DIRECT CALCULATION OF THE S MATRIX IN THE MASSIVE THIRRING MODEL

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Exact quantization of the Thirring model in a pseudoparticle basis is known [1]. Filling of the sea of states with negative energy makes it possible to calculate the observable characteristics of physical particles [2]. In the present paper, this approach is used to calculate the S matrix.

Introduction

The massive Thirring model, in both its classical and its quantum variant, attracts the interest of quantum field theoreticians. Important progress has recently been achieved in the study of this model. Much attention has been devoted to the connection between the quantum Thirring model and the quantum sine–Gordon model [3, 4]. This connection makes it possible to describe the model either in accordance with ordinary perturbation theory, or in accordance with perturbation theory for quantum solitons [5]. In [6], the classical equations of motion were studied and it was shown that these equations are completely integrable. The classical conservation laws are carried over into the quantum variant of the model and significantly simplify the dynamics [7]. The specific structure of scattering made it possible to calculate the S matrix explicitly in the massive Thirring model [8–10].

Study of the massive Thirring model developed independently, and it led to exact calculation of the mass spectrum [11]. The paper [2] is essentially a translation of [11] from lattice language into the language of continuous quantum field theory.

In the present paper, we shall consider the neutral sector of the model in the case of attraction. We use the exact solution of the quantum model in the pseudoparticle basis [1]. The physical vacuum of the model is constructed by filling the sea of negative-energy states. Excitations above this vacuum corresponding to different physical configurations are considered. The energy of the excitations is calculated. Thus, the mass spectrum of the model is calculated. All the calculations are made for fixed spatial cutoff $|x| < L/2$ and for fixed momentum cutoff. More precisely, a rapidity cutoff $|\beta| < \Lambda$ is made. The rapidity is related to the momentum by the formula $p = m \sinh \beta$. The first half of the paper reproduces the results of [2] systematically without reference to [11]. In the second half, the S matrix is calculated in this approach in the massive Thirring model. The method used to calculate the S matrix is very close to the method of calculating the single-loop corrections to the scattering matrix of quantum solitons [5]. The S matrix of two physical particles is factorized into two factors. The first is the S matrix of two bare particles, and the second is the S matrix for scattering of one bare particle on the vacuum polarization produced by the second particle. The dynamical calculation of the S matrix is the main result of the present paper.

In the present paper, we do not discuss the equivalence of the Thirring model and the sine–Gordon model. All the obtained results relate solely to the Thirring model. However, we find it convenient to use the language of the sine–Gordon model. The following correspondence between the Thirring and sine–Gordon models has been established in the literature [3]. A fermion corresponds to a soliton, and bound states of fermions correspond to the quantum states of a periodic soliton or (which is the same thing) bound states of the basic particles. The bound state of fermions with minimal mass corresponds to an ordinary particle in the sine–Gordon model. In this language, it is particularly convenient to compare the S matrix calculated in the present paper with the S matrix known in the literature [8, 10].

We now describe the content of the paper. In Sec. 1, we discuss quantization of the model in the pseudoparticle basis. In Sec. 2, we construct a physical vacuum by filling the sea and we renormalize. We give the eigenfunctions of the Hamiltonian corresponding to different physical configurations. We calculate the energy and momentum of these configurations. In Sec. 3, we calculate the scattering matrix of ordinary particles and their bound states in the sine–Gordon model, i.e., bound states of fermions in the Thirring

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model. In Sec. 4, we calculate the S matrix for scattering of a soliton on a bound state of ordinary particles in the sine–Gordon model, i.e., for scattering of a fermion on a bound state of fermions in the Thirring model. In Sec. 5, we calculate the S matrix of solitons in the sine–Gordon model or fermions in the Thirring model. In the Appendix, we give known results in the sine–Gordon model.

1. Quantization in the Pseudoparticle Basis

3 = 8

The Lagrangian of the Thirring model has the form

$$\mathscr{L}=i\bar{\psi}\gamma^\mu\partial_\mu\psi-m_0\bar{\psi}\psi-{}^1/_2\,\xi:(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi):,\qquad \gamma^0=\sigma^1,\qquad \gamma^1=i\sigma^2. \tag{1}$$

Instead of ξ, it is convenient to consider the quantities

$$\gamma = \quad \omega=\frac{\pi-\xi}{2},\quad 0<\xi<\pi,\quad 0<\omega<\frac{\pi}{2}. \tag{2}$$

The Hamiltonian of the model (1) has the form

$$H=\int dx\{i\psi^+\sigma_3\psi_x+m_0\psi^+\sigma_1\psi+2\,\xi\,\psi_1{}^+\psi_2{}^+\psi_2\psi_1\}. \tag{3}$$

The normal ordering is defined here by means of the pseudoparticle creation and annihilation operators ψ^+ and ψ. The vacuum in the pseudoparticle basis is understood as follows:

$$\psi|0\rangle=0,\quad \langle 0|\psi^+=0,\quad \{\psi^+(x),\psi(y)\}_+=I\delta(x-y). \tag{4}$$

It is obvious that the Hamiltonian does not change the number of pseudoparticles. We seek its eigenfunctions in the form

$$\Psi=\int dx_1\dots dx_l\,\chi^{\alpha_1\dots\alpha_l}(x_1,\dots,x_l)\,\psi^{+\alpha_1}(x_1)\dots\psi^{+\alpha_l}(x_l)\,|0\rangle. \tag{5}$$

The wave function χ is completely antisymmetric, and it satisfies the equation $\mathscr{H}_l\chi=E\chi$. Here, $\mathscr{H}_l$ is the l-particle Dirac operator with delta-functional two-body potential:

$$\mathscr{H}_l=\sum_{j=1}^{l}\left[i\sigma_3{}^j\frac{\partial}{\partial x^j}+m_0\sigma_1{}^j\right]+2\xi\sum_{k<j}\delta(x_k-x_j)I. \tag{6}$$

→ projector

The continuum eigenfunctions of this Hamiltonian were found for the first time in [1]. They can be written in the form [2]

$$\chi^{\alpha_1\dots\alpha_l}(x_1,\dots,x_l|\beta_1,\dots,\beta_l)=\sum_{(k_1\dots k_l)}(-1)^{[k_1\dots k_l]}\varphi^{\alpha_1}(x_1|\beta_{k_1})\dots$$

$$\varphi^{\alpha_l}(x_l|\beta_{k_l})\prod_{j>i}\left[1-i\,\tan(\xi/2)\;\varepsilon(x_j-x_i)\,\mathrm{th}\left(\frac{\beta^{k_j}-\beta^{k_i}}{2}\right)\right];\quad \beta_k\neq\beta_l,\quad k\neq l. \tag{7}$$

Thus, to specify an eigenfunction of the Hamiltonian (3), it is sufficient to specify the set of rapidities $\beta_1\dots\beta_l$. In what follows, we shall see that β is an "unrenormalized" rapidity. The summation on the right-hand side is over permutations of the numbers $1,\dots,l$. The parity of the permutation is $[k_1,\dots,k_l]$. The function φ is an eigenfunction of the single-particle free Dirac Hamiltonian:

$$\varphi(x|\beta)=u(\beta)\exp\{im_0x\,\mathrm{sh}\,\beta\},\quad u(\beta)=\frac{1}{\sqrt{2}}\begin{pmatrix}\exp\left\{-\frac{\beta}{2}\right\}\\ \exp\left\{\frac{\beta}{2}\right\}\end{pmatrix}. \tag{8}$$

The wave function (7) changes sign on the transposition of any pair β. The scattering described by the function (7) is purely elastic. Reflection is absent, and there is only an additional advance of the phase. The many-particle scattering matrix obtained from (7) reduces to a product of two-particle matrices. The wave function (7) belongs to the continuum if $\operatorname{Im}\beta=0$ and $\operatorname{Im}\beta=\pi$. The energy of pseudoparticles with $\operatorname{Im}\beta=0$ is positive and that of pseudoparticles with $\operatorname{Im}\beta=\pi$ is negative. Thus, the Hamiltonian (3) is not bounded below above the vacuum (4).

