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CORRELATION LENGTH OF THE ONE-DIMENSIONAL BOSE GAS

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The exact expression for correlation length in the one-dimensional Bose gas is obtained at any value of coupling constant and temperature.

1. Introduction

Recently the method of calculation of the current correlation function for the one-dimensional Bose gas was created [1-3]. In this paper we consider the one-dimensional Bose gas. The hamiltonian of the system is

$$H = \int_0^L \mathrm{d}x(\partial_x \Psi^+ \partial_x \Psi + c\Psi^+\Psi^+\Psi\Psi - h\Psi^+\Psi)\,,$$

$$[\Psi(x), \Psi^+(y)] = \delta(x-y)\,. \qquad (1)$$

Here L is the length of a box, c a coupling constant $(c>0)$ h a chemical potential $(h>0)$. In the thermodynamical limit $L\to\infty$ and $N\to\infty$ (N the number of the particles), $\rho = N/L$ fixed.

Exact eigenfunctions of H were constructed in [4]. The model was embedded in a quantum inverse scattering method in [7-11]. The zero-temperature case was solved in [4, 5]. The thermodynamical properties of the system were evaluated in the paper [6].

Let us consider an N-particle wave function with periodical boundary conditions. The system of equations for the permitted values of particles momenta looks like [4, 6]

$$\lambda_j L + \sum_{\substack{k=1\\k\neq j}}^{N} \Theta(\lambda_j - \lambda_k) = 2\pi n_j\,. \qquad (2)$$

Here $\Theta(\lambda) = i \ln\{(\lambda+ic)/(\lambda-ic)\} - \pi$, n_j is the set of integer numbers ($n_j \neq n_k$ when $j\neq k$, a consequence of the Pauli principle [14]). It should be mentioned [6] that there exists a one-to-one correspondence for any set $\{n\}$ and eigenfunctions of the hamiltonian (1). Using the symmetry (Bose) of the wave function, we can put

$$n_{j+1} > n_j\,, \qquad \lambda_{j+1} > \lambda_j\,. \qquad (3)$$

Taking the sum of all equations in (2), we find

$$LR = 2\pi \sum_{j=1}^{N} n_j , \qquad R = \sum_{j=1}^{N} \lambda_j . \tag{4}$$

Here R is the total momentum of the system. Further we shall consider the particles in the center-of-mass system, i.e. $R=0$. Eq. (4) then implies

$$\sum_{j=1}^{N} n_j = 0 . \tag{5}$$

In the thermodynamic limit eq. (2) can be rewritten in the form [6]

$$2\pi\rho_t(\lambda) = 2\pi[\rho(\lambda)+\rho_h(\lambda)] = 1 + \int_{-\infty}^{+\infty} K(\lambda,\mu)\rho(\mu)\,d\mu , \tag{6}$$

$$K(\lambda,\mu) = \frac{\partial\Theta(\lambda,\mu)}{\partial\lambda} = \frac{2c}{c^2+(\lambda-\mu)^2} . \tag{7}$$

Here $\rho(\lambda)$ is the destribution function of particles and $\rho_h(\lambda)$ is the distribution function of holes (the exact definition of this function see in [6]) and $\rho_t(\lambda)$ is the distribution of vacancies.

The function $\rho(\lambda)$ is a positive bounded function. The physical density ρ is

$$0 < \rho = \frac{N}{L} = \int_{-\infty}^{+\infty} \rho(\lambda)\,d\lambda . \tag{8}$$

It can be shown that

$$\frac{1}{2\pi} \leqslant \rho_t(\lambda) \leqslant \frac{1}{2\pi}\left(1+\frac{2}{c}\rho\right) . \tag{9}$$

This estimate can be derived from the restriction on the permitted values of the particle momenta in the Dirac sea [12]:

$$|\lambda_{k+1}-\lambda_k| \geqslant \frac{2\pi}{L}\left(1+\frac{2}{c}\rho\right)^{-1} .$$

Now we want to calculate the grand canonical partition function of the model. Let us consider

$$Z = \operatorname{tr} e^{-H/T} = \sum_{N=0}^{\infty} Z_N , \tag{10}$$

where

$$Z_N = \frac{1}{N!} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} \langle\{n\}|\, e^{-H/T} |\{n\}\rangle$$
$$= \frac{1}{N!} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} e^{-E_N/T} . \tag{11}$$

Here $E_N=\sum_{j=1}^{N}(\lambda_j^2-h)$ and $|\{n\}\rangle$ is the eigenfunction of the hamiltonian which corresponds to the set $\{n\}$. Using (3), (5) we can rewrite (11) in the form

$$Z_N=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=n_1+1}^{\infty}\cdots\sum_{n_N=n_{N-1}+1}^{\infty}e^{-E_N/T}$$
$$=\sum_{n_{2,1}=1}^{\infty}\sum_{n_{3,2}=1}^{\infty}\cdots\sum_{n_{N,N-1}=1}^{\infty}e^{-E_N/T}. \tag{12}$$

