HIGHER CONSERVATION LAWS FOR THE QUANTUM NON-LINEAR SCHRODINGER EQUATION

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Abstract:

We construct explicit forms for two non-trivial conservation laws of the quantum non-linear Schrödinger equation and show that they have the correct quasi-classical limit. For H_4 the second quantised form cannot be obtained by normal ordering of the classical conserved quantity H_4^{cl}. We show that the Quantum Inverse Scattering Method also gives the correct higher Hamiltonians H_3 and H_4. The surprising result is that the expansion of fundamental integrals of motion such as $A(\lambda)$, in inverse powers of λ, cannot be recovered by normal ordering of the classical expansion.

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1   INTRODUCTION

The quantum inverse scattering method (QISM) has its origins in attempts to extend the classical inverse scattering method for integrable non-linear systems [1-3] to interacting quantum fields [4-7]. Initial investigations, which were mainly concerned with semi-classical quantisation, were soon followed by a scheme for the exact quantisation of the non-linear Schrödinger equation [8]. In the last ten years there has been rapid progress on the QISM, with many published papers on the subject. Three reviews to which the reader may refer are refs. [9-11]. Most papers have been concerned with the development of this new branch of mathematical physics, although there have been some whose concern is with the mathematical foundations [12-13], or which have raised objections [14-15], particularly for the example of the quantum non-linear Schrödinger equation (QNLS). The most serious of the proposed difficulties relate to the higher conservation laws: it is the main concern of this paper to resolve these questions, at least for the QNLS.

Let us define the model and recall some well known results [16]. We consider the quantum non-linear Schrödinger equation in 1+1 space-time dimensions. In second-quantised form, the Hamiltonian is given as

$$H_2 \;\; = \;\; \int dx \, \{ \, \Psi^\dagger_x(x) \, \Psi_x(x) \, + \, c \, \Psi^\dagger(x) \, \Psi^\dagger(x) \, \Psi(x) \, \Psi(x) \, \} \,. \tag{1.1}$$

Here $\Psi(x,t)$ is a Bose field satisfying the canonical commutation relations

$$[\, \Psi(x), \Psi^\dagger(y) \,] \;\; = \;\; \delta(x{-}y) \,, \qquad [\, \Psi(x), \Psi(y) \,] \;\; = \;\; 0 \,. \tag{1.2}$$

Operators for the number of particles Q and total momentum P are given by

$$Q \;\; = \;\; \int dx \, \Psi^\dagger(x) \Psi(x) \,, \qquad P \;\; = \;\; -i \int dx \, \Psi^\dagger(x) \Psi_x(x) \,. \tag{1.3}$$

They are integrals of motion: $[H_2,P] = [H_2,Q] = [P,Q] = 0$. For the repulsive case $c > 0$, the only case considered herein, a complete set of eigenfunctions of the operators are well-known: viz

$$| \lambda_1,...,\lambda_N \rangle \, = \, (N!)^{-1/2} \int d^N\!x \, \chi_N(x_1,...,x_N | \lambda_1,...,\lambda_N) \, \Psi^\dagger(x_1)...\Psi^\dagger(x_N) \, | 0 \rangle \,, \tag{1.4}$$

where the explicit formula for the functions χ_N is

$$\chi_N(x_1,...,x_N | \lambda_1,...,\lambda_N)$$

$$= \sum_P (-1)^P \prod_{j>k} \{\lambda_{Pj} - \lambda_{Pk} - ic \text{ sgn}(x_j - x_k)\} \exp\left\{ i \sum_{n=1}^N x_n \lambda_{Pn} \right\}. \quad (1.5)$$

Here P runs over the permutations of $(1,...,N)$ and $\text{sgn}(x)$ is the sign of x.

In addition to the second quantised form (1.1) there is a formulation of the QNLS as a non-relativistic many-body problem, in terms of partial differential operators and boundary conditions. In the N-particle sector, the functions χ_N arise as eigenfunctions of the following differential operator H_2:

$$H_2 = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2c \sum_{N \geq j > k \geq 1} \delta(x_j - x_k). \quad (1.6)$$

The delta function interaction may be replaced by boundary conditions at $x_j = x_k$ and this will sometimes be done in the following. The functions χ_N are also eigenfunctions of the momentum differential operator $P = -iH_1$:

$$H_1 = \sum_{j=1}^N \frac{\partial}{\partial x_j}. \quad (1.7)$$

The eigenvalues of these two operators are given by

$$H_1 \chi_N = \left(i \sum_{j=1}^N \lambda_j \right) \chi_N, \qquad H_2 \chi_N = \left(\sum_{j=1}^N \lambda_j^2 \right) \chi_N. \quad (1.8)$$

As we have mentioned, there are claims [14,15] that the higher conservation laws obtained from the QISM are in conflict with those which may be found directly from the above solutions, even for the next two operators H_3 and H_4. We shall show that this is not the case. In the differential equation formulation we have only to correct an error in ref. [14]. This we do in sections 2 and 3, where we construct the operators H_3 and H_4. They have the same eigenfunctions (1.5) with the eigenvalues $i^3 \Sigma \lambda_j^3$ and $\Sigma \lambda_j^4$ respectively. We have broken the

calculation into two sections: in section 2 we deal only with the $N = 2$ and $N = 3$ sectors where the calculations are quite elementary and already reveal the flaw in ref [14]. In section 3 we give the forms in the general N-particle sector.

One must be most careful when writing down the corresponding conservation laws by means of quantum Bose fields because the individual terms in the formal expression (as a sum) involves irregular (undefined) operators, even though the total expression is very well defined. Expressed in the language of second quantisation, the results of sections 2 and 3 for H_3 may be summarised in the expression

$$H_3 = \int dx \{ \Psi^{\dagger}(x) \Psi_{xxx}(x) - (3c/2) \Psi^{\dagger}(x)^2 (\Psi(x))^2)_x \} . \tag{1.9}$$

There is no corresponding expression for H_4. In particular,

$$H_4 \neq \int dx \{ \Psi^{\dagger}_{xx}(x) \Psi_{xx}(x) + 2c (\Psi^{\dagger}(x)^2)_x (\Psi(x)^2)_x$$
$$+ c \Psi^{\dagger}(x)^2 \Psi_x(x)^2 + c \Psi^{\dagger}_x(x)^2 \Psi(x)^2 + 2c \Psi^{\dagger}(x)^3 \Psi(x)^3 \} . \tag{1.10}$$

One may write down a formal second quantised form for H_4 by replacing half of the three-particle interaction term $2c\Psi^{\dagger}(x)^3\Psi(x)^3$ by $c\Psi^{\dagger}(x)^2\Psi(x)\Psi^{\dagger}(x)\Psi(x)^2$, which is not normally ordered. The application of this symbol to a Fock space state such as (1.4) gives rise to the meaningless product $\delta^2(x_1 - x_2)$ of generalised functions, so it is hardly a useful modification. At the same (formal) level of discussion, notice that the numerical coefficient of this interaction term is 2, the same as in the classical case. This tells us that any (formal) quasi-classical limit will be correct, contrary to the claim of ref [14]. We shall see in sections 2 and 3 that H_3 and H_4 are properly represented in terms of irreducible parts J_3 and J_4 together with multinomials in the lower conserved operators. From H_4 on there is no way of regrouping the formulae to give a second quantised form as the one dimensional integral of a density. Thus we cannot obtain H_n ($n \geq 4$) by normal ordering of the classical expressions which are integrals of such densities. In this respect we agree with Gutkin [15], pages 112-114.

