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A THREE-INSTANTON SOLUTION

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ABSTRACT. The Yang-Mills field is studied in the case of the $SU(2)$ algebra. An explicit three-instanton solution is constructed. This solution is a rational function of free real parameters which vary in $R^{16} + R^5_+$.

Bibliography: 13 titles.

Introduction

A duality equation was derived in [1] and [2], and the physical interpretation of its solutions (instantons) was clarified and the simplest one-instanton solution was obtained. The general N-instanton solution was obtained in [3]–[7].

A quaternion formulation of such a solution was given in the lecture notes [8]. We recall that an N-instanton solution for the $SU(2)$ algebra depends on $8N - 3$ independent real parameters (see [9] and [10]). The explicit form of a three-instanton solution was obtained in [11], but the corresponding parametrization was not rational. The general three-instanton solution constructed in the present paper is a rational function of free real parameters which vary in $R^{16} + R^5_+$.

We note that the notation of [12] is most convenient in the present context.

The present paper consists of two sections. The algebraic-geometry construction of instantons is briefly explained in the first section. The explicit parametrization of the three-instanton solution is described in the second section.

§1. The construction of Atiyah, Drinfel'd, Manin, Hitchin and Ward

We shall follow the exposition of [12]. We consider the duality equation

$$F_{\mu\nu}(x) = -{}^*F_{\mu\nu}(x) = -\tfrac{1}{2}\sum \epsilon_{\mu\nu\lambda\gamma}F_{\lambda\gamma}(x)$$

(summation over repeated indices is assumed unless stated otherwise). Here

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + \left[A_\mu(x), A_\nu(x)\right].$$

It follows from the conformal invariance that the field $A_\mu(x)$ can be regarded as defined on the sphere S^4 rather than R^4; the quantities x_μ are the stereographic coordinates on the sphere.

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We consider the Yang-Mills field in the case of SU(2). Let us pass from S^4 to C^4. It is convenient to use quaternions q in writing explicit formulas. They will be regarded as 2×2 matrices defined by

$$q = a_4 + i\sigma_1 a_1 + i\sigma_2 a_2 + i\sigma_3 a_3.$$

Here $a_1, \ldots, a_4$ are real numbers, i is the imaginary unit, and σ_j are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The quantity a_4 will be called the real part of a quaternion. We assign each point in S^4 a quaternion

$$\hat{x} = x_4 + i \sum_{j=1}^{3} \sigma_j x_j.$$

A point in C^4 is assigned two quaternions:

$$q_1 = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}, \quad q_2 = \begin{pmatrix} z_3 & z_4 \\ -\bar{z}_4 & \bar{z}_3 \end{pmatrix}.$$

The quantities z_i are the coordinates of a point in C^4. Each point of C^4 is assigned a point of S^4 by

$$\hat{x} = q_2^{-1} q_1. \tag{1}$$

To clarify which object corresponds to a point in S^4, it is convenient to define an involution σ in C^4 which acts as follows:

$$(z_1, z_2, z_3, z_4) \xrightarrow{\sigma} (-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3).$$

It can be shown that a σ-invariant two-dimensional plane in C^4 passing through the origin corresponds to a point in S^4. The equation of such a plane is given by (1).

Since multiplication of each z_i by a common complex factor does not change x_μ in (1), equation (1) is simply the projection of a three-dimensional projective complex space CP^3 onto S^4.

We define a field $C_i(z, \bar{z})$ in C^4:

$$C_i(z, \bar{z}) = \sum \frac{\partial x_\mu}{\partial z_i} A_\mu(x). \tag{2}$$

The strength of the field

$$F_{ik}(z, \bar{z}) = \frac{\partial C_k}{\partial z_i} - \frac{\partial C_i}{\partial z_k} + [C_i, C_k]$$

can be expressed in terms of $F_{\mu\nu}(x)$:

$$F_{ik}(z, \bar{z}) = \sum \frac{\partial x_\mu}{\partial z_i} \frac{\partial x_\nu}{\partial z_k} F_{\mu\nu}(x) = \sum \frac{D(x_\mu, x_\nu)}{D(z_i, z_k)} F_{\mu\nu}(x),$$

where

$$\frac{D(x_\mu, x_\nu)}{D(z_i, z_k)} = \frac{1}{2}\left(\frac{\partial x_\mu}{\partial z_i} \frac{\partial x_\nu}{\partial z_k} - \frac{\partial x_\nu}{\partial z_i} \frac{\partial x_\mu}{\partial z_k} \right).$$

This quantity has the following important property:

$$\frac{D(x_\mu, x_\nu)}{D(z_i, z_k)} = \frac{1}{2} \sum \epsilon_{\mu\nu\lambda\gamma} \frac{D(x_\lambda, x_\gamma)}{D(z_i, z_k)}.$$

For self-dual solutions $F_{\mu\nu}(x) = -{}^*F_{\mu\nu}(x)$, we obtain

$$F_{ik}(z,\bar{z}) = \frac{1}{2}\sum \frac{D(x_\mu, x_\nu)}{D(z_i, z_k)}\left(F_{\mu\nu}(x) + {}^*F_{\mu\nu}(x)\right) = 0. \tag{3}$$

Equation (3) expresses the content of the Atiyah-Ward theorem, i.e., fields with $F_{ik} = 0$ in CP^3 correspond to self-dual fields in S^4.

The solution of (3) has the form

$$C_i(z,\bar{z}) = \psi(z,\bar{z})\frac{\partial}{\partial z_i}\psi^{-1}(z,\bar{z}).$$

However, we are interested in C_i which can be represented in the form (2), i.e., the above solution should be projected into S^4.