Here in the last term we pass to the new variables

$$n_{j+1,j}\equiv n_{j+1}-n_j\,,\qquad \sum n_j=0\,. \tag{13}$$

Let us calculate the ratio of the number of vacancies and number of particles (in the neighbourhood of given momenta λ_j) in terms of microscopic and macroscopic variables:

$$\frac{n\text{ of vac.}}{n\text{ of part}}=n_{j+1,j}\,,\qquad \frac{n\text{ of vac.}}{n\text{ of part}}=\frac{\rho_t(\lambda_j)}{\rho(\lambda_j)}\,. \tag{14}$$

By means of this formula we can pass now from microscopic variables n_j to macroscopic $\rho_t(\lambda)$, $\rho(\lambda)$. As mentioned by Yang and Yang [6] the given $\rho(\lambda)$ does not define $\{n\}$ in a unique way, for at the fixed $\rho(\lambda)$ there exists

$$\prod_\lambda\frac{[\rho_t(\lambda)\,d\lambda]!}{[\rho(\lambda)\,d\lambda]![\rho_h(\lambda)\,d\lambda]!}$$

different configurations $\{n\}$. Taking into account this fact and formula (14) we can rewrite (10), (12) for the large system ($L\to\infty$) in the form of a functional integral

$$Z=\text{const}\int\left[\prod_\lambda D\frac{\rho_t(\lambda)}{\rho(\lambda)}\right]e^{-X/T}, \tag{15}$$

where X is

$$X=L\int_{-\infty}^{+\infty}(\lambda^2-h)\rho(\lambda)\,d\lambda-LT\int_{-\infty}^{+\infty}[\rho_t(\lambda)\ln\rho_t(\lambda)-\rho(\lambda)\ln\rho(\lambda)$$
$$-\rho_h(\lambda)\ln\rho_h(\lambda)]\,d\lambda\,.$$

When L tends to infinity we may evaluate the integral in (15) by the method of steepest descent. We should minimize the functional X subject to the constraint (6) ($\delta^2X>0$, see [6]); this procedure leads to the equation which defines the state of the thermodynamical equilibrium of the model:

$$\varepsilon(\lambda)=\lambda^2-h-\frac{T}{2\pi}\int_{-\infty}^{+\infty}K(\lambda,\mu)\ln[1+e^{-\varepsilon(\mu)/T}]\,d\mu\,. \tag{16}$$

Here $\varepsilon(\lambda)\equiv T\ln[\rho_h(\lambda)/\rho(\lambda)]$ and T is the temperature. The Fermi factor $\vartheta(\lambda)$

will play an important role below:

$$\vartheta(\lambda)=\frac{1}{1+\exp\{\varepsilon(\lambda)/T\}}. \tag{17}$$

Let us emphasize that the state of thermal equilibrium is not the pure one (it is not the eigenstate of the hamiltonian). This state is a mixture of the eigenstates. Let us denote by $|\phi_T\rangle$ one of these eigenstates.

In our paper we consider the correlation function of the currents $j(x)=\Psi^+(x)\Psi(x)$:

$$\langle j(x)j(0)\rangle=\frac{\operatorname{tr}[\mathrm{e}^{-H/T}j(x)j(0)]}{\operatorname{tr}[\mathrm{e}^{-H/T}]}. \tag{18}$$

For the large system we can again express the trace as the functional integral and evaluate it by the method of steepest descent:

$$\langle j(x)j(0)\rangle=\frac{\langle\phi_T|j(x)j(0)|\phi_T\rangle}{\langle\phi_T|\phi_T\rangle}. \tag{19}$$

Here $|\phi_T\rangle$ is one of the eigenstates of the hamiltonian which corresponds to the state of thermal equilibrium. In [1] we proved that the right-hand side of (19) does not depend on the particular choice of $|\phi_T\rangle$.

In the frame of perturbation theory the correlation functions of the model were studied in [13].