The connection of the QISM to higher conservation laws is a more technical problem. It is claimed in ref [15] that the QISM fails to generate the correct

conservation law even for H_3. We shall show that this is not so: the difficulty lies in the asymptotic analysis. The anchor point of the QISM derivation of higher conservation laws is the fact that the trace of the monodromy operator $\tau(\lambda)$, where λ is the spectral parameter, gives a commuting family [11]:

$$[\tau(\lambda),\tau(\mu)] \quad = \quad 0. \tag{1.11}$$

This is true for both the lattice version and the continuous limit of the QNLS in a finite box: it has its analogue also for an infinite box. (Recall that $\tau(\lambda)$ is the transfer matrix in statistical mechanics.) In section 4 we discuss the quantum trace identities for the lattice QNLS. We show that the higher terms in the λ^{-n} asymptotic expansion are not given by normal ordering of the corresponding classical expressions. Our calculations show that the discrepancies in the asymptotic expansion, as reported in ref [15], are due to the neglect of quantum corrections (contributions from operator reordering). We derive the large λ expansion

$$[(e^{-i\lambda L/2}\, \tau(\lambda)]_{\lambda \to -i\infty} = 1 + \lambda^{-1}A_0 + \lambda^{-2}A_1 + \lambda^{-3}A_2 + \lambda^{-4}A_3 + O(\lambda^{-5}) \tag{1.12}$$

where the commuting constants A_0 to A_3 are

$$A_0 \quad = \quad -icH_0 \quad = \quad -icQ \tag{1.13}$$

$$A_1 \quad = \quad -cH_1 - \frac{c^2}{2}H_0(H_0-1) \tag{1.14}$$

$$A_2 \quad = \quad -icH_2 + ic^2\,(H_0-1)H_1 - \frac{ic^2}{6}H_0(H_0-1)(H_0-2) \tag{1.15}$$

$$A_3 \quad = \quad cH_3 - \frac{c^2}{2}H_1^2 + c^2\,(\frac{3}{2} - H_0)H_2$$
$$\frac{c^3}{2}(H_0-1)(H_0-2)H_1 + \frac{c^4}{24}H_0(H_0-1)(H_0-2)(H_0-3) \tag{1.16}$$

These results differ from those given in ref [15] in a number of respects. A minor difference is that our definition of $A(\lambda)$ follows the usual one of all the preceding literature [7-13] on the QNLS, whereas ref [15] interchanges the meaning of $A(\lambda)$ and its (Hermitean) adjoint $A^{\dagger}(\lambda)$. More important are the differences in the operator A_3. First A_3 is not the normal ordered version of the correspond-

ing classical quantity. Second, we have corrected a numerical error in the coefficient of the third term of A_3: in [15] this is given as $(2-H_0)$. This correction is important because it makes it evident that the difference between the correct result and the normal ordering recipe stems from difficulties with asymptotics, rather than a fundamental flaw in the QISM.

Finding the correct asymptotic expansion directly for the continuous model in an infinite box is an extremely tricky business indeed. The individual operators, $A(\lambda)$, $A^\dagger(\lambda)$, which appear on the diagonal of the monodromy matrix are themselves constants of the motion [12,13]. One needs to expand the operator $A(\lambda)$ in inverse powers of λ. The fact that the higher conservation laws are not normally ordered is equivalent to the surprising result

$$\sum_{n\geq 0} \lambda^{-n-1} A_n \neq \sum_{n\geq 0} \lambda^{-n-1} :A_n^{cl}: , \qquad (1.17)$$

even though $A(\lambda)$ is correctly defined as $:A^{cl}(\lambda):$, the normal ordering of the classical quantity. The QISM has never depended, for its validity, on the normal ordering recipe. However it was difficult to see how this recipe could be broken for the continuous QNLS, even though there is no *a priori* reason to require it. In section 5 we will show that, when we consider $A(\lambda)$ as an integral operation in Fock space and take the asymptotic decomposition in inverse powers of λ, the expansion is non-uniform and this leads to the breakdown of the formal expansion for $n\geq 3$.

2. TWO AND THREE PARTICLE SECTORS

In this section we first discuss two particle wave functions. They are given by

$$\chi_2(x_1,x_2 \mid \lambda_1,\lambda_2) = \{\lambda_2 - \lambda_1 - ic\ \text{sgn}(x_2 - x_1)\} \exp\{i\, x_1\, \lambda_1 + i\, x_2\, \lambda_2\}$$
$$+ \{\lambda_2 - \lambda_1 + ic\ \text{sgn}(x_2 - x_1)\} \exp\{i\, x_1\, \lambda_2 + i\, x_2\, \lambda_1\} . \qquad (2.1)$$

This is a continuous symmetric function of x_1 and x_1; it is an eigenfunction of H_1

$$H_1 \quad = \quad \partial/\partial x_1 + \partial/\partial x_2 , \tag{2.2}$$

$$H_1 \chi_2 \quad = \quad i \, (\lambda_1 + \lambda_2) \, \chi_2 . \tag{2.3}$$

It is also an eigenfunction of the Hamiltonian (1.1)

$$H_2 \quad = \quad -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + 2c \, \delta(x_1 - x_2) , \tag{2.4}$$

$$H_2 \chi_2 \quad = \quad (\lambda_1^2 + \lambda_2^2) \, \chi_2 . \tag{2.5}$$

The operator (2.4) is the free Hamiltonian except at the boundary $x_1 = x_2$ where the interaction is equivalent to the following boundary condition

$$\left[c \, \chi_2 + \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) \chi_2 \right]_{x_2 = x_1 + 0} \quad = \quad 0 . \tag{2.6}$$

In the two particle sector these are the only two independent conservation laws: all higher conserved operators are generated by H_1 and H_2. It is easy to construct them. For convenience, introduce an operator J_2 as

$$J_2 \quad = \quad \frac{\partial^2}{\partial x_1 \partial x_2} + c \, \delta(x_1 - x_2) \quad = \quad \frac{1}{2}(H_1^2 + H_2) . \tag{2.7}$$

The wave function χ_2 is an eigenfunction of J_2

$$J_2 \chi_2 \quad = \quad -(\lambda_1 \lambda_2) \, \chi_2 . \tag{2.8}$$

Now let us construct the operator H_3, with eigenvalues equal to $i^3(\lambda_1^3 + \lambda_2^3)$:

$$H_3 \quad = \quad H_1^3 - 3 \, H_1 \, J_2 . \tag{2.9}$$

From (2.3), (2.5) and (2.7) it follows that the Bethe wave function χ_2 is an eigenfunction of H_3, moreover an elementary calculation shows that it has the expected eigenvalue

$$H_3 \chi_2 \quad = \quad i^3 \, (\lambda_1^3 + \lambda_2^3) \, \chi_2 . \tag{2.10}$$

An explicit formula for H_3 is

$$H_3 = \frac{\partial^3}{\partial x_1^3} + \frac{\partial^3}{\partial x_2^3} - 3c\,\delta(x_1 - x_2)\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right),$$

(2.11)

and this shows that it coincides with the third conservation law constructed in refs [14,15].