We now investigate the question of the discrete spectrum of the Hamiltonian (6). It can be shown that all the eigenfunctions of the discrete spectrum can be obtained from (7) by analytic continuation with respect to β into the complex plane. We illustrate this by the example of two particles. In this case, χ is equal to

$$\chi^{\alpha_1\alpha_2}(x_1, x_2|\beta_1, \beta_2) = \sum_{k_1,k_2} (-1)^{[k_1k_2]} u^{\alpha_1}(\beta_{k_1}) u^{\alpha_2}(\beta_{k_2}) \exp\left\{ im(x_1+x_2)\,\mathrm{sh}\left(\frac{\beta_1+\beta_2}{2}\right) \mathrm{ch}\left(\frac{\beta_{k_1}-\beta_{k_2}}{2}\right) + \right.$$

$$\left. im(x_1-x_2)\,\mathrm{ch}\left(\frac{\beta_1+\beta_2}{2}\right) \mathrm{sh}\left(\frac{\beta_{k_1}-\beta_{k_2}}{2}\right)\right\} \left[1 - i\tan(\xi/2)\,\varepsilon(x_2-x_1)\,\mathrm{th}\left(\frac{\beta_{k_2}-\beta_{k_1}}{2}\right)\right].$$

In this expression, we set $\beta_1 = B+i\omega$, $\beta_2 = B-i\omega$, $\mathrm{Im}\,B=0$. The expression in the square brackets becomes equal to $\theta(x_{k_2}-x_{k_1})$ and suppresses the growing exponential for $x_{k_2}-x_{k_1}>0$.

In the analysis of a bound state of n particles, it is necessary to continue (7) with respect to β_j in such a way that the wave function decreases with respect to all the differences $x_k - x_j$. For this, it is necessary that n factors in the square brackets become θ functions and it is necessary to ensure that the exponentials decrease in the region where the θ functions are not equal to zero. Such an analysis has the consequence that not all n particles can form a bound state. It is easy to show that such a state is formed in two cases.

1. If

$$\sin(\omega p)\sin(\omega(n-p))>0, \quad p=1,\dots,n-1, \tag{9}$$

then there is formed a bound state of n pseudoparticles with mass $M_n = m_0 \dfrac{\sin(\omega n)}{\sin\omega}$.

2. If

$$\sin(\omega p)\sin(\omega(n-p))<0, \quad p=1,\dots,n-1, \tag{10}$$

then there is formed a bound state of n pseudoparticles, $M_n = -m_0 \dfrac{\sin(\omega n)}{\sin\omega}$.

The wave function (7) decreases with respect to the corresponding coordinate differences only if the inequalities (9) and (10) are satisfied. To obtain the eigenfunction of the Hamiltonian corresponding to a bound state of n pseudoparticles in the first or second cases, it is sufficient in (7) to set

$$l=n, \quad \beta_j = B+i\omega(n-1-2j), \quad j=0,1,\dots,n-1, \tag{11}$$

and Im B = 0 in the first and Im B = π in the second case.

The energy and the momentum of such a configuration are E = M_n cosh B and P = M_n sinh B. By analytic continuation of (7) one can obtain a wave function describing the scattering of a bound state of n pseudoparticles on the remaining particles.

We discuss the solution of the inequalities (9) and (10). For fixed ξ, this is an equation for allowed values of n. Obviously,

$$n=1,2,\dots,\left[\frac{\pi}{\omega}\right]+1 \tag{12}$$

is a solution of (9). Here, [ ] denotes the integral part. We call this the principal series. For what follows, it is convenient to divide it into two parts. We denote the n that are less than $\left[\frac{\pi}{\omega}\right]-1$, by n_1 and the n that are equal to $\left[\frac{\pi}{\omega}\right]$ and $\left[\frac{\pi}{\omega}\right]+1$ by n_2:

$$n=n_1 \text{ for } n=1,\dots,\left[\frac{\pi}{\omega}\right]-1, \; n=n_2 \text{ for } n=\left[\frac{\pi}{\omega}\right], \left[\frac{\pi}{\omega}\right]+1. \tag{13}$$

We denote the remaining solutions of (9) by n_I :

$$n=n_I \quad \text{for } \sin\omega p\sin\omega(n-p)>0, \quad n>\left[\frac{\pi}{\omega}\right]+1. \tag{14}$$

We denote the solutions of (10) by n_{II} :

$$n=n_{II} \text{ for } \sin\omega p\sin\omega(n-p)<0. \tag{15}$$

We now consider the phase shifts for scattering of pseudoparticles. They all depend continuously on β. The phase shift of two pseudoparticles with positive energy Φ and scattering matrix S is, respectively, equal to

$$-i\ln S=\Phi(\beta)=-i\ln\left\{-e^{2i\omega}\frac{e^{\beta}-e^{-2i\omega}}{e^{\beta}-e^{2i\omega}}\right\}. \tag{16}$$

Its properties are as follows:

$$\Phi(\beta)+\Phi(-\beta)=0,\quad \Phi(\beta)\xrightarrow[\beta\to-\infty]{}\xi,\quad \Phi(\beta)\xrightarrow[\beta\to\infty]{}-\xi. \tag{17}$$

The phase shift for scattering of a pseudoparticle with positive energy on one with negative energy is

$$\Phi(\beta+i\pi)=\frac{1}{i}\ln\left\{-e^{2i\omega}\frac{e^{\beta}+e^{-2i\omega}}{e^{\beta}+e^{2i\omega}}\right\} \tag{18}$$

and it has the properties

$$\Phi(\beta+i\pi)\xrightarrow[\beta\to\infty]{}2\pi-\xi,\quad \Phi(\beta+i\pi)\xrightarrow[\beta\to-\infty]{}\xi,\quad \Phi(\beta+i\pi)+\Phi(-\beta+i\pi)=2\pi. \tag{19}$$

The S matrix for scattering of a pseudoparticle on a bound state of n pseudoparticles has the form

$$e^{i\tilde{\Phi}_n(\beta)}=(-1)^n e^{2i\omega n}\frac{(e^{\beta}-e^{-i\omega(n+1)})(e^{\beta}-e^{-i\omega(n-1)})}{(e^{\beta}-e^{i\omega(n+1)})(e^{\beta}-e^{i\omega(n-1)})},\quad \tilde{\Phi}_n(\beta)+\tilde{\Phi}_n(-\beta)=0,\quad n\leqslant\left[\frac{\pi}{\omega}\right]+1. \tag{20}$$

The S matrix for scattering of a bound state of n pseudoparticles on a bound state of m pseudoparticles (in the principal series) is equal to

$$S^{n,m}(\beta)=\exp\left\{i\sum_{p=0}^{n-1}\sum_{l=0}^{m-1}\Phi(\beta+i\omega(n-2p-m+2l))\right\}. \tag{21}$$

The phase shift for scattering of a bound state of n pseudoparticles on a negative-energy pseudoparticle has the form

$$\Phi_n(\beta+i\pi)=\frac{1}{i}\sum_{p=0}^{n-1}\ln\left[e^{-i\xi}\left(\frac{e^{\beta+i\omega(2p-n+1)}-e^{i\xi}}{e^{\beta+i\omega(2p-n+1)}-e^{-i\xi}}\right)\right]. \tag{22}$$

We consider first the case $n=n_1\left(n\leqslant\left[\frac{\pi}{\omega}\right]-1\right)$ (13). In this case

$$\Phi_n(\beta+i\pi)\xrightarrow[\beta\to-\infty]{}n\xi,\quad \Phi_n(\beta+i\pi)\xrightarrow[\beta\to\infty]{}n(2\pi-\xi),\quad \Phi_n(\beta+i\pi)+\Phi_n(-\beta+i\pi)=2\pi n,\quad \Phi_n(\beta+i\pi)=\Phi_n(\beta-i\pi). \tag{23}$$

We now consider the case $n = n_2$ (13); then

$$\Phi_n(\beta+i\pi)\xrightarrow[\beta\to-\infty]{}n\xi,\quad \Phi_n(\beta+i\pi)\xrightarrow[\beta\to\infty]{}(n-2)2\pi-n\xi,\quad \Phi_n(\beta+i\pi)+\Phi_n(-\beta+i\pi)=2\pi(n-2),\quad \Phi_n(\beta+i\pi)=\Phi_n(\beta-i\pi). \tag{24}$$

2. Filling of the Sea of Negative-Energy States

In the pseudoparticle basis, the Hamiltonian (3) is not bounded below. Filling of the sea of negative-energy states of the pseudoparticles makes it possible to avoid this difficulty [2]. Thus, the physical vacuum of the model is the following eigenfunction of the Hamiltonian (3):

$$\psi_v=\int d^N x\,\chi^{\alpha_1\ldots\alpha_N}(x_1,\ldots,x_N|\beta_1,\ldots,\beta_N)\,\psi^{+\alpha_1}(x_1)\cdot\ldots\cdot\psi^{+\alpha_N}(x_N)\,|0\rangle. \tag{25}$$