The right-hand side of (19) was calculated in [1-3] in the form of the series

$$\langle\!\langle j(x)j(0)\rangle\!\rangle=\langle :j(x)j(0):\rangle-\langle j(0)\rangle^2=\sum_{k=2}^{\infty}\Gamma_k(x),$$

$$\langle :j(x)j(0):\rangle=\langle j(x)j(0)\rangle-\delta(x)\langle j(0)\rangle. \tag{20}$$

Here

$$\langle j(0)\rangle=\int_{-\infty}^{+\infty}\rho(\lambda)\,\mathrm{d}\lambda=\rho.$$

The first two terms of the decomposition are equal to

$$\Gamma_2(x)=-\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\mathrm{d}\lambda_1\,\omega(\lambda_1)\vartheta(\lambda_1)\int_{-\infty}^{+\infty}\mathrm{d}\lambda_2\,\omega(\lambda_2)\vartheta(\lambda_2)$$
$$\times\left(\frac{\lambda_1-\lambda_2+ic}{\lambda_1-\lambda_2-ic}\right)\left[\frac{p(\lambda_1,\lambda_2)}{\lambda_1-\lambda_2}\right]^2\mathrm{e}^{xp(\lambda_1,\lambda_2)}, \tag{21}$$

$$\Gamma_3(x)=\frac{c}{2\pi^3}\int_{-\infty}^{+\infty}\left\{\prod_{j=1}^{3}\omega(\lambda_j)\vartheta(\lambda_j)\,\mathrm{d}\lambda_j\right\}\left[\frac{p(\lambda_1,\lambda_2)}{\lambda_1-\lambda_2}\right]^2$$
$$\times\left(\frac{\lambda_1-\lambda_2+ic}{\lambda_1-\lambda_2-ic}\right)\left(\frac{\lambda_3-\lambda_2}{\lambda_3-\lambda_1}+\frac{\lambda_3-\lambda_1}{\lambda_3-\lambda_2}\right)\frac{\exp\{xp(\lambda_1,\lambda_2)\}}{(\lambda_3-\lambda_1+ic)(\lambda_2-\lambda_3+ic)}. \tag{22}$$

The principal value of the integral must be taken in (22). The statistical weight $\omega(\lambda)$ is

$$\omega(\lambda)=\exp\left\{-\frac{1}{2\pi}\int_{-\infty}^{+\infty} K(\lambda,\mu)\vartheta(\mu)\,\mathrm{d}\mu\right\},$$

$$0<\omega(\lambda)<1\,. \tag{23}$$

The function $p(\lambda_1,\lambda_2)$ is

$$p(\lambda_1,\lambda_2)=-i(\lambda_1-\lambda_2)+\int_{-\infty}^{+\infty}\mathrm{d}t\,\vartheta(t)P(t,\lambda_1,\lambda_2)\,. \tag{24}$$

The function $P(t,\lambda_1,\lambda_2)$ is defined in a unique way by the dressing nonlinear equation

$$1+2\pi P(t,\lambda_1,\lambda_2)=\left(\frac{\lambda_1-t+ic}{\lambda_1-t-ic}\right)\left(\frac{\lambda_2-t-ic}{\lambda_2-t+ic}\right)\exp\left\{\int_{-\infty}^{+\infty}K(t,s)\vartheta(s)P(s,\lambda_1,\lambda_2)\,\mathrm{d}s\right\} \tag{25}$$

and by inequality $\operatorname{Re}P(t,\lambda_1,\lambda_2)\leqslant 0$. Its domain of definition is $\operatorname{Im}\lambda_1=\operatorname{Im}\lambda_2=\operatorname{Im}t=0$. A detailed investigation of eq. (25) and function P will be given in the next section.

We shall further need the expression for the correlation function at zero temperature. Explicitly it is given in [2, 3]. At $T=0$ eq. (16) becomes

$$\varepsilon_0(\lambda)=\lambda^2-h+\frac{1}{2\pi}\int_{-q}^{q}K(\lambda,\mu)\varepsilon_0(\mu)\,\mathrm{d}\mu\,. \tag{26}$$

The bare Fermi momentum q is defined in a unique way from

$$\varepsilon_0(q)=0\,. \tag{27}$$

(We shall use further the subindex "zero" for the quantities at $T=0$.) The function $\varepsilon_0(\lambda)$ is negative when $-q<\lambda<q$ and is positive when $\lambda>q$, $\lambda<-q$. Using this property it is easy to take the limit $T\to 0$ in (21)-(25) writing

$$\int_{-\infty}^{+\infty}f(t)\vartheta(t)\,\mathrm{d}t\to\int_{-q}^{q}f(t)\,\mathrm{d}t\,. \tag{28}$$

2. Integral equations

Let us consider the integral operator $\hat{K}_T$. If f is any normalized function then

$$(\hat{K}_T f)(\lambda)=\int_{-\infty}^{+\infty}K(\lambda,\mu)\vartheta(\mu)f(\mu)\,\mathrm{d}\mu\,. \tag{29}$$

To get some estimates on its eigenvalues we construct the operator $\tilde{K}$ with the kernel

$$\tilde{K}(\lambda,\mu)=\sqrt{\vartheta(\lambda)}K(\lambda,\mu)\sqrt{\vartheta(\mu)}\,.$$