Now let us construct the operator H_4 with eigenvalues equal to $(\lambda_1^4 + \lambda_2^4)$:

$$H_4 = H_1^4 + 2J_2^2 - 4H_1^2 J_2.$$

(2.12)

We emphasise that, by its very construction in terms of the lower conserved operators, H_4 is well defined as an operator. From (2.3) and (2.8) it follows that

$$H_4\,\chi_2 = (\lambda_1^4 + \lambda_2^4)\,\chi_2.$$

(2.13)

So we have constructed the fourth conservation law for the QNLS in the $N = 2$ sector. It does not coincide with the fourth conservation law constructed in [14]. Let us denote the latter by G_4, it is given in [14] as

$$G_4 = \frac{\partial^4}{\partial x_1^4} + \frac{\partial^4}{\partial x_2^4} - 2c\,\delta(x_1 - x_2)\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_1 \partial x_2}\right)$$
$$- 2c\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_1 \partial x_2}\right)\delta(x_1 - x_2).$$

(2.14)

An elementary (formal) calculation shows that

$$G_4 = H_4 - 2c^2\,\delta^2(x_1 - x_2)$$

(2.15)

This equation has only a formal significance, since $\delta^2(x_1 - x_2)$ is an undefined product. So G_4 is irregular (undefined) because H_4 is regular. This is clear even from (2.14) because $\partial^2/\partial x_1^2\,\chi_2 \approx \delta(x_1 - x_2)\,\chi_2$, so the product $\delta(x_1 - x_2)\,\partial^2/\partial x_1^2$ is not defined as an operator. G_4 does not commute with the Hamiltonian H_2 and the Bethe eigenfunction χ_2 is not an eigenfunction of G_4. We can already see the reason for (1.10). The operator (2.14) is precisely what we recover by using the normal ordered form of (1.10) in the two-particle sector: for $N=2$ there can be no

three particle interaction as contained in the term $\Psi^\dagger(x)^3 \Psi(x)^3$. On the other hand, if we use the normal ordered symbols (1.1) and (1.3) in eqs. (2.7) and (2.12), we find that normal ordering cannot be carried out to rearrange the formula as the one-dimensional integral of a single Hamiltonian density. This is the meaning of the difference between G_4 and H_4.

We have constructed two non-trivial conservation laws in the two-particle sector, and used them to check the consistency of three operators H_3, H_4, G_4. Now we discuss the three particle sector. The Bethe wave function χ_3 is given by

$$\chi_3(x_1, x_2, x_3 \,|\, \lambda_1, \lambda_2, \lambda_3)$$

$$= \sum_P (-1)^P \prod_{3>j>k>1} \{\lambda_{Pj} - \lambda_{Pk} - ic \, \mathrm{sgn}(x_j - x_k)\} \exp\left\{i \sum_{n=1}^{3} x_n \lambda_{Pn}\right\}. \quad (2.16)$$

It is a continuous symmetric function of x_1, x_2, x_3. It is an eigenfunction of the operators H_1, H_2, defined in the three-particle sector as

$$H_1 = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}, \qquad H_2 = -\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{3>j>k>1} \delta(x_j - x_k), \qquad (2.17)$$

The operator $J_2 = (H_1^2 + H_2)/2$ now has the representation

$$J_2 = \sum_{3>j>k>1} \left(\frac{\partial^2}{\partial x_j \partial x_k} + c\, \delta(x_j - x_k) \right). \qquad (2.18)$$

It is well known that χ_3 is an eigenfunction of H_1 and H_2, and that for the Hamiltonian this property is equivalent to the boundary condition

$$\left[c\, \chi_3 + \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}} \right) \chi_3 \right]_{x_{j+1} = x_j + 0} = 0. \qquad (2.19)$$

It follows from our construction that it is a properly defined operator in the three particle sector and that χ_3 is also an eigenfunction of J_2:

$$J_2 \chi_3 = -\left(\sum_{3>j>k>1} \lambda_j \lambda_k\right) \chi_3. \tag{2.20}$$

The third conserved Hamiltonian H_3 was constructed in [14,15] correctly as the differential operator

$$H_3 = \sum_{j=1}^{3} \frac{\partial^3}{\partial x_j^3} - 3c \sum_{3>j>k>1} \delta(x_j - x_k)\left(\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_k}\right), \tag{2.21}$$

Notice that the differentiation in the second term acts in a direction orthogonal to the argument of the delta function. This is important, since the Bethe wave function has discontinuous derivatives at the boundaries $x_j = x_k$. We rewrite H_3 in the form

$$H_3 = H_1^3 - 3 H_1 J_2 + 3 J_3. \tag{2.22}$$

Here we have introduced the new operator J_3 as

$$J_3 = \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} + c\frac{\partial}{\partial x_1}\delta(x_2 - x_3) + c\frac{\partial}{\partial x_2}\delta(x_3 - x_1) + c\frac{\partial}{\partial x_3}\delta(x_1 - x_2). \tag{2.23}$$

From the construction of J_3, we must have the equality

$$J_3 \chi_3 = i^3 (\lambda_1 \lambda_2 \lambda_3) \chi_3. \tag{2.24}$$

and this is equivalent to boundary conditions of the form

$$\frac{\partial}{\partial x_3}\left[c \chi_3 + \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\chi_3\right]_{x_2 = x_1 + 0} = 0. \tag{2.25}$$

which follow immediately from (2.19). This shows directly that (2.24) is valid and that (2.21) is the correct form for H_3. In the three particle sector, all higher conservation laws are generated by H_1, H_2 and H_3. We shall write them as functions of H_1, J_2 and J_3. Let us construct the fourth conservation law for an operator H_4:

$$H_4 = H_1^4 + 2 J_2^2 - 4 H_1^2 J_2 + 4 H_1 J_3. \tag{2.26}$$

From our previous results it follows immediately that this a properly defined operator and that its action on the Bethe eigenstates is

$$H_4 \chi_3 = (\lambda_1^4 + \lambda_2^4 + \lambda_3^4) \chi_3 . \tag{2.27}$$

As in the two particle sector, it does not coincide with the fourth operator G_4 published in ref [14]. Elementary (formal) manipulations show that in the three particle sector,

$$G_4 = H_4 - 2c^2 \sum_{3 \geq j > k \geq 1} \delta^2(x_j - x_k) + 6c^2 \delta(x_1 - x_2) \delta(x_2 - x_3) . \tag{2.28}$$

So G_4 is not an integral of motion, it is not even defined for $N = 2$ and 3. Again the differences between G_4 and H_4 may be (formally) viewed as an ordering problem but it is not a profitable approach.

3   MANY PARTICLE SECTOR

Recall the formula (1.5) for the N particle Bethe eigenstates $\chi_N(x_i | \lambda_j)$. They are continuous symmetric functions of $x_1, ..., x_N$ and $\lambda_1, ..., \lambda_N$, and also eigenstates of the operators H_1, H_2, defined in (1.6) and (1.7). This latter fact is equivalent to boundary conditions of the form (2.6). The third operator H_3 in the sequence of conserved quantities is given in [14,15] as

$$H_3 = \sum_{j=1}^{N} \frac{\partial^3}{\partial x_j^3} - 3c \sum_{N \geq j > k \geq 1} \delta(x_j - x_k) (\partial/\partial x_j + \partial/\partial x_j) , \tag{3.1}$$

Let us write it in the form

$$H_3 = H_1^3 - 3 H_1 J_2 + 3 J_3 . \tag{3.2}$$

as in section 2. Now we have extended the definition of the operators J_2 and J_3 to the N particle sector as

$$J_2 = \frac{1}{2}(H_1^2 + H_2) = \sum_{N \geq j > k > 1} \left(\frac{\partial^2}{\partial x_j \partial x_k} + c\, \delta(x_j - x_k) \right). \tag{3.3}$$

$$J_3 = \sum_{N \geq j > k > l \geq 1} \left(\frac{\partial^3}{\partial x_j \partial x_k \partial x_l} + c \frac{\partial}{\partial x_j} \delta(x_k - x_l) + c \frac{\partial}{\partial x_k} \delta(x_l - x_j) + c \frac{\partial}{\partial x_l} \delta(x_j - x_k) \right) \tag{3.4}$$