The problem of projecting the solution of the equation $F_{ik} = 0$ in CP^3 into self-dual fields in S^4 can be easily solved using the Atiyah-Drinfel'd-Manin-Hitchin construction. It is necessary in this construction to consider a vector space L of dimension $2N+2$ in order to obtain the solution of the duality equation with a topological charge N for the group SU(2). The inner product (L_1, L_2) of two vectors L_1 and L_2 in L is linear in L_2 and antilinear in L_1:

$$\begin{gathered}(L_1, L_2) = \overline{(L_2, L_1)},\\ (L_1, cL_2) = c(L_1, L_2), \quad (cL_1, L_2) = \bar{c}(L_1, L_2), \qquad (L, L) \geqslant 0.\end{gathered} \tag{4}$$

Choosing two orthonormal vectors $E_a(z,\bar{z})$, $a = 1, 2$ in L, $(E_a, E_b) = \delta_{ab}$, we define $C_i^{ab}(z,\bar{z})$ by

$$C_i^{ba}(z,\bar{z}) = -\left(E_a, \frac{\partial}{\partial z_i}E_b\right). \tag{5}$$

It is clear that (5) can be projected provided $E_a(z,\bar{z})$ is constant on σ-invariant planes in C^4 (i.e., if E_a depends only on x). In that case, we obtain

$$C_i^{ba} = -\sum\frac{\partial x_\mu}{\partial z_i}\left(E_a(x), \frac{\partial}{\partial x_\mu}E_b(x)\right), \qquad A_\mu^{ba} = -\left(E_a(x), \frac{\partial}{\partial x_\mu}E_b(x)\right).$$

We first construct the orthogonal complement to the vectors E_a in L. We consider two sequences of linearly independent vectors in L: $\eta_\alpha^1(z)$ and $\eta_\alpha^2(\bar{z})$, $\alpha = 1, \ldots, N$. We assume that they have the following properties:

1) $\eta_\alpha^1(z)$ is a linear function of z, and $\eta_\alpha^2(\bar{z})$ is a linear function of $\bar{z}$.

2) Each vector of the first sequence transforms to a vector of the second sequence under the involution σ:

$$\eta_\alpha^1(\sigma(z)) = \eta_\alpha^2(\bar{z}), \qquad \alpha = 1, \ldots, N.$$

3) Any vector from the first sequence is orthogonal to any vector from the second sequence:

$$\left(\eta_\alpha^1, \eta_\beta^2\right) = 0.$$

Since the dimension of L is equal to $2N+2$, we can find two vectors E_a such that

$$(E_a, E_b) = \delta_{ab}, \qquad (E_a, \eta_\alpha^1) = (E_a, \eta_\alpha^2) = 0. \tag{6}$$

We shall now verify that the two vectors E_1 and E_2 thus obtained are constant on σ-invariant planes; it follows from requirements 1), 2), and 3) that

$$\eta^1_\alpha = z_1 t_{\alpha_1} + z_2 t_{\alpha_2} + z_3 t_{\alpha_3} + z_4 t_{\alpha_4}, \quad \eta^2_\alpha = -\bar{z}_2 t_{\alpha_1} + \bar{z}_1 t_{\alpha_2} - \bar{z}_4 t_{\alpha_3} + \bar{z}_3 t_{\alpha_4}, \tag{7}$$

where the t_{α_i} are constant vectors. We shall combine the vectors η^1_α and η^2_α into pairs

$$\eta_\alpha = \begin{pmatrix} \eta^1_\alpha \\ \eta^2_\alpha \end{pmatrix}.$$

Hence

$$\eta_\alpha = q_1 t^1_\alpha + q_2 t^2_\alpha, \qquad t^1_\alpha = \begin{pmatrix} t_{\alpha_1} \\ t_{\alpha_2} \end{pmatrix}, \; t^2_\alpha = \begin{pmatrix} t_{\alpha_3} \\ t_{\alpha_4} \end{pmatrix}.$$

Multiplying η_α from the left by q_2^{-1}, we obtain a pair of vectors h^1_α, h^2_α which are linear combinations of η^1_α and η^2_α:

$$h_\alpha = \begin{pmatrix} h^1_\alpha \\ h^2_\alpha \end{pmatrix} = \bar{x} t^1_\alpha + t^2_\alpha \tag{8}$$

and depend only on x.

The vectors E_α are orthogonal to h^1_α and h^2_α, and therefore they also depend only on x and are constant on σ-invariant planes in C^4.

Finally, we shall show that the fields $C_i(z, \bar{z})$ defined by (5) constructed from such vectors E_α are solutions of the equation $F_{rk} = 0$.

For $\partial_i E_\alpha$ we obtain

$$\partial_i E_a = -\sum C_i^{ab} E_b + \sum X_i^{a\alpha} \eta^1_\alpha(z) + \sum Y_i^{a\alpha} \eta^2_\alpha(\bar{z}). \tag{9}$$

Since η^1_α depends only on z, we have

$$(\eta^1_\alpha(z), \partial_i E_a) = \frac{\partial}{\partial z_i}(\eta^1_\alpha(z), E_a) = 0.$$

We have made use of the antilinear property of the inner product defined by (4).