The set $\beta_1,\ldots,\beta_N$ is the set of all allowed values of the rapidities of negative-energy pseudoparticles. To regularize the calculations, we place the system in a box of length L with periodic boundary conditions and make a rapidity cutoff:

$$|\beta|<\Lambda,\quad |x|<L/2. \tag{26}$$

Such a set $\{\beta_j\}$ will be finite (34). Subsequently, we first lift the cutoff with respect to x, $L\to\infty$, and then with respect to β, $\Lambda\to\infty$. The allowed values of β_j can be found from the condition of periodicity of the wave function (7) and (25) with respect to each of the arguments:

$$m_0 L\,\mathrm{sh}\,\beta_j=\sum_{k\neq j}\Phi(\beta_j-\beta_k)+2\pi j. \tag{27}$$

Here, β_k ranges over all allowed values in the vacuum except β_j; $\Phi(\beta)$ in (16) is the phase shift for scattering of two negative-energy particles. For comparison, we give the periodicity conditions in the case

$\xi = 0$: g

$$m_0 L \operatorname{sh} \beta_j^0 = 2\pi j. \tag{28}$$

The energy of the vacuum is

$$E_v = -m_0 \sum_k \operatorname{ch} \beta_k. \tag{29}$$

In what follows, it is convenient to use the function

$$\rho(\beta_j) = 1/L(\beta_{j+1} - \beta_j), \quad \rho(\beta) = \rho(-\beta), \tag{30}$$

which has a finite limit as $L \to \infty$. To calculate $\rho(\beta)$ corresponding to (27), we subtract (27) from the equation with number $j + 1$, go to the limit $L \to \infty$, and obtain

$$m_0 \operatorname{ch} \beta = 2\pi\rho(\beta) + \int_{-\Lambda}^{\Lambda} \Phi'(\beta - \alpha)\rho(\alpha)\, d\alpha. \tag{31}$$

Later, we find independently (46) that $m_0 \to 0$ as $\Lambda \to \infty$; $m_0 = a \exp\{-[\xi/(\pi - \xi)]\Lambda\}$. For this form of m_0, the considered equation has one solution that does not depend on Λ:

$0 < \xi = g < \pi$

$$\rho(\alpha) = \left(\frac{\xi}{\pi + \xi}\right) \frac{a}{2 \sin \xi} \operatorname{ch}\left(\frac{\pi}{\pi + \xi}\alpha\right).$$

It is obvious that such $\rho = L^{-1}(\alpha_{j+1} - \alpha_j)^{-1}$ corresponds to the rapidity distribution

$$\frac{\xi}{\sin \xi} aL \operatorname{sh}\left(\frac{\pi}{\pi + \xi}\beta_j\right) = 2\pi j. \tag{32}$$

Thus, we see that the rapidity distribution in the vacuum is quasifree. Indeed, (32) differs from (28) by a renormalization of the mass and the rapidity [5]:

$$\beta \to \theta = \frac{\pi}{\pi + \xi}\beta. \tag{33}$$

In the vacuum (25) all negative-energy pseudoparticle states are filled for which $|\beta| < \Lambda$. The number of pseudoparticles in the vacuum is equal to (32)

$$N = \frac{\xi}{\sin \xi} \frac{aL}{\pi} \operatorname{sh}\left(\frac{\pi}{\pi + \xi}\Lambda\right). \tag{34}$$

Thus, a correct physical vacuum has been constructed.

We consider excitations above this vacuum. The excitations are characterized by the eigenfunctions of the Hamiltonian whose energies are greater than the vacuum energy. From all the excitations, we choose those for which the total number β_j is the same as in the vacuum (34). This corresponds to our considering the sector with zero observable charge. We shall take care to ensure that all the Re β_j are smaller than Λ in modulus. First, we introduce into the vacuum one pseudoparticle with positive energy. In other words, we consider an eigenfunction of the Hamiltonian (3) analogous to (25) in which the set β_j contains not only vacuum pseudoparticles Im $\beta_j = \pi$ but also a pseudoparticle with β_p, Im $\beta_p = 0$. On the plane of the complex β, this set is shown in Fig. 1a. Following [2], we see that this wave function corresponds to a basic particle in the sine-Gordon model. The existence of the new argument of the wave function changes the periodicity conditions. They have the form

$$m_0 L \operatorname{sh} \tilde\beta_j = \sum_{k \neq j} \Phi(\tilde\beta_j - \tilde\beta_k) + \Phi(\tilde\beta_j - \beta_p + i\pi) + 2\pi j. \tag{35}$$

The energy and momentum of this wave function are, respectively, equal to

$$E_{v+p} = -m_0 \sum_k{}' \operatorname{ch} \tilde\beta_k + m_0 \operatorname{ch} \beta_p, \quad P_{v+p} = -m_0 \sum_k{}' \operatorname{sh} \tilde\beta_k + m_0 \operatorname{sh} \beta_p. \tag{36}$$

The set $\{\tilde\beta_j\}$ differs from the vacuum set $\{\beta_j\}$ (32) by a quantity of order $1/L$. To describe $\{\tilde\beta_j\}$, it is convenient to introduce the functions

$$W_p(\beta_j) = L(\beta_j - \tilde\beta_j), \quad F_p(\beta_j) = W(\beta_j)\rho(\beta_j) = \frac{\beta_j - \tilde\beta_j}{\beta_{j+1} - \beta_j} = f_p(\beta - \beta_p), \tag{37}$$

which have a finite limit as $L \to \infty$. From Eq. (27), we subtract Eq. (35), go to the limit $L \to \infty$, and, using (31), obtain

$$0=\Phi(\beta+i\pi-\beta_p)+\int_{-\Lambda}^{\Lambda}\Phi'(\beta-\alpha)F_p(\alpha)d\alpha+2\pi F_p(\beta). \tag{38}$$

Here, we can go to the limit $\Lambda \to \infty$. It is convenient to differentiate this equation:

$$0=\Phi'(\beta+i\pi-\beta_p)+\int_{-\infty}^{\infty}\Phi'(\beta-\alpha)F_p'(\alpha)d\alpha+2\pi F_p'(\beta). \tag{39}$$

As a result of the introduction of the pseudoparticle with positive energy, some of the vacuum pseudoparticles are pushed beyond the cutoff. It is clear that the numbers of pseudoparticles pushed beyond Λ and beyond $-\Lambda$ are, respectively,

$$\Delta N(\Lambda)=-F_p(\Lambda),\quad \Delta N(-\Lambda)=F_p(-\Lambda). \tag{40}$$

More accurately, the integral parts should occur on the right-hand sides.

Thus, the number of pseudoparticles in the sea (the number of β_j with $\operatorname{Im}\beta_j=\pi$, $|\operatorname{Re}\beta_j|<\Lambda$) is $N - \Delta N(\Lambda) - \Delta N(-\Lambda)$ (34). Thus, the total change in the charge as a result of the introduction of the positive pseudoparticle, i.e., the observable charge, is

$$Q=1-\Delta N(\Lambda)-\Delta N(-\Lambda)=1+\int_{-\Lambda}^{\Lambda}F_p'(\beta)d\beta. \tag{41}$$

We calculate the observable value of the energy with allowance for this effect (29), (36):

$$E_p=E_{v+p}-E_v=m_0\operatorname{ch}\beta_p+m_0\sum\operatorname{sh}\beta_j(\beta_j-\tilde{\beta}_j)-m_0\operatorname{ch}\Lambda F(\Lambda)+m_0\operatorname{ch}\Lambda F(-\Lambda).$$

Finally,

$$E_p=m_0\operatorname{ch}\beta_p+m_0\int_{-\Lambda}^{\Lambda}\operatorname{sh}\beta F_p(\beta)d\beta-m_0\operatorname{ch}\beta F_p(\beta)\big|_{-\Lambda}^{\Lambda}=m_0\operatorname{ch}\beta_p-m_0\int_{-\Lambda}^{\Lambda}\operatorname{ch}\beta F_p'(\beta)d\beta. \tag{42}$$

Similarly, for the momentum

$$P_p=m_0\operatorname{sh}\beta_p-m_0\int_{-\Lambda}^{\Lambda}\operatorname{sh}\beta F_p'(\beta)d\beta. \tag{43}$$