The operator $\hat{K}_T$ is similar to the operator $\tilde{K}$. It can be shown that

$$\int_{-\infty}^{+\infty} f^2(\lambda)\left(1-\frac{\vartheta(\lambda)}{2\pi\rho(\lambda)}\right) \geqslant \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} d\mu\, f(\lambda)f(\mu)\tilde{K}(\lambda,\mu)\,. \qquad (30)$$

Here f is an arbitrary function. Thus from (30) and (9) we get the estimate on the eigenvalues $|K|$ of $\hat{K}_T$:

$$0<\frac{1}{2\pi}|K|\leqslant\frac{2\rho}{c+2\rho}\,. \qquad (31)$$

It follows that the eigenvalues of $\hat{K}_T$ range between 0 and 1, or more precisely, they are different from 1 with a gap of order $(1+2\rho/c)^{-1}$. Using these properties of $\hat{K}_T$ we can prove that the solution of the integral equation (25) exists.

Let us rewrite (25) in the form

$$1+2\pi P(t)=a(t)\exp\{(\hat{K}_T P)(t)\}\,, \qquad \operatorname{Re} P(t)\leqslant 0\,. \qquad (32)$$

Here $|a(t)|=1$. Define further the sequence P_n:

$$P_0=0\,,$$

$$P_{n+1}(t)=\frac{a(t)}{2\pi}\exp\{(\hat{K}_T P_n)(t)\}-\frac{1}{2\pi}$$

$$(n=0,1,\ldots,\infty)\,. \qquad (33)$$

We shall prove now that this functional sequence converges. First we show that if $\operatorname{Re} P_n\leqslant 0$ then $\operatorname{Re} P_{n+1}\leqslant 0$. Clearly, we have

$$|a(t)\exp\{\hat{K}_T P_n\}|\leqslant 1 \Rightarrow \operatorname{Re} a(t)\exp\{\hat{K}_T P_n\}\leqslant 1\,.$$

Thus, $\operatorname{Re} P_{n+1}\leqslant 0$. Here we have used the positiveness of the kernel of the operator $\hat{K}_T$. Now we can prove

$$|P_{n+1}(t)-P_n(t)|\leqslant\frac{1}{2\pi}(\hat{K}_T|P_n-P_{n-1}|)(t)\,. \qquad (34)$$

Subtract

$$P_n(t)=\frac{a(t)}{2\pi}\exp\{(\hat{K}_T P_{n-1})(t)\}-\frac{1}{2\pi}$$

from (33) to obtain

$$P_{n+1}(t)-P_n(t)=\frac{a(t)}{2\pi}[e^{(\hat{K}_T P_n)(t)}-e^{(\hat{K}_T P_{n-1})(t)}]\,. \qquad (35)$$

Let us use the well-known inequality

$$|e^{z_1}-e^{z_2}|\leqslant|z_1-z_2|\,.$$

Here z_1 and z_2 are two complex numbers from the left half-plane $\operatorname{Re} z_{1,2}\leqslant 0$. It is now possible to complete the proof

$$|P_{n+1}(t)-P_n(t)|=\frac{1}{2\pi}|e^{\hat{K}_T P_n}-e^{\hat{K}_T P_{n-1}}|$$

$$\leq\frac{1}{2\pi}|\hat{K}_T(P_n-P_{n-1})|\leq\frac{1}{2\pi}(\hat{K}_T|P_n-P_{n-1}|)(t)\,.$$

Since the eigenvalues of $(1/2\pi)\hat{K}_T$ are less than unity but greater than zero (31), the sequence P_n converges in L_2 and its limit satisfy (25). It should be mentioned that if $a(t)$ is real then $P(t)$ is real also.

The uniqueness theorem can be proved similarly. Assume that P_1 and P_2 are two different solutions of (25). Subtracting one from the other we obtain

$$|P_1(t)-P_2(t)|\leq\frac{1}{2\pi}(\hat{K}_T|P_1-P_2|)(t)\,.$$

Now we multiply this relation by $\vartheta(t)|P_1-P_2|$ and integrate it to obtain

$$\int_{-\infty}^{+\infty}f^2(t)\,\mathrm{d}t-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}s\int_{-\infty}^{+\infty}\mathrm{d}t\,f(s)f(t)\tilde{K}(s,t)\leq 0\,,$$

$$f(t)=\sqrt{\vartheta(t)}|P_1(t)-P_2(t)|\,,$$

in contradiction to (30); so $P_1=P_2$.