Multiplying (9) from the left by $\eta^1_\alpha(z)$, we obtain $X_i^{a\alpha} = 0$. Differentiating (9) with respect to z_k, we obtain

$$\partial_k \partial_i E_a = -\sum \partial_k C_i^{ab} E_b + \sum C_i^{ad} C_k^{db} E_b + \sum Y_{i,k}^{a\alpha} \eta^2_\alpha(\bar{z}).$$

Making use of the symmetry $\partial_i \partial_k E_a = \partial_k \partial_i E_a$ and the orthogonality of E_a and η^2_α, we obtain

$$(E_b, \partial_k \partial_i E_a - \partial_i \partial_k E_a) = (F_{ik})^{ab} = 0.$$

It follows that C_i is a solution of the equation $F_{ik} = 0$ and it can be projected onto S^4; the Atiyah-Ward theorem then implies that A_μ is a solution of the duality equation. By the Atiyah-Drinfel'd-Manin-Hitchin theorem, the fields A_μ constructed by this method represent the complete solution of the duality equation. We shall not prove the completeness of the solutions but merely show that they depend on $8N - 3$ parameters.

We choose an orthonormal basis e_i^a in L, where $a = 1, 2$ and $i = 1, \ldots, N + 1$. The pair of vectors η_α can then be written as

$$\eta_\alpha = \sum_{i=1}^{N+1} (q_1 A_{\alpha i} + q_2 B_{\alpha i}) e_i, \qquad e_i = \begin{pmatrix} e_i^1 \\ e_i^2 \end{pmatrix}. \tag{10}$$

Here, A and B are $N \times (N+1)$ matrices with respect to the indices α and i and their matrix elements $A_{\alpha i}$ and $B_{\alpha i}$ are quaternions. The form of (10) ensures that conditions 1) and 2) are satisfied. Condition 3) is satisfied for $\alpha = \beta$ since $A_{\alpha i}$ and $B_{\alpha i}$ are quaternions. The condition $(\eta^1_\alpha, \eta^2_\beta) = 0$ for $\alpha < \beta$ imposes the following restrictions:

$$\sum_{i=1}^{N+1} A_{\alpha i} A^+_{\beta i} = \lambda_{\alpha\beta} I, \quad \sum_{i=1}^{N+1} B_{\alpha i} B^+_{\beta i} = \mu_{\alpha\beta} I, \quad \sum_{i=1}^{N+1} A_{\alpha i} B^+_{\beta i} = \sum_{i=1}^{N+1} A_{\beta i} B^+_{\alpha i}, \tag{11}$$

where $\alpha < \beta$; $\lambda_{\alpha\beta}$ and $\mu_{\alpha\beta}$ are real numbers.

We note that not all the parameters in the matrices $A_{\alpha i}$ and $B_{\alpha i}$ have an effect on E_1 and E_1. The structure of (10) remains unchanged upon the substitution

$$A_{\alpha i} \to A'_{\alpha i} = \sum G_{\alpha\beta} A_{\beta k} \mathscr{V}_{ki}, \qquad G \in GL(N),$$

$$B_{\alpha i} \to B'_{\alpha i} = \sum G_{\alpha\beta} B_{\beta k} \mathscr{V}_{ki}, \qquad \mathscr{V} \in \mathrm{Sp}(N+1),$$

where G is a numerical matrix and $\mathscr{V}$ is an $(N+1) \times (N+1)$ matrix whose matrix elements are quaternions and $\mathscr{V}\mathscr{V}^+ = I$. The group $GL(N)$ corresponds to linear transformations of the vectors η^1_α and η^2_α, and the group $SP(N+1)$ is the group of rotations of the orthonormal basis e_i.

The number of parameters in the matrices A and B is equal to $2 \cdot 4 \cdot N \cdot (N+1) = 8N^2 + 8N$, the number of parameters of the group $GL(N)$ is N^2, and the number of parameters of $Sp(N+1)$ is $2(N+1)^2 + (N+1)$; the number of conditions imposed on the parameters $A_{\alpha i}$ and $B_{\alpha i}$ in (11) is $(5N^2 - 5N)$. Finally, the number of independent parameters in the solution (10) is equal to

$$8N^2 + 8N - N^2 - 2(N+1)^2 - (N+1) - 5N^2 + 5N = 8N - 3,$$

i.e., it follows from the number of parameters that the solution (10) corresponds to the complete solution of the duality equation.

§2. Explicit form of three-instanton solution

Using the transformations from $GL(N)$ and $Sp(N+1)$, we can simplify (11).

Let us consider a quantity u_α defined by

$$u_\alpha = \begin{pmatrix} u^1_\alpha \\ u^2_\alpha \end{pmatrix} = \sum_{i=1}^{N+1} A_{\alpha i} e_i.$$

It is clearly possible to find a matrix $G \in GL(N)$ such that

$$u_\alpha \to \hat{e}_\alpha = \sum_{\beta=1}^{N} G_{\alpha\beta} u_\beta \quad \text{and} \quad (\hat{e}^a_\alpha, \hat{e}^b_\beta) = \delta_{\alpha\beta}\delta_{ab}.$$

We then obtain for the pair η_α

$$\eta_\alpha \to \hat{\eta}_\alpha = q_1 \hat{e}_\alpha + q_2 \sum G_{\alpha\beta} B_{\beta i} e_i. \tag{12}$$

We note that the matrix $\lambda_{\alpha\beta}$ in (11) now reduces to $\delta_{\alpha\beta}$.