Equation (39) can be solved by a Fourier transformation:

$$F_p'(k)=\int_{-\infty}^{\infty}e^{i\beta k}F'(\beta)d\beta=f_p'(k)e^{i\beta_p k},\ f_p'(k)=-\frac{\operatorname{ch}(\omega k)}{\operatorname{ch}((\pi-\omega)k)}=\frac{-\psi'(k)}{2\pi+\Phi'(k)},\quad f_p(\beta)+f_p(-\beta)=\frac{-\pi}{2\omega}, \tag{44}$$

where

$\omega=\gamma$

$$\Phi'(k)=\int_{-\infty}^{\infty}e^{i\beta k}\Phi'(\beta)d\beta=-2\pi\frac{\operatorname{sh}(k\xi)}{\operatorname{sh}(k\pi)},\quad \Psi'(k)=\int_{-\infty}^{\infty}e^{i\beta k}\Phi'(\beta+i\pi)d\beta=2\pi\frac{\operatorname{sh}(2\omega k)}{\operatorname{sh}(k\pi)}. \tag{45}$$

The asymptotic behavior of $F_p'(\beta)$ as $|\beta| \to \infty$ is determined by the singularities of (44) at the points $k = \pm i\pi/2(\pi - \omega)$. It has the form

$$F_p'(\beta)\xrightarrow[|\beta|\to\infty]{}\frac{-1}{\pi+\xi}\sin\left(\frac{\pi\xi}{\pi+\xi}\right)\operatorname{ch}\left(\frac{\pi}{\pi+\xi}(\beta-\beta_p)\right)^{-1}.$$

$k \to \frac{i\pi}{2(\pi-\omega)} = \frac{i\pi}{\pi+g}$

Hence and from (42) it can be seen that E_p is finite only if

$$m_0=a\exp\left\{-\frac{\xi}{\pi+\xi}\Lambda\right\},\quad \frac{da}{d\Lambda}=0. \tag{46}$$

$\omega = \frac{\pi-g}{2} = \gamma$

Using this, we obtain for the energy and the momentum

$$E_p=2m_h\sin\left(\frac{\pi}{2}\cdot\frac{\omega}{(\pi-\omega)}\right)\operatorname{ch}\left(\frac{\pi}{2(\pi-\omega)}\beta\right),\quad P_p=2m_h\sin\left(\frac{\pi}{2}\frac{\omega}{(\pi-\omega)}\right)\operatorname{sh}\left(\frac{\pi}{2(\pi-\omega)}\beta\right). \tag{47}$$

The first terms in (42) and (43) do not contribute to (46). The "bare" rapidity β has been renormalized and

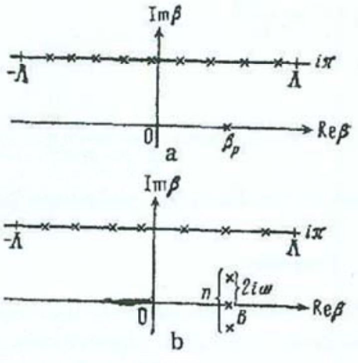

Fig.1

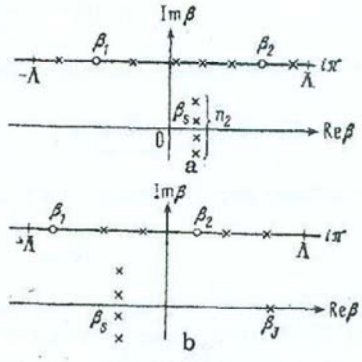

Fig.2

transformed into the physical rapidity θ (33). The quantity m_h, which in what follows will be the soliton mass, is equal to

$$m_h = a\xi^{-1}\,\mathrm{tg}\left(\frac{\pi\xi}{\pi+\xi}\right). \tag{48}$$

The observable charge is equal to

$$Q = 1 + F_p'(k)\,|_{\lambda=0} = 0, \tag{49}$$

as must be the case for a basic particle in the sine–Gordon model. $n = 1, 2, .. \left[\frac{\pi}{\omega}\right] - 1$

We introduce into the vacuum a bound state n_t of pseudoparticles (13). This means that we consider the wave function (7), (25), which is characterized by the following set of rapidities. In this set we have besides the vacuum $\mathrm{Im}\,\beta_j = \pi$ a further $n = n_t$ rapidities distributed in accordance with formula (11):

$$\beta_l = B + i\omega(n-1-2l), \quad l = 0, 1, \ldots, n-1. \tag{50}$$

On the complex rapidity plane, this set is shown in Fig. 1b. The new periodicity conditions have the form

$$m_0 L\,\mathrm{sh}\,\beta_j^n = \sum \Phi(\beta_j^n - \beta_k^n) + \Phi_n(\beta_j^n + i\pi - B) + 2\pi j \tag{51}$$

(see (22)).

As in the case of (37), we introduce the function

$$F_n(\beta_j) = \frac{\beta_j - \bar{\beta}_j}{\beta_{j+1} - \beta_j} = f_n(\beta - B). \tag{52}$$

For it, we obtain an equation analogous to (38) and (39):

$$0 = \Phi_n(\beta - B + i\pi) + \int_{-\infty}^{\infty} \Phi'(\beta-\alpha) F_n(\alpha)\,d\alpha + 2\pi F_n(\beta), \quad 0 = \Phi_n'(\beta - B + i\pi) + \int_{-\infty}^{\infty} \Phi'(\beta-\alpha) F_n'(\alpha)\,d\alpha + 2\pi F_n'(\beta). \tag{53}$$

The expressions for the observables are similar to (41), (42), and (43):

$$Q_n = n + \int_{-\infty}^{\infty} F_n'(\beta)\,d\beta, \quad E_n = -m_0 \int_{-\Lambda}^{\Lambda} \mathrm{ch}\,\beta F_n'(\beta)\,d\beta, \quad P_n = -m_0 \int_{-\Lambda}^{\Lambda} \mathrm{sh}\,\beta F_n'(\beta)\,d\beta. \tag{54}$$

In accordance with (22),

$$\Phi_n(\beta + i\pi) = \sum_{j=0}^{n-1} \Phi(i\pi + \beta + i\omega(n-1-2j)).$$

Hence and from (53) it can be seen that

$$F_n(\beta) = \sum_{j=0}^{n-1} f_p(\beta - B + i\omega(n-1-2j)) = f_n(\beta - B). \tag{55}$$

The Fourier transform $f_n'(\beta)$ has the form

$n=1,2 .. \frac{\pi}{\omega}-1$

$$f_n'(k) = -\frac{\mathrm{ch}(k\omega)\,\mathrm{sh}(k\omega n)}{\mathrm{ch}(k(\pi-\omega))\,\mathrm{sh}(k\omega)}, \quad f_n(\beta)+f_n(-\beta)=\frac{-\pi n}{2\omega}. \tag{56}$$

Calculating the observables (54) using (56), we obtain

$< \frac{\pi}{2}$

$$E_n=M_n\,\mathrm{ch}\left(\frac{\pi}{2(\pi-\omega)}B\right), \quad P_n=M_n\,\mathrm{sh}\left(\frac{\pi}{2(\pi-\omega)}B\right), \quad Q_n=0, \quad M_n=2m_h\sin\left(\frac{\pi}{2}\,\frac{\omega}{\pi-\omega}n\right). \tag{57}$$

These expressions and (79) enable us to identify the coupling constants in the Thirring and sine–Gordon models:

$$\gamma/8=\omega, \quad \gamma'/8=\pi\omega/(\pi-\omega), \quad 0<\gamma'<8\pi, \quad 0\leq\gamma<4\pi. \tag{58}$$

It can be seen from these expressions that all observables which commute with the Hamiltonian and the Hamiltonian itself have, when calculated on the constructed wave functions, the same eigenvalues as on the bound state of n particles in the sine–Gordon model. The wave function corresponding to the introduction of one particle n = 1 with respect to all quantum numbers is identical with that of a basic particle. We now consider other eigenfunctions of the Hamiltonian (3).