Therefore if $a(t)=1$ then $P=0$. Notice that if $|a(t)|$ in (32) is less than unity the uniqueness theorem and the theorem of existence can be proved similarly. It means that the function P can be analytically continued with respect to λ. In the next section we shall need the analytical continuation with respect to λ_2 into the upper half-plane and with respect to λ_1 into the lower one. It follows from

$$|a(t)|\leq 1\,,\qquad a(t)=\left(\frac{\lambda_1-t+ic}{\lambda_1-t-ic}\right)\left(\frac{\lambda_2-t-ic}{\lambda_2-t+ic}\right),$$

that P could be analytically continued without singularities into the domain

$$\operatorname{Im} t=0\,,\qquad \operatorname{Im}\lambda_1\leq 0\,,\qquad \operatorname{Im}\lambda_2\geq 0\,. \tag{36}$$

We want now to investigate some other properties of the P-function. Let us prove that in the domain of definition

$$|P(t,\lambda_1,\lambda_2)|\leq\frac{1}{\pi}\,. \tag{37}$$

Clearly

$$|P(t)|\leq\frac{1}{2\pi}|a(t)\,e^{(\hat{K}_T P)(t)}|+\frac{1}{2\pi}\leq\frac{1}{\pi}\,.$$

This inequality and (9) give

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, \vartheta(t) P(t, \lambda_1, \lambda_2) \leq \frac{1}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}t\, \vartheta(t) \leq 2\rho . \tag{38}$$

This means that $P(\lambda_1, \lambda_2)$ slightly differs from $-i(\lambda_1-\lambda_2)(p(\lambda_1, \lambda_2) \sim -i(\lambda_1-\lambda_2))$.

The rest of the properties of the P-function we shall enumerate without proof:

(i) if $a(t) \neq 1$, t, λ_1, λ_2 are finite, then $\operatorname{Re} P(t, \lambda_1, \lambda_2) \neq 0$;

(ii) $\bar{P}(t, \lambda_1, \lambda_2) = P(\bar{t}, \bar{\lambda}_2, \bar{\lambda}_1)$, $\bar{p}(\lambda_1, \lambda_2) = p(\bar{\lambda}_2, \bar{\lambda}_1)$;

(iii) $P(t, \lambda, \lambda) = 0$;

(iv) $P(-t, -\lambda_1, -\lambda_2) = P(t, \lambda_2, \lambda_1)$;

(v) when $\lambda_1 = \bar{\alpha}$, $\lambda_2 = \alpha$ $(\operatorname{Im} \alpha > 0, \operatorname{Im} t = 0) P(t, \bar{\alpha}, \alpha)$ is real since $a(t)$ is real; $p(\bar{\alpha}, \alpha)$ is real;

(vi) for $c \to \infty$

$$P(t, \lambda_1, \lambda_2) = \frac{1}{i\pi c}(\lambda_1-\lambda_2)\left(1+\frac{2}{c}\rho\right) - \frac{1}{\pi c^2}(\lambda_1-\lambda_2)^2 + \mathrm{O}\left(\frac{1}{c^3}\right), \tag{39}$$

$$\rho(\lambda) = \frac{1}{2\pi}\left(1+\frac{2}{c}\rho\right)\vartheta(\lambda) + \mathrm{O}\left(\frac{1}{c^3}\right), \tag{40}$$

$$p(\lambda_1, \lambda_2) = -i(\lambda_1-\lambda_2)\left(1+\frac{2}{c}\rho\right) - \frac{2}{c^2}(\lambda_1-\lambda_2)^2\rho + \mathrm{O}\left(\frac{1}{c^3}\right). \tag{41}$$

We shall use these properties to calculate the asymptotics of correlator in the next section.

To conclude this section let us analyze the equation for $\varepsilon(\lambda)$ (16). This equation has a unique solution [6] with the properties

$$\varepsilon(\lambda) = \varepsilon(-\lambda), \tag{42}$$

$$\bar{\varepsilon}(\bar{\lambda}) = \varepsilon(\lambda), \tag{43}$$

$$\varepsilon(\lambda) \xrightarrow[\lambda \to \pm\infty]{} \lambda^2 . \tag{44}$$

$\varepsilon(\lambda)$ has no singularities on the real axis.