We introduce a basis vector $\hat{e}_{N+1}$ normalized to unity (i.e., $(\hat{e}^a_{N+1}, \hat{e}^b_{N+1}) = \delta_{ab}$) and orthogonal to all the $\hat{e}_\alpha$. We can then expand the second term in (12) in terms of the orthonormal basis $(\hat{e}_1, \ldots, \hat{e}_{N+1})$, which yields

$$\hat{\eta}_\alpha = q_1 \hat{e}_\alpha + q_2 \sum_{i=1}^{N+1} \hat{B}_{\alpha i} \hat{e}_i. \tag{13}$$

where the $\hat{B}_{\alpha i}$ are new quaternions. For $\hat{\eta}_\alpha$ expressed in the notation of (10), we obtain

$$\hat{A}_{\alpha i} = \delta_{\alpha i}. \tag{14}$$

For $\hat{A}_{\alpha i}$ defined by (14), the first equation of (11) is satisfied trivially with $\lambda_{\alpha\beta} = 0$ for $\alpha < \beta$, the third equation in (11) yields

$$\hat{B}_{\alpha\beta} = \hat{B}_{\beta\alpha},$$

and the second yields

$$\sum_{i=1}^{N+1} \hat{B}_{\alpha i}\hat{B}_{\beta i}^{+} = \hat{\mu}_{\alpha\beta} I. \tag{15}$$

We omit $\alpha < \beta$ since (15) becomes an identity for $\alpha = \beta$ (for $\alpha = \beta$, the left-hand side of (15) is proportional to a unit matrix; the proportionality factor is denoted by $\hat{\mu}_{\alpha\alpha}$). The matrix $\hat{\mu}$ is a real symmetric $N \times N$ matrix with numerical matrix elements. We note that this is the form of the equations satisfied by the parameters of the N-instanton solution which was quoted in [8].

Let us now discuss the remaining symmetry groups. Firstly, the group $GL(N)$ has been reduced to $O(N)$, but, to preserve the form of $\hat{\eta}_\alpha$ given by (13), it is necessary not only to rotate $\hat{\eta}_\alpha$ but also to perform the same rotation of the basis (the group $Sp(N+1)$ contains $O(N)$ as a subgroup):

$$\hat{\eta}_\alpha \to \eta_\alpha = \sum_{\beta=1}^{N} V_{\alpha\beta}\hat{\eta}_\beta, \qquad V \in O(N),$$

$$\hat{e}_\alpha \to e_\alpha = \sum_{\beta=1}^{N} V_{\alpha\beta}\hat{e}_\beta, \qquad \hat{e}_{N+1} \to \hat{e}_{N+1} = e_{N+1}.$$

As a result of such rotations, we obtain

$$\hat{\eta}_\alpha \to \eta_\alpha = q_1 e_\alpha + q_2 \sum_{i=1}^{N+1} B_{\alpha i} e_i, \tag{16.1}$$

$$B_{\alpha\beta} = \sum V_{\alpha\gamma}\hat{B}_{\gamma\lambda}V_{\lambda\beta}^{-1}, \qquad B_{\alpha, N+1} = \sum V_{\alpha\beta}\hat{B}_{\beta, N+1}. \tag{16.2}$$

It follows from (15) that the new parameters $B_{\alpha i}$ satisfy

$$\sum_{i=1}^{N+1} B_{\alpha i}B_{\beta i}^{+} = \mu_{\alpha\beta} I, \qquad \mu = V\hat{\mu}V^{-1}.$$

We now choose the matrix V. Let V be a matrix which diagonalizes the real symmetric matrix $\hat{\mu}$:

$$\mu = V\hat{\mu}V^{-1} = \mathrm{diag}(\mu_1, \ldots, \mu_N).$$

Such a choice reduces $O(N)$ to the group of reflections, i.e., the transformations defined by (16) with a diagonal matrix

$$V_{\alpha\beta} = \delta_{\alpha\beta}(-1)^{k(\alpha)}. \tag{17}$$

are admissible and the diagonal matrix elements of the matrix V are powers of (-1). We shall employ the group of reflections to transform the real parts of certain quaternions into positive quantities.

In the basis defined by (16), the group $Sp(N+1)$ reduces to the group of rotations of the $(N+1)$th axis

$$G = I, \qquad \mathscr{V} = \begin{pmatrix} I & 0 \\ 0 & m \end{pmatrix}, \qquad e_{N+1} \to e'_{N+1} = m e_{N+1},$$

where m is a quaternion whose modulus is equal to 1, $mm^+ = I$. The aforementioned group acts on the matrix B_{ai} as follows:

$$B_{ai} \to B'_{ai} = \begin{cases} B_{a,N+1} m^{-1}, & i = N+1, \\ B_{ai}, & i = 1, \ldots, N. \end{cases} \tag{18}$$

We choose m from the condition

$$B'_{1,N+1} = b_{1,N+1} I,$$

where $b_{1,N+1}$ is a real positive number. Having eliminated the arbitrariness due to symmetry groups, we find that the system (11) assumes the form

$$\sum_{i=1}^{N+1} B_{ai} B^+_{\beta i} = 0, \quad \alpha < \beta, \qquad B_{\alpha\beta} = B_{\beta\alpha}, \qquad B_{1,N+1} = b_{1,N+1} I. \tag{19}$$

Explicit parametrization of instantons can be obtained from the solution of (19). It is then necessary to take care that the vectors ψ^1_a and ψ^2_a defined by (7) are indeed linearly independent. The dimension of the space spanned by these vectors should be $2N$.