We introduce into the vacuum a bound state of n_2 pseudoparticles (13). In this case, all calculations are as before. However, in calculating the Fourier transform of $\Phi_n'(\beta+i\pi)$ (22), (53) we find that the imaginary part of the rapidity of terms with j = 0, n − 1 is too large. The pole of the integrand passes beyond the cut, and the expression (56) is no longer valid. Independent calculations lead to the expression

$\frac{\pi-\omega n}{\omega}$ $k\to 0$

$$f_n'(k)=\frac{\mathrm{sh}(k(\pi-\omega n))}{\mathrm{sh}(k\omega)}=\int_{-\infty}^{\infty} e^{i\beta k} f_n'(\beta)\,d\beta, \quad F_n(\beta)=f_n(\beta-B), \quad f_n(\beta)+f_n(-\beta)=\frac{-\pi}{2\omega}(n-2). \tag{59}$$

Calculation of Q, E, and P leads to the results

$n+f_n'|_{k=0} =$

$$Q=\left[\frac{\pi}{\omega}\right], \quad E_n=P_n=0. \tag{60}$$

This result is explained by the fact that the obtained $f_n'(k)$ does not have singularities at the points $k=\pm i\pi/2(\pi-\omega)$. This state can be regarded as an analog of the vacuum with nontrivial topological charge and definite parity. We recall however that in the present paper we are interested in states with Q = 0.

Introduction into the vacuum of a bound state of n_I (14) pseudoparticles leads to a state of a charged-vacuum type:

$$E=0, \quad P=0, \quad Q=\left[\frac{\pi}{\omega}(2l+1)\right], \quad l>0. \tag{61}$$

Introduction into the vacuum of a bound state of n_{II} pseudoparticles (15) leads to a state of the type containing a soliton and a charged vacuum:

$$E=m_h\,\mathrm{ch}\left(\frac{\pi\beta}{2(\pi-\omega)}\right), \quad P=m_h\,\mathrm{sh}\left(\frac{\pi}{2(\pi-\omega)}\beta\right), \quad Q=\left[\frac{2\pi n}{\omega}+\frac{3}{2}\,\frac{\pi}{\omega}\right], \quad n>0. \tag{62}$$

Let us consider a hole. It is described by the wave function (7), (25), which is characterized by the following set of rapidities. The set differs from a vacuum set in that there is no filling of the state $\beta=\beta_{n_0}$. The new periodicity conditions are

$$m_0L\,\mathrm{sh}\,\bar\beta_j=\sum_{h\neq n_0}\Phi(\bar\beta_j-\bar\beta_h)+2\pi j, \quad j\neq n_0. \tag{63}$$

In going to the limit $L\to\infty$ we shall change n_0 to achieve $\beta_{n_0+1}\to\beta_h$, $\beta_{n_0-1}\to\beta_h$. We shall call β_h the unrenormalized rapidity of the hole. We make calculations as before and obtain for

$$F_h(\beta_j)=\frac{\bar\beta_j-\beta_j}{\beta_{j+1}-\beta_j}=f_h(\beta-\beta_h) \tag{64}$$

the equation

$$0=-\Phi(\beta-\beta_h)+\int_{-\infty}^{\infty}\Phi'(\beta-\alpha)F_h(\alpha)\,d\alpha+2\pi F_h(\beta). \tag{65}$$

The observables can be expressed in terms of F in accordance with the old formulas analogous to (54). The solution of this equation has the form

$\omega = \gamma$

$$f_h(\beta)+f_h(-\beta)=0, \quad f_h'(k)=-\frac{\operatorname{sh}(k(\pi-2\omega))}{2\operatorname{sh}(k\omega)\operatorname{ch}(k(\pi-\omega))}. \tag{66}$$

The observables, calculated on the hole, have the form

$$E_h=m_h\operatorname{ch}\left(\frac{\pi}{2(\pi-\omega)}\beta_h\right), \quad P_h=m_h\operatorname{sh}\left(\frac{\pi}{2(\pi-\omega)}\beta_h\right), \quad Q=-\frac{1}{2}\frac{\pi}{\omega}. \tag{67}$$

hole

This is a state consisting of a charged vacuum and a soliton.

It can be seen from our calculations that the Hamiltonian (3) is positive in the basis of physical particles (above the vacuum (25), (32)).

The treatment of scattering states is trivial, since all the equations are linear and all the observables are additive.

It is easy to construct the wave function corresponding to the state of a soliton plus an antisoliton in the continuum. To do this, we introduce into the vacuum two holes with rapidities β_1 and β_2 and a bound state of n_2 pseudoparticles (13) with rapidity $\beta_s=(\beta_1+\beta_2)/2$. The set of unrenormalized rapidities β characterizing the wave function of the soliton–antisoliton configuration is shown in Fig. 2a. The value of β_s is chosen such that the wave function in the rest frame, $\beta_1+\beta_2=0$, has a definite parity. The deformation of the vacuum produced by the holes and the bound state is additive (see (55) and (64)):

$$\begin{gathered} m_0 L \operatorname{sh}\tilde{\beta}_j=\sum_n \Phi(\tilde{\beta}_j-\tilde{\beta}_n)-\Phi(\tilde{\beta}_j-\beta_1)-\Phi(\tilde{\beta}_j-\beta_2)+\Phi_n(\beta_j+i\pi-(\beta_1+\beta_2)/2)+2\pi j; \\ F_{as}=\frac{\beta_l-\tilde{\beta}_l}{\beta_{l+1}-\beta_l}=f_h(\beta_l-\beta_1)+f_h(\beta_l-\beta_2)+f_n\left(\beta_l-\frac{\beta_1+\beta_2}{2}\right). \end{gathered} \tag{68}$$

All observable quantities are also additive. They have the form (60), (67):

$$E=m_h\operatorname{ch}\left(\frac{\pi}{2(\pi-\omega)}\beta_1\right)+m_h\operatorname{ch}\left(\frac{\pi}{2(\pi-\omega)}\beta_2\right), \quad P=m_h\operatorname{sh}\left(\frac{\pi}{2(\pi-\omega)}\beta_1\right)+m_h\operatorname{sh}\left(\frac{\pi}{2(\pi-\omega)}\beta_2\right), \quad Q=0. \tag{69}$$

The spatial parity of this state is [12]

$$(-1)^{n_2}. \tag{70}$$

Thus, the eigenvalues of all observables that commute with the Hamiltonian calculated on such eigenfunctions are equal to the values of the observables calculated on the soliton-plus-antisoliton state.

Note that the functions $F'(\beta)$ (37), (52), (68) can be called the form factors of the corresponding particles, i.e., $F'(\beta)$ is the density of the bare particles from the sea in the physical particle (42), (43), (54), (73).

3. S Matrix of the Basic Particles

In one-dimensional potential scattering, the S matrix can be defined as follows. We place the system in a box and consider two cases: the potential is equal to zero and the potential is equal to a given V. The phase collected by a wave function on passing through the box in the first case is $\varphi_1 = kL$ and in the second it is $\varphi_2 = kL + (1/i)\ln S(k)$.

Thus, for the S matrix in the case of motion of a particle in the potential V we obtain the definition

$$\frac{1}{i}\ln S(k)=\varphi_2(k)-\varphi_1(k). \tag{71}$$

In the case of periodic boundary conditions, as argument on the right-hand side we shall take, for example, the allowed values of k in the free case. Then $\varphi_1(k)$ will be a multiple of 2π and $\varphi_2(k)$ will not be a multiple of 2π. This formula is literally correct in the absence of reflection. If reflection is possible, it is necessary to consider symmetrized and antisymmetrized wave functions for which the S matrix is diagonal (reduces to a pure phase). We consider only even potentials for which the above is true.

We apply this definition to the scattering of basic particles in the sine–Gordon model.

We shall calculate the phase shift of the basic particles as follows. We consider first the wave function of the Hamiltonian corresponding to one particle with bare rapidity β_1 (see Fig. 1a). We calculate the phase φ_1 which the wave function collects when its argument varies from 0 to L (along the contour). The phase φ_1 is made up of the free phase $m_0L \sinh\beta$ and the phase shift resulting from scattering on pseudoparticles in the sea:

$$\varphi_1 = m_0 \operatorname{sh}\beta_1 L + \sum_j \Phi(\beta_1-\hat{\beta}_j+i\pi) = m_0 L \operatorname{sh}\beta_1 + \sum_j \Phi(\beta_1-\beta_j+i\pi) + \int_{-\infty}^{\infty} \Phi'(\beta_1-\alpha+i\pi) f_p(\alpha-\beta_1)\, d\alpha. \tag{72}$$

Here, we have used (30) and (37).

