATURE CITED

1. W. Heisenberg, "Zur Theorie des Ferromagnetismus," Z. Phys., 49, 619-636 (1928).
2. H. Bethe, "Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette," Z. Phys., 71, 205-226 (1931).
3. L. Hulthen, "Über das Austauschproblem eines Kristalles," Arkiv Mat., Astron. Fysik, 26A, No. 11, 1-106 (1938).
4. J. Des Cloizeaux and J. J. Pearson, "Spin-wave spectrum of the antiferromagnetic linear chain," Phys. Rev., 128, No. 5, 2131-2135 (1962).
5. R. Orbach, "Linear antiferromagnetic chain with anisotropic coupling," Phys. Rev., 112, No. 2, 309-316 (1958).
6. C. N. Yang and C. P. Yang, "One-dimensional chain of anisotropic spin—spin interaction. I. Proof of Bethe's hypothesis for ground state in finite system," Phys. Rev., 150, No. 1, 321-327 (1966).
7. C. N. Yang and C. P. Yang, "One-dimensional chain of anisotropic spin—spin interaction. II. Properties of the ground state energy per lattice site for an finite system," Phys. Rev., 150, No. 1, 327-339 (1966).
8. R. J. Baxter, "One-dimensional anisotropic Heisenberg chain," Ann. Phys., 70, No. 2, 323-327 (1972).
9. M. Gaudin, "Modeles exacts en mecanique statistique: La methode de Bethe et ses generalisations," Centre d'Etudes Nucleaires de Saclay, CEA-N-1559(1), Septembre, 1972.
10. M. Takahashi, "One-dimensional Heisenberg model at finite temperatures," Progr. Theor. Phys., 46, No. 2, 401-415 (1971).
11. A. A. Ovchinnikov, "The spectrum of excitations of the antiferromagnetic Heisenberg chain," Zh. Eksp. Teor. Fiz., 56, No. 4, 1354-1365 (1969).
12. P. P. Kulish and N. Yu. Reshetikhin, "Generalized Heisenberg ferromagnet and Gross—Neveu model," Zh. Eksp. Teor. Fiz., 80, No. 1, 214-228 (1981).
13. L. D. Faddeev, "Quantum completely integrable models in field theory," Soviet Scientific Reviews, Harvard Academic, London, 107-155 (1980).
14. E. K. Sklyanin, L. A. Takhtadzhyan, and L. D. Faddeev, "Quantum method of the inverse problem," Teor. Mat. Fiz., 40, No. 2, 194-220 (1979).
15. L. A. Takhtadzhyan and L. D. Faddeev, "The quantum method of the inverse problem and the Heisenberg XYZ model," Usp. Mat. Nauk, 34, No. 5, 13-63 (1979).
16. A. A. Belavin, "Exact solution of the two-dimensional model with asymptotic freedom," Phys. Lett., 87B, Nos. 1-2, 117-121 (1979).
17. N. Andrei and J. H. Lowenstein, "Diagonalization of the chiral invariant Gross—Neveu Hamiltonian," Phys. Rev. Lett., 43, No. 23, 1698-1700 (1979).
18. N. Andrei and J. H. Lowenstein, "Derivation of the chiral Gross—Neveu spectrum for arbitrary SU(N)-symmetry," Phys. Lett., 90B, No. 3, 106-110 (1980).
19. M. Gaudin, "Etude d'un modele a une dimension pour un system de fermions en interaction," Centre d'Etude Nucleaires de Saclay, Rapport CEA-R-3569 (1968).
20. C. N. Yang, "Some exact results for the many-body problem in one dimension with repulsive delta-function interaction," Phys. Rev. Lett., 19, No. 23, 1312-1314 (1967).
21. E. H. Lieb and V. F. Wu, "Absence of Mott transition in an exact solution of short-range 1-band model in 1-dimension," Phys. Rev. Lett., 20, No. 25, 1445-1448 (1968).
22. J. Hubbard, "Electron correlations in narrow energy bands," Proc. R. Soc., A276, No. 1365, 238-257 (1963).
23. P. P. Kulish and E. K. Sklyanin, "Quantum inverse scattering method and the Heisenberg ferromagnet," Phys. Lett. A, 70A, Nos. 5-6, 461-463 (1979).
24. M. Gaudin, B. M. McCoy, and T. T. Wu, "Normalization sum for the Bethe's hypothesis wave functions of the Heisenberg—Ising chain," Phys. Rev. D, 23, No. 2, 417-419 (1981).
25. J. von Neumann, "On infinite direct products," Comp. Math., 6, No. 1, 1-77 (1938).
26. E. K. Sklyanin, "Quantum version of the method of the inverse scattering problem," Zap. Nauchn. Sem. Leningr. Otd. Mat. Inst., 95, 55-128 (1980).
27. E. K. Sklyanin and L. D. Faddeev, "Quantum-mechanical approach to completely integrable models of field theory," Dokl. Akad. Nauk SSSR, 243, 1430-1433 (1978).
28. L. E. Thomas, "Ground state representation of the infinite one-dimensional Heisenberg ferromagnet. I," J. Math. Anal. Appl., 59, No. 2, 392-414 (1977).

29. D. Babbitt and L. Thomas, "Ground state representation of the infinite one-dimensional Heisenberg ferromagnet. II. An explicit Plancherel formula," Commun. Math. Phys., 54, No. 3, 255-278 (1977).
30. D. Babbitt and L. Thomas, "Ground state representation of the infinite one-dimensional Heisenberg ferromagnet. III. Scattering theory," J. Math. Phys., 19, No. 8, 1699-1704 (1978).
31. A. B. Zamolodchikov and A. B. Zamolodchikov, "Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models," Ann. Phys., 120, No. 2, 253-291 (1979).
32. V. E. Korepin, "Direct calculation of the S-matrix in the thirring model," Teor. Mat. Fiz., 41, No. 2, 169-189 (1979).
33. P. Fazekas and A. Sütö, "On the spectrum of singlet excitations of the S = 1/2 linear Heisenberg antiferromagnet," Solid-State Commun., 19, No. 11, 1045-1048 (1976).
34. A. Sütö, "Excitations in the antiferromagnetic Heisenberg chain," Solid-State Commun., 20, No. 11, 681-682 (1976).
35. H. Bateman and A. Erdelyi, Higher Transcendental Functions, Vol. 3, McGraw-Hill, New York (1955).
36. B. Berg, M. Karowskii, P. Weisz, and V. Kurak, "Factorized U(n) symmetric S-matrices in two dimensions," Nucl. Phys., B134, No. 1, 125-132 (1978).