Let us first consider the case of two instantons: $N = 2$. The system (19) assumes the form

$$B_{11} B^+_{12} + B_{12} B^+_{22} + b_{13} B^+_{23} = 0. \tag{20}$$

The quaternions B_{12}, B_{22}, and B_{23} and the number b_{13} can be regarded as free parameters. The quaternion B_{11} can be obtained explicitly from (20):

$$B_{11} = -(B_{12} B^+_{22} + b_{13} B^+_{23})(B^+_{12})^{-1}.$$

An explicit expression for $A_\mu(x)$ is discussed at the end of this section. We have thus constructed the complete two-instanton solution depending on 13 real parameters. It is shown in Appendix 1 that the corresponding vectors ψ^1_a and ψ^2_a are linearly independent in the most general situation. The dimension of the space of parameters for which the vectors ψ_a are linearly dependent is equal to 9. It follows that the corresponding codimension is equal to 4, which is a sufficient condition for the set of parameters to be doubly connected.

We now discuss the true three-instanton case. The system (19) assumes the form

$$B_{11} B^+_{12} + B_{12} B^+_{22} + B_{13} B^+_{23} + b_{14} B^+_{24} = 0, \tag{21}$$

$$B_{11} B^+_{13} + B_{12} B^+_{23} + B_{13} B^+_{33} + b_{14} B^+_{34} = 0, \tag{22}$$

$$B_{12} B^+_{13} + B_{22} B^+_{23} + B_{23} B^+_{33} + B_{24} B^+_{34} = 0. \tag{23}$$

We choose the quaternions

$$B_{12}, \quad B_{13}, \quad B_{23}, \quad B_{22}, \quad B_{24} \tag{24}$$

and a number

$$b_{14} \geq 0 \tag{25}$$

as independent parameters. The remaining quaternions B_{11}, B_{33}, and B_{34} can be termined from the above system of equations. The quaternion B_{11} can be calcu from (21)

$$B_{11} = -(B_{12}B_{22}^{-} + B_{13}B_{23}^{-} + b_{14}B_{24}^{-})(B_{12}^{-})^{-1}$$

We note that the remaining equations (22) and (23) are linear in the quaternions B_{3}^{-} B_{14}^{-}, and also that these two quaternions are multiplied by other quaternions only fro left and, therefore, can be readily determined:

$$B_{33} = \left[(B_{23}B_{12}^{+} + B_{13}B_{11}^{-})\frac{B_{24}^{-}}{b_{14}} - B_{23}B_{22}^{-} - B_{13}B_{12}^{-}\right]\left(B_{23}^{+} - \frac{B_{13}^{+}B_{24}^{+}}{b_{14}}\right)^{-1},$$

$$B_{34} = -\frac{1}{b_{14}}(B_{23}B_{12}^{+} + B_{33}B_{13}^{+} + B_{13}B_{11}^{+}).$$

We note that the group of reflections defined by (17) and (16.2) can be used to achie

$$\operatorname{Re} B_{24} \geq 0, \qquad \operatorname{Re} B_{11} \geq 0.$$

Since each quaternion depends on 4 real parameters, we find that the total num independent real parameters is equal to 21, where three parameters assume values real semiaxis and the remaining 18 parameters assume values on the real axis. Equ (26)–(28) represent a parametrization in the general situation. Degenerate cases these equations are not applicable (for example, $B_{12} = 0$) are discussed in Append can be shown that the dimension of such a manifold of parameters is equal to 17. corresponding codimension is equal to 4. Therefore we can claim that the mani parameters is doubly connected.

It remains to verify that the corresponding vectors η_a defined by (7) are independent. We show in Appendix 1 that the vectors η_a are linearly independen general situation. The dimension of the manifold of parameters for which the ve are linearly independent is equal to 17. It follows that the corresponding codime equal to 4, which implies that the manifold of parameters remains doubly connecte

Finally, following the general method of constructing instanton solutions, we q explicit form of their coordinate representation:

$$A_{\mu}^{ab}(x) = \frac{1}{2i}\sum (A_{\mu}^{k}(x)\sigma_{k})^{ab} = -(E_{b}, \partial_{\mu}E_{a}), \qquad k = 1, 2, 3;\ a, b = 1, 2$$

where E_a is given by (6).

Let us now construct the vectors h_a using (8)

$$h_{n} = q_{2}^{\dagger}\eta_{a} = \bar{x}e_{n} + \sum_{r=1}^{4} B_{ar}e_{r}.$$

We consider a pair of vectors $E = \binom{E_1}{E_2}$:

$$E = \frac{\sum_{a=1}^{3} K_{a}e_{a} + e_{N+1}}{\sqrt{1 + \sum_{a=1}^{3}\det K_{a}}}, \qquad K_{a} = K_{a}^{4} + i\sum_{j=1}^{3}\sigma_{j}K_{a}^{j}.$$

Here the K_n are quaternions.

We assume that E_1 and E_2 are orthogonal to the vectors h_a^1 and h_a^2. This following equation for the matrix K_n:

$$\sum_{\beta=1}^{3}(\bar{x}\delta_{n\beta} + B_{n\beta})K_{\beta}^{+} = -B_{n4}.$$

The orthogonality of E_1 and E_2 is trivial since the K_α are quaternions. An explicit solution of (31) can be obtained; clearly such a solution is a rational function of independent parameters. The Yang-Mills fields are given by

$$A_\mu^\nu(x) = 2\sum \eta_{\mu\lambda\nu} \frac{\partial_\mu K_\alpha^\lambda \cdot K_\alpha^\nu}{1 + \Sigma K_\beta^\rho K_\beta^\rho}. \tag{32}$$

Here $\eta_{\mu\lambda\nu}$ is the 't Hooft tensor [13] given by

$$\eta_{\mu\lambda\nu} = \begin{cases} \varepsilon_{\mu\lambda\nu}, & \lambda, \nu = 1, 2, 3, \\ \delta_{\mu\lambda}, & \nu = 4, \\ -\delta_{\mu\nu}, & \lambda = 4, \\ 0, & \lambda = \nu = 4. \end{cases}$$

Equations (26)–(28) and (30)–(32) define the general form of the three-instanton solution in terms of independent parameters defined by (24), (25), and (29). The construction of the 't Hooft solution is given in Appendix 3. We note that the general solution $A_\mu(x)$ is a rational function of independent real parameters.