We now turn to the subtraction of φ_2. For this, we consider a wave function corresponding to two basic particles with unrenormalized rapidities β_1 and β_2.

The periodicity conditions for such a wave function have a form analogous to (35),

$$m_0 L \operatorname{sh}\hat{\beta}_j = \sum_k \Phi(\hat{\beta}_j-\hat{\beta}_k) + \Phi(\hat{\beta}_j+i\pi-\beta_1) + \Phi(\hat{\beta}_j+i\pi-\beta_2) + 2\pi j.$$

The function $F_{12}(\beta)$, like (37), satisfies an equation analogous to (38),

$$0 = \Phi(\beta+i\pi-\beta_1) + \Phi(\beta+i\pi-\beta_2) + 2\pi F_{12}(\beta) + \int_{-\infty}^{\infty} \Phi'(\beta-\alpha) F_{12}(\alpha)\, d\alpha.$$

The solution of the equation has the form (44) $F_{12}(\beta) = (\beta_j-\hat{\beta}_j)/(\beta_{j+1}-\beta_j) = f_p(\beta-\beta_1) + f_p(\beta-\beta_2)$.

We now calculate the phase φ_2 collected by the first particle in the presence of the second on the passage around the contour (as its argument changes from 0 to L). The calculation of φ_2 is similar to that of φ_1:

$$\varphi_2 = m_0 L \operatorname{sh}\beta_1 + \pi + \Phi(\beta_1-\beta_2) + \sum \Phi(\beta_1-\hat{\beta}_k+i\pi) = m_0 L \operatorname{sh}\beta_1 + \sum_j \Phi(\beta_1-\beta_j+i\pi) + \int \Phi'(\beta_1-\alpha+i\pi)\,(f_p(\alpha-\beta_1) +$$

$$f_p(\alpha-\beta_2))\, d\alpha + \pi + \Phi(\beta_1-\beta_2).$$

Here, $\Phi(\beta_1-\beta_2)$ is the direct phase shift resulting from scattering of two pseudoparticles with rapidities β_1 and β_2 (16), and π has arisen from permutation of the arguments of the antisymmetric wave function. We use formula (71) and obtain

$$\frac{1}{i}\ln S^{11}(\beta_1-\beta_2) = \pi + \Phi(\beta_1-\beta_2) + \int \Phi(\beta_1-\beta_2-\alpha+i\pi) f_p'(\alpha)\, d\alpha = \pi + \Phi(\beta_1-\beta_2) + \int_{-\infty}^{\infty} \Phi'(\beta_1-\beta_2-\alpha+i\pi) f_p(\alpha)\, d\alpha -$$

$$[\Phi(\Lambda+i\pi)\Delta N(\Lambda) + \Phi(-\Lambda+i\pi)\Delta N(-\Lambda)]. \tag{73}$$

It is easy to explain the meaning of each term on the right-hand side. The first two terms are simply the phase shift for scattering of pseudoparticles with rapidities β_1 and β_2. The introduction of the second pseudoparticle deforms the vacuum. This deformation is described by the function $f_p(\alpha-\beta_2)$ (44). It can be seen that the third term is the phase shift for scattering of the first pseudoparticle on the deformation of the vacuum produced by the second pseudoparticle. The expression in the square brackets is the contribution of the particles pushed beyond the cutoff; one can show that this expression is equal to 2π. Using (17), (40), and (44), we can show that $\ln S^{11}(\beta)$ is odd.

We calculate the derivative of the phase shift:

$$\frac{1}{i}\frac{d}{d\beta}\ln S^{11}(\beta) = \Phi'(\beta) + \int_{-\infty}^{\infty} \Phi'(\beta-\alpha+i\pi) f_p'(\alpha)\, d\alpha.$$

The Fourier transform of the derivative has a particularly simple form:

$$\int e^{i\beta k}\frac{d}{d\beta}\frac{1}{i}\ln S^{11}(\beta)\, d\beta = \Phi'(k) + \Psi'(k) f_p'(k) = -2\pi\frac{\operatorname{ch}(k(\pi-3\omega))}{\operatorname{ch}(k(\pi-\omega))}$$

(see (44), (45)).

Calculating the inverse Fourier transform, we obtain finally

$$\frac{\pi\omega}{\pi-\omega}=\frac{\gamma'}{8}=\frac{\omega}{1-\omega/\pi}$$

$$S^{11}(\beta)=\frac{\operatorname{sh}\left(\frac{\pi\beta}{2(\pi-\omega)}\right)+i\sin\left(\frac{\pi\omega}{\pi-\omega}\right)}{\operatorname{sh}\left(\frac{\pi\beta}{2(\pi-\omega)}\right)-i\sin\left(\frac{\pi\omega}{\pi-\omega}\right)}=\frac{\operatorname{sh}\theta+i\sin\frac{\gamma'}{8}}{\operatorname{sh}(\theta)-i\sin\frac{\gamma'}{8}}.$$

Thus, we have calculated the S matrix of two ordinary particles in the sine–Gordon model. This expression is equal to the known (80).

We consider the phase shift for scattering of bound states of the basic particles. We consider the scattering of a bound state of n_1 particles (13), $\beta_p{}^1=\beta_1+i\omega(n_1-1-2p)$, $p=0,1,\ldots,n_1-1$, on a bound state of m_1 particles, $\beta_l{}^2=\beta_2+i\omega(m_1-1-2l)$, $l=0,1,\ldots,m_1-1$. After calculations similar to those above, we obtain

$$\frac{1}{i}\ln S^{n,m}(\beta_1-\beta_2)=nm\pi+\sum_{p=0}^{n-1}\sum_{l=0}^{m-1}\left\{\Phi(\beta_1-\beta_2+i\omega(n-2p-m+2l))+\right.$$
$$\left.\int_{-\infty}^{\infty}\Phi'(\beta_1+i\omega(n-1-2p)-\alpha+i\pi)f_p(\alpha-\beta_2+i\omega(2l+1-m))\,d\alpha\right\}.$$

It can be seen from this that

$$S^{n,m}(\theta)=\prod_{p=0}^{n-1}\prod_{l=0}^{m-1}S^{11}\left(\theta+i\frac{\gamma'}{16}(n-m-2p+2l)\right)$$

We have obtained a result agreeing with the known (81). It is here important that, on account of (33), the expressions (11) and (84) are identical.

It follows from the linearity of all the equations that the S matrix factorizes and the phase shift of several particles is equal to the sum of the two-particle phase shifts.

4. S Matrix for Scattering of Bound States of Ordinary Particles on a Soliton

We calculate first the S matrix for scattering of an ordinary particle in the sine–Gordon model on a soliton. For this, we consider the scattering of an ordinary particle on a soliton–antisoliton state. The set of rapidities characterizing the wave function describing this scattering differs from the vacuum set by the presence of a pseudoparticle with positive energy $\beta=\beta_3$, two holes with rapidities β_1 and β_2, and the bound state of n_2 (13) pseudoparticles with rapidity $\beta_s=(\beta_1+\beta_2)/2$ (see Fig. 2b). For this configuration, the periodicity conditions are

$$m_0L\operatorname{sh}\beta_j'=2\pi j+\Phi(\beta_j'+i\pi-\beta_3)+\Phi_n\left(\beta_j'-\frac{\beta_1+\beta_2}{2}\right)+\sum_{\beta_k\neq\beta_1,\beta_2}\Phi(\beta_j'-\beta_k').$$

It is obvious that (44), (68)

$$F_{ss}{}^p(\beta_j)=\frac{\beta_j-\beta_j'}{\beta_{j+1}-\beta_j}=F^p(\beta_j)+F^{ss}(\beta_j).$$

We now turn directly to the S matrix. The linearity of all the relations makes it obvious that the phase shift for scattering of an ordinary particle on a soliton–antisoliton state is equal to the sum of three terms: the phase shift for scattering of the particle on the first and the second holes, $\delta_h(\beta_3-\beta_1)$, $\delta_h(\beta_3-\beta_2)$, and the phase shift δ_n for scattering on the bound state of n_2 pseudoparticles. For the S matrix S^1_{ss} of the considered process, using calculations similar to those in the previous section, we obtain

$$\frac{1}{i}\ln S_{ss}{}^1=\delta_h(\beta_3-\beta_1)+\delta_h(\beta_3-\beta_2)+\delta_n\left(\beta_3-\frac{\beta_1+\beta_2}{2}\right)-\Phi(\Lambda+i\pi)\Delta N(\Lambda)-\Phi(-\Lambda+i\pi)\Delta N(-\Lambda),\quad\beta_3>\beta_1,\quad\beta_3>\beta_2.\tag{74}$$