From (1.8) it follows that

$$J_2\, \chi_N = -\left(\sum_{N \geq j > k \geq 1} \lambda_j \lambda_k \right) \chi_N. \tag{3.5}$$

To prove that χ_N is also an eigenfunction of H_3 we may first prove that it is an eigenfunction of J_2:

$$J_3\, \chi_N = i^3 \left(\sum_{N \geq j > k > l \geq 1} \lambda_j \lambda_k \lambda_l \right) \chi_N, \tag{3.6}$$

and this is readily reduced to the following boundary condition, analogous to (2.25),

$$\left(\sum_{\substack{l=1 \\ l \neq j, j+1}}^{N} \frac{\partial}{\partial x_l} \right) \left[c\, \chi_N + \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}} \right) \chi_N \right]_{x_{j+1} = x_j + 0} = 0. \tag{3.7}$$

So we have proved that

$$H_3\, \chi_N = i^3 \left(\sum_{j=1}^{N} \lambda_j^3 \right) \chi_N. \tag{3.8}$$

and H_3 is the correct operator for the third conservation law.

Let us construct the fourth conservation law in a similar manner. We commence with the definition

$$H_4 = H_1^4 + 2J_2^2 - 4 H_1^2 J_2 + 4 H_1 J_3 - 4 J_4. \tag{3.9}$$

Here we have introduced the operator

$$
\begin{aligned}
J_4 \;=\; \sum_{N \geq j > k > l > m \geq 1} & \left(\frac{\partial^4}{\partial x_j \partial x_k \partial x_l \partial x_m} + c \left[\frac{\partial^2}{\partial x_j \partial x_k} \delta(x_l - x_m) \right. \right. \\
& + \frac{\partial^2}{\partial x_j \partial x_l} \delta(x_k - x_m) + \frac{\partial^2}{\partial x_j \partial x_m} \delta(x_k - x_l) + \frac{\partial^2}{\partial x_k \partial x_l} \delta(x_j - x_m) \\
& + \frac{\partial^2}{\partial x_k \partial x_m} \delta(x_j - x_l) + \frac{\partial^2}{\partial x_l \partial x_m} \delta(x_j - x_k) \right] + c^2 \left[\delta(x_j - x_k) \delta(x_l - x_m) \right. \\
& \left. \left. + \delta(x_j - x_l) \delta(x_k - x_m) + \delta(x_j - x_m) \delta(x_k - x_l) \right] \right).
\end{aligned}
\tag{3.10}
$$

To prove that

$$
H_4 \, \chi_N \;=\; \left(\sum_{j=1}^{N} \lambda_j^4 \right) \chi_N
\tag{3.11}
$$

is equivalent to showing that

$$
J_4 \, \chi_N \;=\; \left(\sum_{N \geq a > b > c > d \geq 1} \lambda_a \lambda_b \lambda_c \lambda_d \right) \chi_N \, ,
\tag{3.12}
$$

and this reduces to the following boundary conditions:

$$
\sum_{N \geq j > k \geq 3} \left(\frac{\partial^2}{\partial x_j \partial x_k} + c \, \delta(x_j - x_k) \right) \left[c \, \chi_N + \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) \chi_N \right]_{x_2 = x_1 + 0} = 0 \, .
\tag{3.13}
$$

Again these are valid, and follow immediately from the simpler boundary condition (2.19) because the differentiations are always in an orthogonal direction to the planes on which the delta functions have their support. In this way we have constructed the fourth conserved quantity H_4. It does not coincide with the operator of refs [14, eq.(2.20)], which we shall call G_4. We repeat the formula here:

$$G_4 \;=\; \sum_j \frac{\partial^4}{\partial x_j^4} + 18c^2 \sum_{N \geq j > k > l \geq 1} \delta(x_j - x_k)\, \delta(x_k - x_l) \qquad (3.14)$$

$$- 2c \sum_{N \geq j > k \geq 1} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial x_j \partial x_k} \right) \delta(x_j - x_k) + \delta(x_j - x_k) \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial x_j \partial x_k} \right)$$

Comparing this with H_4 we see that

$$G_4 \;=\; H_4 - 2c^2 \sum_{N \geq j > k \geq 1} \delta^2(x_j - x_k) + 6c^2 \sum_{N \geq j > k > l \geq 1} \delta(x_j - x_k)\, \delta(x_k - x_l) \qquad (3.15)$$

So we have constructed H_3 and H_4. We would like to discuss their second quantised form and compare these with the corresponding classical conserved observables H_3^{cl} and H_4^{cl}. The latter are given in eqs. (4.7). The operators J_2, J_3, J_4 are well defined as differential operators, and also in second quantised form. The square of J_2 is also a well defined operator, but to carry out the desired comparison we need to write it as the sum of irregular terms (the whole sum is well defined). One of the terms involves (formally) the square of a delta function, another $\delta(x_1 - x_2)\, \partial^2/\partial x_1^2$, and the singularities cancel. So we can write H_3 and H_4 in the form of a multi-dimensional integral of the field operators and their derivatives: this will be well defined but the individual terms in a rearranged one-dimensional expression may not be so well defined. With this caveat, for H_3 we recover the result of ref [14]: we could find the same result by applying the recipe of normal ordering to H_3^{cl} - see eq. (3.16). In the case of H_4 however, we do not reproduce the result of ref [14], which we called G_4 above. H_4 is not the normal ordering of H_4^{cl}, although its (formal) quasi-classical limit is correct. This is an important correction to the result claimed in ref [14]. As we have already observed, the QISM does not depend on the recipe of normal ordering, and in fact it is broken at the operator H_4. However, it is required that the quasi-classical limit of an integrable quantum theory (when problems of ordering go away) should be correct.

4. QUANTUM TRACE IDENTITIES.

First let us discuss the trace identities as they were constructed in refs. [2-3]. We commence with the classical U operator

$$U(x|\lambda) = \frac{d}{dx} + \frac{i\lambda}{2}\sigma_3 + Q(x),$$

$$Q(x) = \begin{bmatrix} 0 & i\sqrt{c}\,\psi^\dagger(x) \\ -i\sqrt{c}\,\psi(x) & 0 \end{bmatrix}. \tag{4.1}$$

The transition matrix $T(x,y|\lambda)$ is defined as the solution of an initial value problem:

$$U(x|\lambda)\,T(x,y|\lambda) = 0, \qquad T(y,y|\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.2}$$

Suppose we impose periodic boundary conditions in a box of length L. Then the monodromy matrix $T(\lambda)$ and its trace $\tau(\lambda)$ are defined as

$$T(\lambda) = T(L,0|\lambda), \qquad \tau(\lambda) = \text{tr}\,T(\lambda). \tag{4.3}$$