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Appendix 1

The dimension of the space spanned by the vectors η_α should be $2N$. However, the vectors η_α become linearly dependent for certain values of the free parameters defined by (24) and (25). We now determine the dimension of such a space of parameters.

First we discuss general problems. Since the vectors η_α^1 of the first series are orthogonal to the vectors η_α^2 of the second series, vectors belonging to different series cannot be linearly dependent. We shall now state the condition of linear dependence (we consider the case of n-fold degeneracy, i.e., a space spanned by the vectors η_α^1 is $(N-n)$-dimensional):

$$\sum_{\alpha=1}^{N} a_\alpha^{(k)} \eta_\alpha^1 = 0, \qquad k = 1, \dots, n.$$

There are only n different sets of the coefficients a_α, and they are labelled by the upper index k. We can bring the coefficients $a_\alpha^{(k)}$ to the canonical form $a_\alpha^{(k)} = \delta_\alpha^k$ for $\alpha = 1, \dots, n$. Such a normalization uniquely determines all the other coefficients $a_\alpha^{(k)}$. We can obtain other sets of a_α from the canonical set using arbitrary linear combinations $\Sigma a_\alpha^{(k)} c_k$ (the c_k are complex numbers).

It can be easily shown that the vectors η_α^2 also become linearly dependent:

$$\sum_{\alpha=1}^{N} \bar{a}_\alpha^{(k)} \eta_\alpha^2 = 0.$$

Here the $a_\alpha^{(k)}$ are complex numbers, and $\bar{a}_\alpha^{(k)}$ are their complex conjugates.

The above conditions can be written in the form

$$\sum_{\alpha=1}^{N} \hat{a}_\alpha^{(k)} \eta_\alpha = 0. \tag{33}$$

Here the $\hat{a}_\alpha^{(k)}$ are quaternions of the form

$$\hat{a}_\alpha^{(k)} = \begin{pmatrix} a_\alpha^{(k)} & 0 \\ 0 & \bar{a}_\alpha^{(k)} \end{pmatrix}. \tag{34}$$

It is easy to show that the linear dependence condition expressed in terms of the vectors h_a is analogous:

$$\sum_{a=1}^{N} \hat{b}_a^{(k)} h_a = 0. \quad (35)$$

Here the $\hat{b}_a^{(k)}$ are quaternions of the form (34). In fact, we can easily show that all the first components of the vectors h_a are orthogonal to all the second components of h_β. We also note that the vectors h_a are n times degenerate, i.e., there are only n sets of b_a such that condition (35) is satisfied together with the normalization condition $b_a^{(k)} = \delta_a^k$ (here, k assumes n different values).

We now show that the quaternions $\hat{a}_a^{(k)}$ and $\hat{b}_a^{(k)}$ are real. Equation (33) represents conditions on $a_a^{(k)}$ and on z_j, i.e., we require coefficients $a_a^{(k)}$ and points z_j in C^4 satisfying (33). Let us consider all the points in C^4 belonging to the same σ-invariant plane. If the coefficients $a_a^{(k)}$ can be found for at least one point of the plane, they exist for all the other points since all such points are projected onto the same point in S^4.

Let us consider a σ-invariant plane defined by

$$\hat{x} = q_2^{-1} q_1.$$

Here $\hat{x}$ is fixed. We shall parametrize the points of such a plane by means of the quaternion q_2. We use the relation $h_a = q_2^{-1} v_a$ to rewrite (33) in the form

$$\sum_{a=1}^{N} q_2^{-1} \hat{a}_a^{(k)} q_2 h_a = 0. \quad (36)$$

Comparing (36) and (35), we obtain $b_a^{(k)} = \delta_a^k$ for $a = 1, \ldots, n$, and

$$\hat{b}_a^{(k)} = q_2^{-1} \hat{a}_a^{(k)} q_2.$$

Here $\hat{a}_a^{(k)}$ and $\hat{b}(k)_a$ are fixed quaternions of the form (34), and q_2 is an arbitrary quaternion. Such a condition can be satisfied only if the quaternions $\hat{a}_a^{(k)}$ and $\hat{b}_a^{(k)}$ are real.

Finally, we find that the condition of linear dependence assumes the form

$$\sum_{a=1}^{N} a_a h_a = 0,$$

where the a_a are real numbers (we shall omit the index k).