Here, ΔN is the number of particles pushed beyond the cutoff of the soliton–antisoliton configuration. The values of δ_h and δ_n are, respectively,

$$\delta_h(\beta)=-\Phi(\beta+i\pi)+\int_{-\infty}^{\infty}\Phi'(\beta-\alpha+i\pi)f_h(\alpha)\,d\alpha+\pi,\quad\delta_n(\beta)=\Phi_n(\beta)+\int_{-\infty}^{\infty}\Phi'(\beta-\alpha+i\pi)f_n(\alpha)\,d\alpha+n\pi.=0\tag{75}$$

Direct calculations by means of (20), (18), (45), and (56) show that $\delta_n(\beta)-\Phi(\Lambda+i\pi)\Delta N(\Lambda)-\Phi(-\Lambda+i\pi)\Delta N(-\Lambda)$ are multiples of 2π. One can show that (19) and (66)) $\delta_h(\beta)$ is odd. It is easiest to calculate the Fourier transform of $\delta_h'(\beta)$ (45), (66):

$$\delta_h'(k) = -\Psi'(k) + \Psi'(k) f_h'(k) = -2\pi \frac{\mathrm{ch}(\omega k)}{\mathrm{ch}(k(\pi-\omega))}. \tag{76}$$

Calculation of the inverse Fourier transform leads us to

$$S_{s1}^{1} = \left(\frac{\mathrm{sh}(\theta_3-\theta_1) + i\cos\frac{\gamma'}{16}}{\mathrm{sh}(\theta_3-\theta_1) - i\cos\frac{\gamma'}{16}} \right) \left(\frac{\mathrm{sh}(\theta_3-\theta_2) + i\cos\left(\frac{\gamma'}{16}\right)}{\mathrm{sh}(\theta_3-\theta_2) - i\cos\left(\frac{\gamma'}{16}\right)} \right).$$

Here, we have used Eqs. (76), (74), (33), (58) $\gamma' = 8\pi\omega/(\pi-\omega)$, $\theta = \pi\beta/2(\pi-\omega)$. Thus, the S matrix for scattering of an ordinary particle on a soliton is equal to the known (82)

$$S^1(\theta) = -[\mathrm{sh}\,\theta + i\cos(\gamma'/16)]/[\mathrm{sh}\,\theta - i\cos(\gamma'/16)].$$

One can again calculate $\delta_h(\beta)$ as the phase shift for the scattering of a hole β_3 on a particle β_1. We generalize the definition (71) as follows. We consider those pseudoparticles in the vacuum whose rapidities are closest to β_3, i.e., pseudoparticles with rapidities β_{n_0-1}, β_{n_0+1} (63), and we calculate the additional phase that they acquire in the presence of the introduced pseudoparticle: $\varphi = \varphi_2 - \varphi_1$. Since the momenta of a hole and a particle in the sea are opposite, the sign on the right-hand side of (71) must be changed. Thus, in the case of a hole

$$\frac{1}{i} \ln S = \varphi_1 - \varphi_2. \tag{77}$$

Direct calculation in the framework of this definition leads to the above expression for $S^1(\theta)$.

The S matrix for scattering of several particles on a soliton factorizes because of the linearity. From this there already follows the correct formula for the S matrix for scattering of bound states on a soliton [12]. Calculations similar to those made at the end of Sec. 3, using (44), (45), and (55), lead to the following form for the S matrix for scattering of a bound state of $n = n_1$ particles on a soliton:

$$S^n(\theta) = \prod_{p=0}^{n-1} S^1\left(\theta + i\frac{\gamma'}{16}(n-1-2p)\right).$$

This expression agrees with (83).

5. S Matrix of Solitons

The phase shift for scattering of a soliton on an antisoliton, $(1/i)\ln S$ (77), is the additional phase that the soliton acquires in the presence of the antisoliton. We consider the eigenfunction of the Hamiltonian corresponding to the soliton–antisoliton configuration (see Fig. 2a). We recall that this configuration consists of two holes β_1 and β_2 and a bound state of n_2 (13) pseudoparticles, $\beta_* = (\beta_1+\beta_2)/2$. To calculate the phase shift, we use the definition (77). In accordance with the definition, $(1/i)\ln S$ is the difference between the two phases φ_1 and φ_2, where φ_1 is the phase collected by a free hole on the passage around the contour. The phase φ_2 is the phase which the hole collects on the passage around the contour in the presence of the other hole and the bound state of n_2 pseudoparticles. By the phase of the hole, we mean the phase that is collected by a pseudoparticle from the negative sea whose rapidity tends to β_1.

We calculate first φ_2:

$$\varphi_2 = -m_0 L\,\mathrm{sh}\,\beta_1 + \sum_{\beta_k \neq \beta_1, \beta_2} \Phi(\beta_1 - \tilde{\beta}_k) + \Phi_n\left(\beta_1 + i\pi - \frac{\beta_1+\beta_2}{2}\right).$$

Here, $\tilde{\tilde{\beta}}_k$ are solutions of Eq. (68). The first term here is the free phase; the second term is due to scattering on the particles that constitute the sea; the third term is the direct phase shift for scattering on the bound state of $n = n_2$ particles. The quantity $\tilde{\tilde{\beta}}_k$ differs from β_k by an amount of order $1/L$. We expand the second term in a series in this difference:

$$\varphi_2 = -m_0 L\,\mathrm{sh}\,\beta_1 + \sum_k \Phi(\beta_1 - \beta_k) + \Phi_n\left(\beta_1 + i\pi - \frac{\beta_1+\beta_2}{2}\right) - \Phi(\beta_1 - \beta_2) + \sum_k \Phi'(\beta_1 - \beta_k)(\beta_k - \tilde{\beta}_k).$$

The last term here can be written as (68):

$$\sum_k \Phi'(\beta_1-\beta_k)(\beta_k-\tilde{\beta}_k)=\int \Phi'(\beta_1-\alpha)F_{**}(\alpha)\,d\alpha.$$

Finally (68)

$$\varphi_2=\left\{-m_0L\,\mathrm{sh}\,\beta_1+\sum \Phi(\beta_1-\beta_k)+\int \Phi'(\beta_1-\alpha)f_h(\alpha-\beta_1)\,d\alpha\right\}+\delta_1(\beta_1-\beta_2)+\delta_2(\beta_1-\beta_2). \tag{78}$$

The expression in the curly brackets depends only on β_1:

$$\delta_1(\beta)=\Phi(\beta)-\int\limits_{-\infty}^{\infty} d\alpha\,\Phi(\beta-\alpha)f_h'(\alpha),\quad \delta_2(\beta)=-\Phi_n\left(\frac{\beta_1-\beta_2}{2}+i\pi\right)-\int \Phi'(\beta_1-\alpha)f_n\left(\alpha-\frac{\beta_1+\beta_2}{2}\right)d\alpha=2\pi f_n\left(\frac{\beta_1-\beta_2}{2}\right)$$

(see (16), (22), (56), (66), and (53)).

Thus, we have have calculated φ_2. We shall not calculate φ_1 but use the fact that φ_1 depends only on β_1, since this is the free phase. We differentiate the expression $-\mathrm{i}\ln S = \varphi_1 - \varphi_2$ (77) with respect to β_2. Then φ_1 will not contribute to the derivative, as in the case of the expression in the curly brackets in (78). Finally, we obtain

$$\delta_1'(\beta)=\Phi'(\beta)-\int\limits_{-\infty}^{\infty} d\alpha\,\Phi'(\beta-\alpha)f_h'(\alpha),\quad \delta_2'(\beta)=2\pi\frac{d}{d\beta}f_n\left(\frac{\beta}{2}\right).$$

The phase δ_1 is the hole–hole phase shift. The phase δ_2 is the phase shift for scattering of a hole on the bound state of $n = n_2$ (13) pseudoparticles. The value of δ_2 can be calculated to the end (59):

$$e^{i\delta_2(\beta)}=\frac{\mathrm{ch}\left(\frac{\pi}{4\omega}\beta+i\frac{\pi^2}{2\omega}\right)}{\mathrm{ch}\left(\frac{\pi}{4\omega}\beta-i\frac{\pi^2}{2\omega}\right)},\quad n\ \text{even},\qquad e^{i\delta_2(\beta)}=\frac{\mathrm{sh}\left(\frac{\pi\beta}{4\omega}+i\frac{\pi^2}{2\omega}\right)}{\mathrm{sh}\left(\frac{\pi\beta}{4\omega}-i\frac{\pi^2}{2\omega}\right)},\quad n\ \text{odd}.$$

If we bear in mind that $\theta=\pi\beta/2(\pi-\omega)$, $\gamma'=8\pi\omega/(\pi-\omega)$ (33), (58), it is clear that $\exp\{i\delta_2(\beta)\}$ is equal to $U(\theta)$ (85). With regard to $\delta_1(\beta)$, it is easiest to calculate its Fourier transform:

$$\int e^{i\beta k}\frac{d\delta_1(\beta)}{d\beta}d\beta=\Phi'(k)-\Phi'(k)f_h'(k)=-\pi\frac{\mathrm{sh}((\pi-2\omega)k)}{\mathrm{sh}(k\omega)\,\mathrm{ch}((\pi-\omega)k)}$$

(see (45), (66)).