We are interested in the decomposition of $\exp(-i\lambda L/2)\tau(\lambda)$ in inverse powers of λ, as $\lambda \to -i\infty$. Let us make a gauge transformation which diagonalises $T(x,y|\lambda)$:

$$T(x,y|\lambda) = V(x|\lambda)\,D(x,y|\lambda)\,V^{-1}(y|\lambda). \tag{4.4}$$

Here D is a diagonal matrix while $V(x|\lambda)$ and $V^{-1}(y|\lambda)$ depend only on one space variable. $V(x|\lambda)$ can be represented in the form

$$V(x|\lambda) = \begin{bmatrix} 1 & f(x,\lambda) \\ \bar{f}(x,\lambda) & 1 \end{bmatrix}. \tag{4.5}$$

We note that $f(x,\lambda) \to 0$ in the limit $\lambda \to -i\infty$. If we substitute (4.4) and (4.5) into (4.2) we get an equation for $f(x,\lambda)$ and an equation for $D(x,y|\lambda)$, from which it is very easy to get the $1/\lambda$ decomposition of $f(x,\lambda)$ and for $\tau(\lambda)$. Here we simply quote the results [2,3]

$$\left[e^{-i\lambda L/2} \tau(\lambda) \right]_{\lambda \to -i\infty} \approx 1 - \frac{ic}{\lambda} H_0^{cl} - \frac{1}{\lambda^2} \left(c H_1^{cl} + \frac{c^2}{2} (H_0^{cl})^2 \right)$$

$$+ \frac{1}{\lambda^3} \left(-ic H_2^{cl} + ic^2 H_0^{cl} H_1^{cl} + \frac{ic^3}{6} (H_0^{cl})^3 \right) + \frac{1}{\lambda^4} \left(c H_3^{cl} + \frac{c^2}{2} (H_1^{cl})^2 \right.$$

$$\left. - c^2 H_0^{cl} H_2^{cl} + \frac{c^3}{2} (H_0^{cl})^2 H_1^{cl} + \frac{c^4}{24} (H_0^{cl})^4 \right) + \mathcal{O}(1/\lambda^5) . \qquad (4.6)$$

We have introduced the following notation so as to keep the classical and quantum cases parallel

$$H_0^{cl} = \int dx \, \bar{\psi}(x) \, \psi(x) ,$$

$$H_1^{cl} = \int dx \, \bar{\psi}(x) \, \psi_x(x) ,$$

$$H_2^{cl} = \int dx \, \{ \bar{\psi}_x(x) \, \psi_x(x) + c \, \bar{\psi}(x)^2 \, \psi(x)^2 \} , \qquad (4.7)$$

$$H_3^{cl} = \int dx \, \{ \bar{\psi}(x) \, \psi_{xxx}(x) - (3c/2) \, \bar{\psi}(x)^2 \, (\psi(x)^2)_x \}$$

$$H_4^{cl} = \int dx \, \{ \bar{\psi}_{xx}(x) \, \psi_{xx}(x) + 2c \, (\bar{\psi}(x)^2)_x \, (\psi(x)^2)_x$$

$$+ c \, \bar{\psi}(x)^2 \, \psi_x(x)^2 + c \, \bar{\psi}_x(x)^2 \, \psi(x)^2 + 2c \, \bar{\psi}(x)^3 \, \psi(x)^3 \} .$$

It is well known that the asymptotic expansion of the logarithm takes a simple form: in our present notation it is

$$\left[\log \left(e^{-i\lambda L/2} \tau(\lambda) \right) \right]_{\lambda \to -i\infty}$$

$$\approx 1 - \frac{ic}{\lambda} H_0^{cl} - \frac{c}{\lambda^2} H_1^{cl} - \frac{ic}{\lambda^3} H_2^{cl} + \frac{c}{\lambda^4} H_3^{cl} + \mathcal{O}(1/\lambda^5) . \qquad (4.8)$$

Now let us discuss the quantum case. One can repeat similar calculations in the quantum case to those above, for example this is done in ref [21]. Now one has the problem of non-commutation together with the fact that some of the necessary operations involve repeated differentiation of formally defined objects. The expansion up to $\mathcal{O}(\lambda^{-4})$ is

$$\left[e^{-i\lambda L/2} \tau(\lambda) \right]_{\lambda \to -i\infty} \approx 1 - \frac{ic}{\lambda} :H_0^{cl}: - \frac{1}{\lambda^2} : \left(c H_1^{cl} + \frac{c^2}{2} (H_0^{cl})^2 \right) :$$

$$+ \frac{1}{\lambda^3} : \left(-ic H_2^{cl} + ic^2 H_0^{cl} H_1^{cl} - \frac{ic^2}{6} (H_0^{cl})^3 \right) : + \mathcal{O}\left(\frac{1}{\lambda^4} \right) . \qquad (4.9)$$

It is well known that everything is correct to this point. Performing the indicated normal ordering on the coefficients found thus far (with the help of the canonical commutation relations) we find that the first three commuting constants are given by eqs. (1.13) - (1.15). Now let us take the next coefficient from the classical expansion (4.6) and apply the same normal ordering prescription: this gives

$$
\begin{aligned}
B_3 &= \ : \left(cH_3^{cl} - \frac{c^2}{2}(H_1^{cl})^2 - c^2 \, H_0^{cl} \, H_2^{cl} + \frac{c^3}{2}(H_0^{cl})^2 \, H_1^{cl} + \frac{c^4}{24}(H_0^{cl})^4 \right): \\
&= \ A_3 \ - \ \frac{c^3}{2} \int dx \ \Psi^\dagger(x)^2 \Psi(x)^2
\end{aligned}
\tag{4.10}
$$

One can see that B_3 does not commute with A_0, A_1 and A_2. There were quantum corrections even for A_1 and A_2, exhibited in the replacement of $(H_0^{cl})^2$ and $(H_0^{cl})^3$ by $H_0(H_0-1)$ and $H_0(H_0-1)(H_0-2)$. The corrections are expressed in terms of an operator which was already a generator of the sub-algebra of commuting operators. The same treatment of B_3 has given an operator which differs from H_3 by a non-commuting part because one of the quantum corrections does not arise with the correct coefficient. One might observe that it corresponds to a delta function interaction: that is, in Fock space, its effect is only felt at the boundaries $x_i = x_j$. This does not explain the term away, but the explanation is closely related as we shall see in the next section. It is useful also to consider the quantum expansion of $\log[\exp(-i\lambda L/2)\tau(\lambda)]$. It may be obtained by taking the logarithm of (1.12), or by direct computation, which will be given below. Either way the result is

$$
\begin{aligned}
\left[\log\left(e^{-i\lambda L/2} \tau(\lambda) \right) \right]_{\lambda \to -i\infty} &\approx 1 - \frac{ic}{\lambda} N - \frac{c}{\lambda^2}\left(H_1 - \frac{c}{2}N \right) \\
&\quad - \frac{ic}{\lambda^3}\left(H_2 + cH_1 - \frac{c^2}{3}N \right) + \frac{c}{\lambda^4}\left(H_3 + \frac{3c}{2}H_2 + c^2 H_2 - \frac{c^3}{4}N \right) + O(\lambda^{-5}).
\end{aligned}
\tag{4.11}
$$

This is not the normal ordered form of the classical expansion (4.8)! Whilst it is true that each of the coefficients obtained by the normal ordering recipe is a conserved quantity, up to H_3, there are quantum corrections to the asymptotic expansion beginning even with the H_1 term. Moreover, the expansion (4.11) is equivalent to (1.12) by simple exponentiation, so the normal order of the terms in (4.11) implies the lack of normal order for A_3.