Using the relation

$$h_a = \hat{x} e_a + \sum_{j=1}^{N+1} B_{aj} e_j,$$

we obtain

$$a_\beta \hat{x} + \sum_{a=1}^{N} a_a B_{a\beta} = 0, \qquad \sum_{a=1}^{N} a_a B_{a,N+1} = 0. \quad (37)$$

Let us first consider the case of two instantons. The system (37) assumes the form

$$a_1(\hat{x} + B_{11}) + a_2 B_{12} = 0, \quad a_1 B_{12} + a_2(\hat{x} + B_{22}) = 0, \quad a_1 b_{13} + a_2 B_{23} = 0. \quad (38)$$

We first consider the case $a_1 a_2 \neq 0$. We introduce a variable $c = a_1 a_2^{-1}$. Equation (38) yields

$$\hat{x} = -B_{22} - cB_{12}, \qquad B_{23} = -cb_{13}, \qquad B_{22} = B_{11} + (c^{-1} - c) B_{12}.$$

Substituting these expressions in (20), we obtain

$$B_{11}B_{12}^+ + B_{12}B_{11}^+ + B_{12}B_{12}^+(c^{-1} - c) - b_{13}^2 c = 0.$$

Regarding this as an equation for c, we find that it has two real solutions. We have thus found the solution of the orthogonality condition (20) and of the linear dependence condition (38); we choose the quaternions B_{11}, B_{12} and the number b_{13} as free parameters. It can be seen that these equations have a 9-parameter family of solutions. The complete two-instanton solution depends on 13 parameters. It follows that the corresponding codimension is equal to 4. The cases $a_1 = 0$ or $a_2 = 0$ can be discussed separately, and it can be shown that the corresponding codimension is also equal to 4.

We shall now consider the case of three instantons. Then (37) assumes the form

$$a_1(\dot{x} + B_{11}) + a_2 B_{12} + a_3 B_{13} = 0, \tag{39}$$

$$a_1 B_{12} + a_2(\dot{x} + B_{22}) + a_3 B_{23} = 0, \tag{40}$$

$$a_1 B_{13} + a_2 B_{23} + a_3(\dot{x} + B_{33}) = 0, \tag{41}$$

$$a_1 b_{14} + a_2 B_{24} + a_3 B_{34} = 0. \tag{42}$$

We shall seek the solution of this system together with the orthogonality conditions (21)–(23). We shall first consider the case $a_1 a_2 a_3 \neq 0$. Let us rewrite (39)–(41) in the form

$$-\dot{x} = B_{11} + \frac{a_2}{a_1}B_{12} + \frac{a_3}{a_1}B_{13} = B_{22} + \frac{a_1}{a_2}B_{12} + \frac{a_3}{a_2}B_{23} = B_{33} + \frac{a_1}{a_3}B_{13} + \frac{a_2}{a_3}B_{23}. \tag{43}$$

We now transform (22) and (23); we multiply (22) by a_3/a_2 and add it to (21); using (42), (43), and (22), we obtain

$$b_{14}^2 + |B_{12}|^2 + |B_{13}|^2 + |B_{11}|^2 = |\dot{x}|^2. \tag{44}$$

Similarly, we multiply (21) by a_1/a_2 and add it to the equation conjugate to (23), which yields

$$|B_{12}|^2 + |B_{22}|^2 + |B_{23}|^2 + |B_{24}|^2 = |\dot{x}|^2. \tag{45}$$

We now require the solution of (43), (42), (21), (44), and (45). We shall show that such a system of equations is equivalent to a single real equation involving 18 real parameters. We shall choose such parameters to be the four quaternions

$$B_{11},\quad B_{12},\quad B_{13},\quad B_{23}$$

and two real numbers

$$\frac{a_1}{a_2},\quad \frac{a_1}{a_3}.$$

We can determine B_{22} and B_{33} from (43). The quantity b_{14} can be calculated from (44). We evaluate B_{24} from (21). Using (42), we determine B_{34}. It follows that we have determined all the required quantities, and one real equation, namely (45), still remains. This procedure proves that the solution of the system (39)–(42), (21)–(23) depends on 17 real parameters. The general three-instanton solution depends on 21 parameters. It follows that the corresponding codimension is equal to 4. The cases when several a_α vanish can be treated separately. Such situations do not alter the value of the codimension.

Appendix 2

As already discussed, equations (26)–(28) determine the parametrization of the problem in the general situation. They contain singularities in the cases when any of the quaternions B_{12}, $B_{23} - (1/b_{14})B_{24}B_{13}$, and $b_{14}I$ are equal to zero. It is therefore necessary to investigate all the degenerate cases.

We shall discuss in detail the case when $B_{12} = 0$ and $b_{14} \neq 0$. Then (21)–(23) assume the form

$$B_{13}B_{23}^{+} + b_{14}B_{24}^{+} = 0,$$
$$B_{11}B_{13}^{+} + B_{13}B_{33}^{+} + b_{14}B_{34}^{+} = 0,$$
$$B_{22}B_{13}^{+} + B_{23}B_{33}^{+} + B_{24}B_{34}^{+} = 0.$$

It is easy to solve the above system: when $B_{23} \neq 0$, we can write

$$B_{34}^{+} = -\frac{1}{b_{14}}(B_{11}B_{13}^{+} + B_{13}B_{33}^{+}),$$
$$B_{22} = -(B_{23}B_{33}^{+} + B_{24}B_{34}^{+})(B_{13}^{+})^{-1}, \tag{46}$$
$$B_{24}^{+} = -\frac{1}{b_{14}}B_{13}B_{23}^{+},$$

if $B_{23} = 0$, then

$$B_{24} = 0, \qquad B_{34}^{+} = -\frac{1}{b_{14}}(B_{11}B_{13}^{+} + B_{13}B_{33}^{+}). \tag{47}$$

The quaternions

$$B_{11}, \quad B_{13}, \quad B_{23}, \quad B_{33}, \quad b_{14}I,$$

represent independent parameters for the solution defined by (46), and the quaternions

$$B_{11}, \quad B_{13}, \quad B_{22}, \quad B_{33}, \quad b_{14}I,$$

are independent parameters for the solution (47). In both cases, the number of free real parameters is equal to 17.