If we compare these expressions with (85) and (86), we see that we have obtained an expression identical to the known expression for the soliton S matrix. It is clear that from our calculations we have not found the constant factor of the S matrix. The common factor can be obtained either by means of Levinson's theorem [12] or by analyzing the residues of the S matrix. The S matrix of several solitons will be factorized because of the linearity of all the obtained equations.

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Appendix

We describe the well-known results in the sine–Gordon model. The Lagrangian of the sine–Gordon model has the form

$$\mathscr{L}=\frac{1}{\gamma}\left[\frac{1}{2}(\partial_\mu u)^2-\mu^2(1-\cos u)\right].$$

To describe the quantum results in this model, it is convenient to introduce $\gamma' = 8\pi\gamma/(8\pi - \gamma)$. We denote the soliton mass by M_s. The mass of the bound state of n basic particles has the form

$$M_n=2M_s\sin\left(\frac{\gamma'}{16}n\right). \tag{79}$$

The state with $n = 1$ is a basic particle.

The S matrix for two basic particles has the form [10] basic particles

$$S^{11}(\theta_1-\theta_2)=\frac{\operatorname{sh}(\theta_1-\theta_2)+i\sin\left(\frac{\gamma'}{8}\right)}{\operatorname{sh}(\theta_1-\theta_2)-i\sin\left(\frac{\gamma'}{8}\right)},\quad \theta_1>\theta_2. \tag{80}$$

The energy and momentum of these particles are $E = M_1 \cosh\theta_1 + M_1\cosh\theta_2$ and $P = M_1\sinh\theta_1 + M_1 \sinh\theta_2$. The S matrix for scattering of a bound state of n basic particles on a bound state of m basic particles [9, 8] can be represented in the form

$$S^{n,m}(\theta_{12})=\prod_{p=0}^{n-1}\prod_{q=0}^{m-1}S^{11}\left(\theta_{12}+i\frac{\gamma'}{16}(n-m+2q-2p)\right),\quad \theta_{12}=\theta_1-\theta_2>0. \tag{81}$$

The S matrix for scattering of a basic particle on a soliton has the form

$$S_1(\theta_{12})=-\frac{\operatorname{sh}\theta_{12}+i\cos\left(\frac{\gamma'}{16}\right)}{\operatorname{sh}\theta_{12}-i\cos\left(\frac{\gamma'}{16}\right)}. \tag{82}$$

The S matrix for scattering of a bound state of n basic particles on a soliton [8, 9] can be represented in the form

$$S^n(\theta_{12})=\prod_{p=0}^{n-1}S^1\left(\theta_{12}+i\frac{\gamma'}{16}(n-1-2p)\right). \tag{83}$$

The S matrix for scattering on an antisoliton is the same. The expressions (79), (81), and (83) indicate that the bound state of n basic particles can be represented as n noninteracting particles propagating together with complex rapidities

$$\theta_p=B+i\frac{\gamma'}{16}(n-1-2p),\quad \operatorname{Im}B=0,\quad p=0,1,\ldots,n-1. \tag{84}$$

We consider the soliton–antisoliton S matrix. In this case, the reflection coefficient r is nonzero. We denote the transmission coefficient by t. It is easiest to describe the S matrix by considering a soliton–antisoliton state with definite parity (symmetrized or antisymmetrized wave function). In each of these states, the S matrix reduces to a pure phase shift. We denote these S matrices by $S_\pm$. It is known that $S_\pm = t \pm r$. In the considered model,

$$S_\pm(\theta)=u_\pm(\theta)S(\theta),\quad u_+(\theta)=\frac{\operatorname{sh}\frac{4\pi}{\gamma'}(i\pi+\theta)}{\operatorname{sh}\frac{4\pi}{\gamma'}(i\pi-\theta)},\quad u_-(\theta)=-\frac{\operatorname{ch}\frac{4\pi}{\gamma'}(\theta+i\pi)}{\operatorname{ch}\frac{4\pi}{\gamma'}(\theta-i\pi)},$$
$$S(\theta)=\exp\left\{-\int_0^\infty\frac{dx}{x}\frac{\operatorname{sh}\left(\left(\frac{4\pi}{\gamma'}-\frac{1}{2}\right)x\right)\operatorname{sh}\left(i\theta\frac{8}{\gamma'}x\right)}{\operatorname{sh}\left(\frac{x}{2}\right)\operatorname{ch}\left(\frac{4\pi}{\gamma'}x\right)}\right\}. \tag{85}$$

The Fourier transform of the logarithmic derivative of $S(\theta)$ with respect to the unrenormalized rapidity has the particularly simple form (33)

$$\frac{1}{i}\int_{-\infty}^{\infty}e^{i\beta k}\frac{d\ln S(\theta)}{d\beta}d\beta=-\pi\frac{\operatorname{sh}((\pi-2\omega)k)}{\operatorname{sh}(k\omega)\operatorname{ch}((\pi-\omega)k)}. \tag{86}$$

Finally, we note that all the many-particle S matrices reduce to products of two-particle S matrices.

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ANOMALIES AND ELLIPTIC OPERATORS

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The coefficients of the asymptotic expansion of Sp exp(−tA) in the limit $t \to 0$ for the operators of quantum field theory are calculated and it is shown how the obtained results can be applied to the calculation of the axial and conformal anomalies, charge renormalization in gauge theories, and to the investigation of two-dimensional electrodynamics.

1. The so-called anomalies in quantum field theory have recently attracted much interest. It is customary to speak of an anomaly if the quantum expectation value of some quantity differs from the naively assumed value and the difference cannot be eliminated by renormalizations.

In the present paper, we shall investigate anomalies and renormalizations in the single-loop approximation by means of the methods of the theory of elliptic operators. Namely, we use the fact that for an elliptic operator in the limit $t \to 0$ there holds the asymptotic expansion

$$\langle x|\exp(-tA)|x\rangle \approx \sum_k \psi_k(x|A)t^{-k}.$$

The coefficients $\psi_k(x|A)$ can be calculated by quasiclassical methods; for their calculation, it is convenient to use the algorithm developed in [1]. This algorithm will be briefly described in Sec. 2. In Sec. 3, we give the coefficients $\psi_k(x|A)$ for the operators encountered in quantum field theory (the calculation of these coefficients must be regarded as the main result of the paper). In Sec. 4 we show how it is possible to calculate the anomaly of the axial current and conformal anomalies by means of the results of Sec. 3. Finally, these results will be used to calculate the renormalization of the coupling constant in gauge theories (Sec. 5) and to study the Schwinger model – two-dimensional massless electrodynamics (Sec. 6).

In the present paper, all fields are assumed to be smooth and defined, not in the physical space with pseudo-Riemannian metric, but on compact manifolds with positive-definite metric (in other words, we assume that the so-called Euclidean rotation has been made). This enables us to go over from hyperbolic operators in the pseudo-Riemannian space to elliptic operators on Riemannian manifolds.

2. Let A be a differential operator of m-th order on the compact n-dimensional manifold M:

$$A=\sum_{|\alpha|\leqslant m} a_\alpha(x)D^\alpha,$$

where $x \in M$, $x_1, \ldots, x_n$, are local coordinates, $\alpha=(\alpha_1,\ldots,\alpha_n)$ is a multiple index, $|\alpha|=\alpha_1+\ldots+\alpha_n$,

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