The best way to control this ordering problem is to use lattice regularisation. That is, we solve the QNLS model on a lattice exactly using the QISM and this will allow us to calculate the quantum corrections directly. To do this we need the results of ref [18] for the inverse scattering scheme for both the classical and quantum cases. The L operator for the lattice version has the form

$$L(n\,|\,\lambda) = \begin{bmatrix} 1-i\lambda\Delta/2+(c\Delta^2/2)\psi_n^\dagger\psi_n & -i\Delta\sqrt{c}\psi_n^\dagger\rho_n \\[2mm] +i\Delta\sqrt{c}\rho_n\psi_n & 1+i\lambda\Delta/2+(c\Delta^2/2)\psi_n^\dagger\psi_n \end{bmatrix}.$$

$$\rho_n = \sqrt{1+(c\Delta^2/4)\,\psi_n^\dagger\psi_n}\,, \tag{4.12}$$

Here Δ is a length (the step length) which we use to take the continuous limit. Also we are using a canonical Bose field ψ_n on the lattice. For the classical model we have the Poisson brackets

$$\{\psi_m,\psi_n^\dagger\} = i\,\Delta^{-1}\delta_{mn}\,, \quad \{\psi_m,\psi_n\} = \{\psi_m^\dagger,\psi_n^\dagger\} = 0\,, \tag{4.13}$$

where δ_{mn} is the Kronecker delta symbol. For the quantum case the equivalent commutation relations are

$$[\psi_m,\psi_n^\dagger] = \Delta^{-1}\delta_{mn}\,, \quad [\psi_m,\psi_n] = [\psi_m^\dagger,\psi_n^\dagger] = 0\,. \tag{4.14}$$

The transition matrix $T(m,n\,|\,\lambda)$ is given by the usual formula for the QISM:

$$T(m,n\,|\,\lambda) = L(m\,|\,\lambda)\,L(m-1\,|\,\lambda)\,...\,L(n+1\,|\,\lambda)\,L(n\,|\,\lambda)\,, \quad (m>n)\,. \tag{4.15}$$

The monodromy matrix $T(\lambda)$ is well defined for a one-dimensional lattice of N sites: it is

$$T(\lambda) = T(N,1\,|\,\lambda)\,. \tag{4.16}$$

The transfer matrix which incorporates periodic boundary conditions is simply the trace:

$$\tau(\lambda) = \mathrm{tr}\,T(\lambda)\,. \tag{4.17}$$

The central identity is that

$$\{\tau(\lambda),\tau(\mu)\} \;=\; 0, \qquad \text{(classical)},$$

$$[\tau(\lambda),\tau(\mu)] \;=\; 0, \qquad \text{(quantum)}, \tag{4.18}$$

It is important that it is still valid in the continuous limit. In fact the classical r–matrix, which is

$$r(\lambda,\mu) \;=\; \left(\frac{c}{\lambda-\mu}\right) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.19}$$

and the quantum R matrix, which is

$$R(\lambda,\mu) \;=\; \begin{bmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{bmatrix}, \tag{4.20}$$

$$f(\mu,\lambda) \;=\; \left(\frac{\mu-\lambda+ic}{\mu-\lambda}\right), \qquad\qquad g(\mu,\lambda) \;=\; \left(\frac{ic}{\mu-\lambda}\right), \tag{4.21}$$

do not depend on Δ and this is a simplifying factor in taking the limit $\Delta \to 0$. We see immediately from these formulae that normal ordering is lost on the lattice. That is,

$$T(\lambda) \;\neq\; :T^{\mathrm{cl}}(\lambda): , \qquad \tau(\lambda) \;\neq\; :\tau^{\mathrm{cl}}(\lambda): . \tag{4.22}$$

The crucial point is that exact integrability remains in the continuous limit $\Delta \to 0$, and expansions around $\Delta = 0$ are easy to get. Moreover, in any such expansions the first few terms will be normally ordered. This is exactly the behaviour noted above.

Now we proceed to the calculation for the continuous case in a finite box. The eigenvalues $\theta(\lambda)$ of the transfer matrix $\tau(\lambda) = A(\lambda) + D(\lambda)$ are known from the QISM, they are

$$\theta(\lambda) = e^{-i\lambda L/2} \prod_{j=1}^{N} \left(1 + \frac{ic}{\lambda-k_j}\right) + e^{i\lambda L/2} \prod_{j=1}^{N} \left(1 - \frac{ic}{\lambda-k_j}\right). \tag{4.23}$$

Here the momenta k_j must satisfy the Bethe Ansatz system of equations:

$$e^{ik_l L} = \prod_{\substack{j=1 \\ j \neq l}}^{N} \left(\frac{k_l - k_j + ic}{k_l - k_j - ic}\right) \tag{4.24}$$

Decomposition of $\theta(\lambda)$ in the $\lambda \to -i\infty$ limit may now be made by expanding the (finite) product of eigenvalues (4.23). This is similar to eq. (9.3.27) of ref [15], here we correct two numerical errors in the coefficients. The result is

$$\begin{aligned}
\left[e^{-i\lambda L/2} \theta(\lambda)\right]_{\lambda \to -i\infty} &\approx 1 - \frac{ic}{\lambda}N - \frac{ic}{\lambda^2}\left(\sum_{j=1}^{N} k_j + \frac{ic}{2}N(N-1)\right) \\
&\quad - \frac{ic}{\lambda^3}\left(\sum_{j=1}^{N} k_j^2 + ic\,(N-1)\sum_{j=1}^{N} k_j - \frac{c^2}{6}N(N-1)(N-2)\right) \\
&\quad - \frac{ic}{\lambda^4}\left(\sum_{j=1}^{N} k_j^3 - ic\,(N-\tfrac{3}{2})\sum_{j=1}^{N} k_j^2 - \frac{ic}{2}\left(\sum_{j=1}^{N} k_j\right)^2\right. \\
&\quad \left. - \frac{c^2}{2}(N-1)(N-2)\sum_{j=1}^{N} k_j + \frac{ic^3}{24}N(N-1)(N-2)(N-3)\right)
\end{aligned} \tag{4.25}$$

In section 3 we already constructed operators with these eigenvalues, on a complete set of states. This identification leads to the following decomposition of $\tau(\lambda)$ itself:

$$\left[e^{-i\lambda L/2} \tau(\lambda)\right]_{\lambda \to -i\infty} \approx 1 + \lambda^{-1}A_0 + \lambda^{-2}A_1 + \lambda^{-3}A_2 + \lambda^{-4}A_3 + O(\lambda^{-5}) \tag{4.26}$$

where A_3 carries the required quantum correction, viz

$$A_3 = :\left(cH_3^{cl} - \frac{c^2}{2}(H_1^{cl})^2 - c^2 H_0^{cl} H_2^{cl} + \frac{c^3}{2}(H_0^{cl})^2 H_1^{cl} + \frac{c^4}{24}(H_0^{cl})^4 \right):$$

$$+ \frac{c^3}{2} \int dx\, \Psi^\dagger(x)^2 \Psi(x)^2 \tag{4.27}$$

This is the result quoted in eq. (1.15). It follows from the commutativity of the family $\tau(\lambda)$. A similar decomposition may be made for the logarithm of the eigenvalues. Corresponding to (4.25) we have

$$\left[\log\left(e^{-i\lambda L/2}\, \theta(\lambda) \right) \right]_{\lambda \to -i\infty}$$

$$\approx 1 - \frac{ic}{\lambda}N - \frac{ic}{\lambda^2}\left(\sum_{j=1}^N k_j + \frac{ic}{2}N \right) - \frac{ic}{\lambda^3}\left(\sum_{j=1}^N k_j^2 + ic\sum_{j=1}^N k_j - \frac{c^2}{3}N \right)$$

$$- \frac{ic}{\lambda^4}\left(\sum_{j=1}^N k_j^3 + \frac{3ic}{2}\sum_{j=1}^N k_j^2 - c^2\sum_{j=1}^N k_j - \frac{ic^3}{4}N \right) + O(\lambda^{-5}) \tag{4.28}$$

from which we obtain (4.11).