Let us try to continue $\varepsilon(\lambda)$ into the upper half-plane. It is easily seen that with the help of (16) we can continue $\varepsilon(\lambda)$ up to $\operatorname{Im} \lambda = c$. Within this region $\varepsilon(\lambda)$ has the asymptotic λ^2 when $\lambda \to \pm\infty$. At $\operatorname{Im} \lambda = c$ the kernel K becomes singular. If we want to continue $\varepsilon(\lambda)$ further than $\operatorname{Im} \lambda > c$ it is sufficient to shift the contour of integration into the upper half-plane $0 < \operatorname{Im} \mu < c$. It is clear that we can do so up to the point α, where

$$\vartheta^{-1}(\alpha) = 1 + \mathrm{e}^{\varepsilon(\alpha)/T} = 0, \qquad \operatorname{Im} \alpha > 0 . \tag{45}$$

The function $\varepsilon(\lambda)$, however, can be continued further with the help of (16). We thus obtain that the first singularity of $\varepsilon(\lambda)$ is $\alpha + ic$ (the contour of integration is locked by singularities of $\ln(1+\exp\{-\varepsilon(\lambda)/T\})$ and $K(\lambda, \mu)$). It is easy to prove that the solution of (45) necessarily exists. In fact, if $\varepsilon(\lambda)$ is continued into the complex plane so that $1+\exp\{\varepsilon(\lambda)/T\}$ has no zeros we could continue $\varepsilon(\lambda)$ into an entire complex plane without singularities with the help of (16). The function $\varepsilon(\lambda)$ would be an entire function with polynomial asymptotics λ^2. It is possible only if $\varepsilon(\lambda)$ is polynomial. But the polynomial does not satisfy (16). So the zeros of $\vartheta^{-1}(\lambda)$ exist and they form the quadrangle

$$\alpha, -\alpha, \bar{\alpha}, -\bar{\alpha}, \qquad \operatorname{Im}\alpha > 0 \qquad (\text{when } T>0). \tag{46}$$

The function $\varepsilon(\lambda)$ could be continued up to these zeros without singularities. This property will help us in calculating the asymptotics of correlation function.

The zero α necessarily lies in the complex plane. On the real axis $\varepsilon(\lambda)$ is real and $\vartheta^{-1}(\lambda)$ has no zeros.

The statistical weight $\omega(\lambda)$ also could be continued into the complex plane. Its singularity nearest to the real axis is $\lambda = \alpha + ic$.

The analytical continuation of these functions has the properties

$$\bar{\varepsilon}(\lambda) = \varepsilon(\bar{\lambda}), \qquad \varepsilon(-\lambda) = \varepsilon(\lambda),$$
$$\bar{\omega}(\lambda) = \omega(\bar{\lambda}), \qquad \omega(-\lambda) = \omega(\lambda). \tag{47}$$

3. Asymptotic behaviour of the correlation function

We now consider

$$\langle\langle j(x)j(0)\rangle\rangle = \langle :j(x)j(0): \rangle - \langle j(0)\rangle^2. \tag{48}$$

The first term is given by (21). To analyze this expression when x tends to infinity we shall shift the contour of integration with respect to λ_1 into the lower half-plane and with respect to λ_2 into the upper one. The nearest barriers, when shifting, are the singularities of the Fermi factor $\vartheta(\lambda)$ which are situated at the points α, $-\alpha$, $\bar{\alpha}$, $-\bar{\alpha}$. These points are simple poles of $\vartheta(\lambda)$:

$$\vartheta(\lambda)|_{\lambda\to\alpha} \to -\frac{T}{\varepsilon'(\alpha)(\lambda-\alpha)}, \qquad e^{\varepsilon(\alpha)/T} = -1. \tag{49}$$

The contribution of these poles to $\Gamma_2(x)$ is

$$2T^2\left|\frac{\omega(\alpha)}{\varepsilon'(\alpha)}\right|^2\left(\frac{2\operatorname{Im}\alpha - c}{2\operatorname{Im}\alpha + c}\right)\left(\frac{p(\bar{\alpha},\alpha)}{2\operatorname{Im}\alpha}\right)^2 \exp\{xp(\bar{\alpha},\alpha)\} \tag{50}$$

$$+2T^2\operatorname{Re}\left[\left(\frac{\omega(\alpha)}{\varepsilon'(\alpha)}\right)^2\left(\frac{2\alpha - ic}{2\alpha + ic}\right)\left(\frac{p(-\alpha,\alpha)}{2\alpha}\right)^2 \exp\{xp(-\alpha,\alpha)\}\right]. \tag{51}$$

To calculate the contribution of other singularities to (21) we must shift the contour still further from the real axis. This will lead to the expressions decreasing with respect to x faster than (50), (51). The considerations based on the perturbation theory show us that (51) decreases faster than (50) when $x \to \infty$.

The asymptotics of the first term (21) at large distances are

$$\langle\!\langle j(x)j(0)\rangle\!\rangle \to \mathrm{e}^{-x/r_c}, \tag{52}$$

where r_c is the correlation length

$$\frac{1}{r_c} = -p(\bar{\alpha}, \alpha) = 2\,\mathrm{Im}\,\alpha - \int_{-\infty}^{+\infty} \mathrm{d}t\, \vartheta(t) P(t, \bar{\alpha}, \alpha) \geqslant 2\,\mathrm{Im}\,\alpha \geqslant 0\,. \tag{53}$$

It should be noted that the function p is real and

$$-\int_{-\infty}^{+\infty} \mathrm{d}t\, \vartheta(t) P(t, \bar{\alpha}, \alpha) \geqslant 0$$

is positive.