A similar approach can be used to study the case $b_{14} = 0$. It is possible to make the quaternion B_{24} (or B_{34}) real, $B_{24} = b_{24}I$, using the method that was applied in (18) to make B_{14} real; the number of parameters then reduces by 3. Solving (21)–(23) with $b_{14} = 0$ and $B_{24} = b_{24}I$, we find that the number of free parameters is then at most 17.

Finally, if

$$B_{23} = \frac{B_{24}B_{13}}{b_{14}},$$

we can determine from (21) and (22) the quaternions B_{22} and B_{34}:

$$B_{22} = -\left[B_{24}\left(\frac{|B_{13}|^2}{b_{14}} + b_{14}\right) + B_{12}B_{11}^{+}\right](B_{12}^{+})^{-1},$$
$$B_{34} = -\frac{1}{b_{14}}\left[B_{33}B_{13}^{+} + B_{13}B_{11}^{+} + \frac{1}{b_{14}}B_{24}B_{13}B_{12}^{+}\right].$$

The free parameters

$$B_{11}, \quad B_{12}, \quad B_{13}, \quad B_{24}, \quad B_{33}, \quad b_{14}.$$

which involve 21 real variables then satisfy the following quaternion equation (provided $B_{11} \neq 0$) which can be derived from (23):

$$B_{11}^{+} B_{12} B_{13}^{+} B_{24}^{+} (B_{13}^{+})^{-1} + B_{12}^{+} B_{24} B_{11} - b_{14} |B_{12}|^2 + \frac{1}{b_{14}} B_{12}^{-1} B_{24} B_{12} B_{13}^{+} B_{24}^{+} (B_{13}^{+})^{-1} (|B_{13}|^2 + |B_{12}|^2 + b_{14}^2) = 0.$$

We can use this equation, which is a linear equation for the components of B_{11}, to determine B_{11}^{+}, where $B_{11} = B_{11}^{+} + i\Sigma\sigma_j B_{11}^j$. It follows from our discussion that the number of free real parameters is at most 17. It can be verified that this assertion holds for all the degenerate cases.

We thus conclude that the solution of the duality equation at singular points of the general solution (26)–(28) depends on at most 17 free real parameters, i.e., the codimension is greater than or equal to four.

Appendix 3

The 't Hooft three-instanton solution depends on 15 parameters. We need to reduce our 21-dimensional manifold to a 15-dimensional manifold. The parameters B_{12}, B_{13}, B_{23}, B_{22}, B_{24}, and b_{14} then clearly become functions of 15 independent parameters.

Let $\hat{x}_\alpha$ be quaternions and λ_α real numbers, $\alpha = 1, 2, 3$. We introduce numbers ρ_α and a matrix W by

$$\rho_\alpha = \det \hat{x}_\alpha, \qquad W = \begin{pmatrix} \rho_1 + \lambda_1^2, & \lambda_1\lambda_2, & \lambda_1\lambda_3 \\ \lambda_1\lambda_2, & \rho_2 + \lambda_2^2, & \lambda_2\lambda_3 \\ \lambda_1\lambda_3, & \lambda_2\lambda_3, & \rho_3 + \lambda_3^2 \end{pmatrix}.$$

We define a matrix V as follows:

$$VWV^T = \text{diag}, \qquad VV^T = I.$$

Let us now parametrize B_{12}, B_{13}, B_{23}, B_{24}, B_{22}, and b_{14} as follows:

$$\begin{aligned} B_{12} &= -\sum V_{1\alpha} \hat{x}_\alpha V_{2\alpha}, \qquad B_{13} = -\sum V_{1\alpha} \hat{x}_\alpha V_{3\alpha}, \\ B_{23} &= -\sum V_{2\alpha} \hat{x}_\alpha V_{3\alpha}, \qquad B_{22} = -\sum V_{2\alpha} \hat{x}_\alpha V_{2\alpha}, \\ B_{24} &= \sum V_{2\alpha} \cdot \lambda_\alpha \frac{|V_{1\beta}\lambda_\beta|}{V_{1\gamma}\lambda_\gamma}, \qquad b_{14} = \left|\sum V_{1\alpha}\lambda_\alpha\right|. \end{aligned} \tag{48}$$

Substituting (48) in the general solution (26)–(28), we obtain

$$B_{11} = -\sum V_{1\alpha} \hat{x}_\alpha V_{1\alpha}, \quad B_{33} = -\sum V_{3\alpha} \hat{x}_\alpha V_{3\alpha}, \quad B_{34} = \sum V_{3\alpha} \lambda_\alpha \frac{|V_{1\beta}\lambda_\beta|}{V_{1\gamma}\lambda_\gamma}.$$

After some trivial operations, we obtain the vector potential in the coordinate form

$$A_\mu^a(x) = -\sum \eta_{a\mu\nu} \partial_\nu \ln(1 + K_\alpha^\lambda K_\alpha^\lambda) = -\sum \eta_{a\mu\nu} \partial_\nu \ln\left(1 + \sum_{\alpha=1}^{3} \frac{\lambda_\alpha^2}{(x - x_\alpha)^2}\right).$$

where

$$K_a^\lambda = -\frac{\lambda_a}{(x-x_a)^2}(x-x_a)\lambda$$

(there is no summation over repeated indices), i.e., in the parametrization (48), the general solution yields the 't Hooft solution.

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