We conclude by mentioning that the calculations can also be performed using the methods of ref [18], in which a staggered lattice model is introduced in order to make $\lambda = -2i/\Delta$ a special point where the local transition operators become one-dimensional projectors. The higher Hamiltonians may then be extracted directly using logarithmic differentiation of the transfer matrix at this point. On the lattice, the failure of normal ordering is no surprise since as we already noted in eq. (4.22). The calculations are very long and will not be given here.

5 DIFFICULTIES WITH DIRECT ASYMPTOTIC EXPANSION

We have demonstrated in the previous three sections that when due correction of errors is made to formulae given in refs [14,15], there is no problem with the generation of higher conservation laws for the QNLS using either the differential equation formulation or the QISM. We now address the question of what goes wrong with the direct asymptotic expansion of the operator $A(\lambda)$ for the continuous QNLS in an infinite box. First we give some definitions and make some general observations. For a classical field theory involving the field

$\psi(x,t)$, and for given functions $a_{mn}(x_1,...,x_m;y_1,...,y_n)$, we define a functional $A^{cl}(\bar\psi,\psi)$ of the form

$$A^{cl}(\bar\psi,\psi) = \sum_{mn} \int d^m x \, d^n y \, a_{mn}(x_1,...,x_m;y_1,...,y_n) \, \bar\psi(x_1)..\bar\psi(x_m) \, \psi(y_1)..\psi(y_n) \,. \quad (5.1)$$

In a quantum theory, involving the field $\Psi(x,t)$, operators A may be constructed similarly . In second quantised form we write

$$A = \sum_{mn} \int d^m x \, d^n y \, a_{mn}(x_1,...,x_m;y_1,...,y_n) \, \Psi^\dagger(x_1)..\Psi^\dagger(x_m) \, \Psi(y_1)..\Psi(y_n) \,. \quad (5.2)$$

Because we must specify the ordering of operators, there are many possibilities for A. Here we have shown the "normal ordered" form: we write $A = :A^{cl}:$ to indicate normal ordering.

There seems to be a folk theorem which says that, whenever we have a Poisson bracket relation for classical observables (for instance, a conservation law) then the corresponding quantum version must use the normal ordered form of the classical functional. Such a connection is not a necessary ingredient for exact integrability of a quantum theory. It is well known [9-13] that the classical and quantum coefficients $A^{cl}(\lambda)$, $B^{cl}(\lambda)$, $A(\lambda)$ and $B(\lambda)$ of the Zakharov-Shabat scheme for the NLS, in an infinite box, satisfy

$$A(\lambda) \quad = \quad :A^{cl}(\lambda): \qquad (\mathrm{Im}\,(\lambda) < 0)\,,$$

$$B^\dagger(\lambda) \quad = \quad :B(^{cl}\lambda): \qquad (\mathrm{Im}\,(\lambda) = 0)\,, \qquad\qquad (5.3)$$

For the CNLS in an infinite box it is the expansion of the logarithm of $A^{cl}(\lambda)$ which generates the higher Hamiltonians in a simple (linear) way. Since the logarithm is non-linear, we would expect quantum corrections in this expansion. Thus it causes no difficulty for the QISM that we should have

$$H_2 \quad = \quad :H_2^{cl}: \,, \qquad\qquad H_4 \quad \neq \quad :H_4^{cl}: \qquad\qquad (5.4)$$

What is important is that H_4^{cl} can be recovered from H_4 in the quasiclassical limit, and this is so because we have shown that there is no discrepancy in the various

terms of a_{mn}. Again, the expansion of $A(\lambda)$ itself leads to products of quantities H_n^{cl} in the higher coefficients A_n^{cl}: the difference between normal ordering the H_n^{cl} or the A_n^{cl} once more involves quantum corrections.

In ref [15] some problems are indicated with the expansion of $A(\lambda)$. We have already mentioned that some of these are computational errors, and we have given the corrected formula for A_3 in (1.16). The substantial argument given in ref [15] is that, if the asymptotic methods used for the decomposition of $A^{cl}(\lambda)$ are repeated with $:A^{cl}(\lambda):$ with the normal ordering retained at each step, then we should get the expansion coefficients as $A_n = :A_n^{cl}:$ for all n. This is not so, and we need to see why the analysis fails for $n \geq 3$. We stress that the manipulations used in the quoted analysis are purely formal, and the calculation of the quantum corrections (which appear from the normal ordering) depends on using the canonical commutation relations for fields $\Psi^\dagger(x)$ and $\Psi(y)$ in integrals which have $x = y$ as one limit. While this kind of formal analysis may work well in many cases, we have no right to expect this. Any proper asymptotic analysis will depend on the action of $A(\lambda)$ in Fock space as an integral operation. In the QISM, $A(\lambda)$ is formally defined by its formula in the second quantised form [12]: viz

$$A(\lambda) \quad = \quad \sum_{n \geq 0} c^n \int dx^n \, dy^n \, \theta(x_1 < y_1 < ... < x_N < y_N)$$

$$\exp \left[i\lambda(x_1 - y_1 + ... + x_N - y_N) \right] \Psi^\dagger(x_1) ... \Psi^\dagger(x_N) \, \Psi(y_1) ... \Psi(y_N) \qquad (5.5)$$

Here $\theta(x_1 < y_1 < ... < x_N < y_N)$ stands for the indicator function of the set $\{x_1 < y_1 < ... < x_N < y_N\}$. The action of this operator in Fock space was given in ref [13]. Let $|f\rangle$ and $|g\rangle$ be two N particle states specified by symmetric functions $f(x_1, ..., x_N)$ and $g(x_1, ..., x_N)$ via

$$|f\rangle \quad = \quad \int dx^n \, f(x_1, ..., x_N) \, \Psi^\dagger(x_1) ... \Psi^\dagger(x_N) |0\rangle \,, \qquad (5.6)$$

with a similar equation for $|g\rangle$, then the action

$$|g\rangle \quad = \quad A(\lambda)|f\rangle \qquad (5.7)$$

is given by the following integral operator:

$$g(x_1 < ... < x_N) = f(x_1,...,x_N) + \sum_{n=1}^{N} c^n \sum_{i_1 < ... < i_n} \int_{x_{i_n}}^{\infty} d\xi_n ... \int_{x_{i_1}}^{x_{i_2}} d\xi_1$$

$$\exp\left[i\lambda(x_{i_1}-\xi_1+ ... +x_{i_n}-\xi_n)\right] f(\xi_1,...,\xi_N | \xi_1 \rightarrow x_{i_1}, ... ,\xi_n \rightarrow x_{i_n}) \qquad (5.8)$$

Here the notation for the integrand means that the indicated changes of variables are made in the function f. Also the evaluation of $g(x_1,...,x_N)$ for orderings other than $x_1 < ... < x_N$ is by symmetrisation.