Thus we have analyzed the first term of the sequence for the correlation function [1]. The tracing of the others allows us to make the conjecture that the expression

$$r_c = -\frac{1}{p(\bar{\alpha}, \alpha)} \tag{54}$$

is correct for any value of the coupling constant. This is the principal formula of our work.

Let us analyze now different special cases. Consider the correlation length at $T \to 0$ (the point $T=0$ is the phase transition point). It is easy to show that the solution of the equation

$$\vartheta^{-1}(\alpha) = 0\,, \qquad \varepsilon(\alpha) = i\pi T, \tag{55}$$

is

$$\alpha = q + \frac{i\pi T}{\varepsilon_0'(q_T)}, \tag{56}$$

where q_T is defined by $\varepsilon(q_T) = 0$, $q_T > 0$ (as $T \to 0$, $q_T \to q$). Thus when $T \to 0$ the difference between α and $\bar{\alpha}$ becomes small. The factor $a(t)$ in (32), (25) tends to 1, so the solution of (32) is $P(t, \bar{\alpha}, \alpha) \to 0$. More precisely

$$P(t) = -\frac{T}{\varepsilon_0'(q)} F(t)\,,$$

where $F(t)$ satisfies the linear equation

$$F(t) - \frac{1}{2\pi}\int_{-q}^{q} K(t, s) F(s)\, \mathrm{d}s = \frac{2c}{c^2 + (t-q)^2}\,. \tag{57}$$

The correlation length tends to infinity:

$$\frac{1}{r_c} = \frac{2\pi T}{\varepsilon_0'(q)} + \frac{T}{\varepsilon_0'(q)}\int_{-q}^{q} F(t)\, \mathrm{d}t = \frac{2\pi T}{\varepsilon_0'(q)}\left[1 + \frac{1}{2\pi}\int_{-q}^{q} F(t)\, \mathrm{d}t\right]. \tag{58}$$

The coefficient on the right-hand side has a distinct physical sense; the velocity of sound. So

$$r_c = \frac{v}{2\pi T}. \tag{59}$$

The velocity of sound v is the derivative of physical energy with respect to physical momentum on the Fermi surface [5]:

$$v = \left.\frac{\mathrm{d}\varepsilon_0(\lambda)}{\mathrm{d}k_0(\lambda)}\right|_{\lambda=q} = \left.\frac{\mathrm{d}\varepsilon_0(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=q}\left[1+\frac{1}{2\pi}\int_{-q}^{q} F(t)\,\mathrm{d}t\right]^{-1}. \tag{60}$$

The physical momentum $k_0(\lambda)$ is

$$k_0(\lambda) = \lambda + \int_{-q}^{q} \Theta(\lambda-\mu)\rho_0(\mu)\,\mathrm{d}\mu.$$

Substituting (60) into (58) we obtain (59).

The same result was obtained in [13]. The correlations disintegrate when $T\to\infty$ (see the appendix).

Let us consider now the limit $c\to\infty$. We have

$$\varepsilon(\lambda) = \lambda^2 - A + \mathrm{O}\left(\frac{1}{c^3}\right), \qquad A>0, \qquad A = h + \frac{2}{c}\mathcal{P}.$$

Here $\mathcal{P}$ is the pressure [6]:

$$\mathcal{P} = \frac{T}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\lambda \,\ln\left(1+\mathrm{e}^{-\varepsilon(\lambda)/T}\right).$$

Changing $K\to 2/c$ it is easy to find the $1/c$ series expansion of A. We have

$$\alpha = \sqrt{A+i\pi T} = \sqrt{h+\frac{2}{c}\mathcal{P}+i\pi T}, \qquad \operatorname{Im}\alpha > 0.$$

Substituting this expression into (53) and (41) we get

$$\frac{1}{r_e} = 2\operatorname{Im}\alpha\left(1+\frac{2}{c}\rho\right) + \frac{2\rho}{c^2}4(\operatorname{Im}\alpha)^2.$$

Let us emphasise that in the strong coupling limit the two terms (50) and (51) begin to compete. The term (51) contains an additional decreasing factor $\exp\{-8\rho(\operatorname{Re}\alpha)^2 x/c^2\}$. For $c=\infty$ the sum of the two terms (50) and (51) gives asymptotics and contains oscillations [1].