An integral operator typically represents a boundary value problem. Direct computation from (5.8) shows that this is so for $A(\lambda)$. The functions $f(x_1,...,x_N)$ and $g(x_1,...,x_N)$ are related by the boundary value problem

$$\prod_{j=1}^{N}\left(\lambda + i\frac{\partial}{\partial x_j}\right)g(x_1,...,x_N) = \prod_{j=1}^{N}\left(\lambda + i\frac{\partial}{\partial x_j} - ic\right)f(x_1,...,x_N) \qquad (5.9)$$

$$\left[cg + (\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}})g\right]_{x_{j+1}=x_j+0} = \left[cf + (\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}})f\right]_{x_{j+1}=x_j+0}. \qquad (5.10)$$

It is shown in ref [15] that the various operators of the QNLS theory ($c \neq 0$) are intertwinings of corresponding free operators ($c=0$) restricted to a domain in which appropriate boundary conditions are satisfied. Eq. (5.10) tells us that $A(\lambda)$ may be restricted to the appropriate domain: that is, $A(\lambda)$ preserves just the correct boundary conditions. In fact we can see this directly from ref [13] where it is shown that the integral operator (5.8) is diagonal on the Bethe-Ansatz eigenstates. The latter are a complete set among precisely those functions which satisfy the boundary conditions required to define the commuting operators H_n for the interacting case as restrictions of the free Hamiltonians.

The decomposition of the operator (5.8) in inverse powers of λ, using the usual techniques of integration by parts, gives a non-uniform asymptotic expansion in the variables $x_1,...,x_N$, which fails exactly at the boundaries $x_j = x_k$. The two particle sector will suffice to illustrate the ideas and in fact we only need go to $1/\lambda^3$ to see how things work:

$$g(x,y) \approx f(x,y)$$

$$+ c\left(\frac{2}{i\lambda} f(x,y) + \frac{1}{(i\lambda)^2} \{f_x(x,y) + f_y(x,y)\} + \frac{1}{(i\lambda)^3} \{f_{xx}(x,y) + f_{yy}(x,y)\} + \ldots \right)$$

$$+ c^2\left(\frac{1}{(i\lambda)^2} [f(x,y) - e^{i\lambda(x-y)} f(y,y)]\right.$$

$$\left. + \frac{1}{(i\lambda)^3} [\{f_x(x,y) + f_y(x,y)\} - e^{i\lambda(x-y)} \{f_x(y,y) + f_y(y,y)\}] \right) + \ldots \qquad (5.11)$$

Away from the boundaries $x = y$, this expansion correctly gives the differential parts of the operators in the asymptotic expansion, since we may neglect the exponentially small corrections when $x < y$ and $\lambda \to -i\infty$. The results then are the same as for the non-interacting theory. To complete the expansion we must find out what happens at the boundaries and this cannot be deduced from (5.11). However, our comments above show us that we may identify the operators in the expansion of $A(\lambda)$ by their differential parts found from the asymptotic expansion away from the boundaries $x_i = x_j$. These give us the free Hamiltonians: the intertwining property takes them into the corresponding interacting Hamiltonians. The details of this calculation are equivalent to the calculations involving eigenvalues given in section 4, so we do not repeat them here. The result is therefore that the correct asymptotic expansion of $A(\lambda)$ in terms of higher Hamiltonians is given by eqs. (1.12) - (1.16), in agreement with the calculations made from a lattice limit.

6   CONCLUSIONS

As we mentioned in the introduction, there have been a number of papers which have raised various mathematical questions about the QISM solution of the QNLS. In this paper we have been concerned with the most serious objections, which suggested that the conservation laws are flawed. We have shown that they are not. However, it must be stressed that explicit formulas for the higher conserved quantities are difficult to get and to use because one must go through singular calculations. While these difficulties may impair their practical utility, it it certainly not a flaw in the QISM, and that is the chief concern of this paper. Fortunately, there exists a well behaved lattice regularisation of the model

which can control these problems. The same comments apply to the quantum trace identities [21]. In this view, everything in the continuous case is understood as the appropriate limit from the lattice. This controls the ordering problem for these laws, and shows that normal ordering is not correct beyond H_3: also that there exist quantum corrections beginning with A_3 in the expansion of $A(\lambda)$.

Our conclusion is that the Bethe Ansatz solution and the QISM give the same (valid) conservation laws: also that the quasi-classical limit is correct. So there is no failure of the QISM for the quantum non-linear Schrödinger equation. This was a most important point to resolve now that the QISM seems poised to solve the long-standing problem of the construction of correlation functions for solvable models [22].

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REFERENCES

1.   M. Ablowitz, D Kaup, A Newell & H. Segur, Studies in. Appl. Math., **53**, (1974), 249.

L. D. Faddeev & L. A. Takhtajan, "Hamiltonian methods in the theory of Solitons" (Springer, New York, 1987).

2.   V. E. Zakharov & A. B. Shabat, Zh. Eksp. Teor. Fiz., **61**, (1971), 116.

3.   V. E. Zakharov & S. V. Manakov, Teor. Mat. Fiz., **19**, (1974), 332.

4.   V. E. Korepin & L. D. Faddeev, Theor. Math. Phys., **25**, (1975), 1039.

5.   D. J. Kaup, J. Math. Phys., **16**, (1975), 2036.

6.   P. P. Kulish, S. V. Manakov & L. D. Faddeev, Theor. Math. Phys., **28**, (1976), 615.

7.   L. D. Faddeev & V. E. Korepin, Phys. Rep., **42**, (1978), 1.

8.   L. D. Faddeev, & E. K. Sklyanin, Dokl. Akad. Nauk. SSSR, **243**, (1978), 1430.
     E. K. Sklyanin, L. A. Takhtajan & L. D. Faddeev, Teor. Mat. Fiz **40**, (1979), 194.

9.   L. D. Faddeev, Sov. Sci. Rev. Math. Phys., **C1**, (1981), 107.

10   H. B. Thacker, Rev. Mod. Phys., **53**, (1981), 253.

11. N. M. Bogoliubov, A.G. Izergin & V. E. Korepin, Lecture notes in Physics,
     242, (Springer, Berlin, 1985), 220.

12. B. Davies & T. D. Kieu, Inverse Problems **2**, (1986), 141
     B Davies, Inverse Problems **4**, (1988), 47.

13. B. Davies & E. Gutkin, Physica **A151**, (1988), 167.

14. E. Gutkin, Ann. Inst. Henri Poincare, Anal. nonlin., **2**, (1985), 67.

15. E. Gutkin, Physics Reports **167**, (1988), 1.

16. E. Lieb & W. Liniger, Phys. Rev., **130**, (1963), 1605: F. A. Berezin, G. P. Pohil
     & V. M. Finkelberg, Vetn. Mosk. Univ. Ser 1, (1964), 1: J. B. McGuire J. Math.
     Phys., **5**, (1964), 622: C. N. Yang, Phys. Rev. Lett., **19**, (1967), 1312.

17. A. G. Izergin & V. E. Korepin, Dokl. Akad. Nauk. SSSR, **259**, (1981), 76.

18. A. G. Izergin & V. E. Korepin, Nucl. Phys., **B205 [FSS]**, (1981), 401.

19. N. M. Bogoliubov & V. E. Korepin, Theor. Mat. Fiz. **66**, (1986), 300.

20. V. O. Tarasov, L. A. Takhtajan & L. D. Faddeev, Teor. Mat. Fiz. **57**, (1988), 163.

21. A. G. Izergin, V. E. Korepin & F. A. Smirnov, Teor. Math. Fiz **48**, (1981), 319.

22. A. R. Its, A. G. Izergin, V. E. Korepin, & N. A. Slavnov, CMA preprint.