For $c=\infty$ the correlation function is given by the following explicit formula:

$$\langle\!\langle j(x)j(0)\rangle\!\rangle = -\frac{1}{4\pi^2}\left[\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{i\lambda x}\,\mathrm{d}x}{1+\exp\{(\lambda^2-h)/T\}}\right]^2.$$

4. Asymptotic behaviour of the correlator at zero temperature

At zero temperature the correlation function

$$\langle\langle j(x)j(0)\rangle\rangle = \sum_{k=2}^{\infty} \Gamma_k^0(x) \tag{61}$$

was calculated in [2, 3] in the form of a series. Let us write down its first term:

$$\Gamma_2^0(x) = -\frac{1}{4\pi^2}\int_{-q}^{q} d\lambda_1\, \omega(\lambda_1) \int_{-q}^{q} d\lambda_2\, \omega(\lambda_2)\left(\frac{\lambda_1-\lambda_2+ic}{\lambda_1-\lambda_2-ic}\right)\left[\frac{p(\lambda_1,\lambda_2)}{\lambda_1-\lambda_2}\right]^2 e^{xp(\lambda_1,\lambda_2)}. \tag{62}$$

We can find $\Gamma_3^0(x)$ from (22) using the rule (28). Let us analyze this expression when x tends to infinity. Integrating by parts we shall get the leading term of the asymptotics:

$$-\frac{\omega^2(q)}{2\pi^2}\frac{1}{x^2}, \tag{63}$$

taking into account that $p(\lambda, \lambda) = 0$. Among the correction terms we have

$$\text{const}\,\frac{1}{x^2}\,e^{xp(q,-q)}. \tag{64}$$

This term contains oscillations. When $0<c<\infty$, Re $P<0$, so this term decreases exponentially with respect to x. When $c=\infty$, Re $P=0$ and (64) should be added to the leading term (63). So, when $c=\infty$ the asymptotic contains oscillations. This fact explains the results of [3]. If we analyze the rest of the terms of (61) we shall see that the asymptotics of the correlator at $0<c<\infty$ are equal to

$$\langle\langle j(x)j(0)\rangle\rangle \xrightarrow[x\to\infty]{} \frac{a}{x^2},$$

where a is the dimensionless constant. This formula was previously obtained in [13].

We see that the representation of the correlation function which has been obtained in [1-3] is very effective. Really, to calculate the asymptotic behaviour of the correlator it is sufficient to deal with its first two terms.

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Appendix

Let us analyze the behaviour of the correlation function at the high-temperature limit $T\to\infty$. It is difficult to investigate the expression (54) in this limit, so we shall solve here a more simple problem. We shall fix the distance x and study (21) when T tends to infinity.

To do this, let us rewrite eq. (16) using the following notation:

$$\tilde{\varepsilon}(\tilde{\lambda})=\frac{\varepsilon(\lambda)}{T}, \qquad \tilde{\lambda}=\frac{\lambda}{\sqrt{T}}, \qquad \tilde{\mu}=\frac{\mu}{\sqrt{T}},$$

$$\tilde{c}=\frac{c}{\sqrt{T}}, \qquad \tilde{h}=\frac{h}{T}, \qquad \tilde{K}(\tilde{\lambda},\tilde{\mu})=\frac{2\tilde{c}}{\tilde{c}^2+(\tilde{\lambda}-\tilde{\mu})^e}. \tag{A.1}$$

$$\tilde{\varepsilon}(\tilde{\lambda})=\tilde{\lambda}^2-h-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{K}(\tilde{\lambda},\tilde{\mu})\ln[1-\exp\{-\tilde{\varepsilon}(\tilde{\mu})\}]\,d\tilde{\mu}. \tag{A.2}$$

As $T\to\infty$ we have $\tilde{c}\to 0$, $\tilde{h}\to 0$ and $\tilde{K}(\tilde{\lambda},\tilde{\mu})\to 2\pi\delta(\tilde{\lambda}-\tilde{\mu})$. Thus (A.2) gives (case $c=0$ in [6])

$$\vartheta(\lambda)=\frac{1}{1+\exp\{\varepsilon(\lambda)/T\}}=e^{-\lambda^2/T}. \tag{A.3}$$

This leads to the following: $\omega(\lambda)\to e^{-1}$, $P(t,\lambda_1,\lambda_2)\to 0$ (see (25)) and

$$\Gamma_2(x)\xrightarrow[T\to\infty]{}\frac{T}{4\pi e^2}e^{-Tx^2/2}. \tag{A.4}$$

So we find that correlation of the currents $\langle\langle j(x)j(0)\rangle\rangle$ disintegrates at a distance of order $x\sim 1/\sqrt{T}$. It should be noted that expression (A.4) is correct for not very large x (the pre-asymptotic region). When x tends to infinity $\Gamma_2(x)$ decreases exponentially (see (52)). But for high temperatures correlations disintegrate now in the pre-asymptotic region.

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