The Construction of Supergravity Theories and their symmetries.

- Review the methods to construct a supergravity theory.

- shows that $D = 11$ supergravity is a non-linear realization. hep-th/05270

- argue that $D = 11$, IIA and IIB supergravity theories have a large Kac-Moody symmetry (E_{11}). hep-th/10104081
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Methods for Constructing Supergravity Theories

- Noether method using an action
- Noether method using the algebra
- On-shell superspace
- Dimensional reduction
- Gauging a space-time group
  Chamseddine West. MacDowell Mansouri
- Off-shell superspace
  Wess Zumino Siegel.

The Noether method using an action

Construction of off-shell supergravity. (...., Ferrara, Freedman, van Nieuwenhuizen)

Start from the linearized theory

$$h_{\mu\nu},\ \psi_{\mu\alpha};\quad M,\ N,\ b_\mu$$

The action

$$A^{(0)} = \int d^4x \left\{ h\partial^2 h + \psi\partial\psi + M^2 + N^2 + b_\mu^2 \right\}$$

is invariant under

$$\delta h_{\mu\nu} = \varepsilon \gamma_{(\mu}\psi_{\nu)}, \ldots\ldots$$

and

$$\delta\psi_{\mu\alpha} = \partial_\mu \eta_\alpha .$$

Let $\varepsilon \longrightarrow \varepsilon(x^\mu)$ then

$$\delta A^{(1)} = \int d^4x\ \partial_\mu \varepsilon_\alpha \, j^{\mu\alpha}$$

$$j_{\mu\alpha} = (\partial h\psi + \cdots)_{\mu\alpha}$$

An action invariant to order κ^0 is

$$A^{(1)} = A^{(0)} - \frac{\kappa}{2}\int d^4x\ \bar{\psi}^{\mu\alpha} j_{\mu\alpha}.$$

provided $\eta_\alpha = \frac{2}{\kappa}\varepsilon(x)_\alpha$

Now add to action and transformations to gain invariance order by order in κ.

$$A = \int d^4x \left\{ \frac{e}{2\kappa^2} R - \frac{1}{2} \psi_\mu R^\mu - \frac{e}{3} \left[M^2 + N^2 - b_\mu b^\mu \right] \right\}$$

invariant under

$$\delta \psi_\mu = \frac{2}{\kappa} D_\mu \varepsilon + i \gamma_5 \left(b_\mu - \frac{1}{3} \gamma_\mu \not{b} \right) \varepsilon$$
$$- \frac{1}{3} \gamma_\mu \left(M + i \gamma_5 N \right) \varepsilon$$

etc

stelle West
Ferrara
van N ieuwenhuizen

Noether Method on the Algebra

Construction of IIB supergravity

field content

$$e_\mu^a,\ A,\ A_{\mu\nu},\ A_{\mu,\dots\mu_4};\ \psi_{\mu\alpha},\ \lambda_\alpha$$ Nahm

The linearized equations of motion are invariant under the rigid supersymmetry transformation

$$\delta\lambda_\alpha = \gamma^\mu \partial_\mu A\, \varepsilon^* + \dots$$

$$\delta A = \bar{\varepsilon}^*\lambda \quad \text{etc.}$$

Let $\varepsilon \to \varepsilon(x)$ then

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\,\lambda_\alpha = \bar{\varepsilon}_1^*\gamma^\mu\varepsilon_2\,\partial_\mu\lambda_\alpha$$

$$+\ \gamma^\mu(\partial_\mu\bar{\varepsilon}_1^*\lambda)\,\varepsilon_2^* - (1\leftrightarrow 2)$$

provided $\not{\partial}\lambda = 0$

Closure to order κ^0 is given by

$$\delta\lambda_\alpha = \gamma^\mu(\partial_\mu A - \tfrac{\kappa}{2}\bar{\psi}_\mu\lambda)\varepsilon^*$$

provided $\delta\psi_{\mu\alpha} = \frac{2}{\kappa}\partial_\mu\varepsilon + \dots$

Proceeding order by order in κ one finds the field transformations and equations of motion of IIB supergravity Schwarz West Schwarz

On-Shell Superspace.

The IIB supergravity content is

$$e_m{}^a \,,\quad A \,,\quad A_{mn} \,,\quad A_{m_1\cdots m_4} \,,\quad \psi_{m\alpha} \,,\quad \lambda_\alpha$$

acts as	$\partial\partial e$	∂A	$\partial_{[m}A_{n_1n_2n_3]}$	$\partial_{[m}A_{n_1\cdots n_5]}$	$\partial_{[m}\psi_{n]\alpha}$	λ_α
dimension	2	1	1	1	3/2	1/2
U(1) charge	0	4	2	0	1	3

Superspace x^μ, θ^α, $\bar\theta^{\dot\alpha}$ contains the supervielbein $E_m{}^A$ and super connection $\Omega_m{}^{ab}$. The invariant objects are

$$T_{AB}{}^C = E_A{}^M \partial_M E_B{}^N E_N{}^C - \Omega_{[AB]}{}^C$$

and $R_{AB}{}^{cd}$, ...

On grounds of U(1) charge and dimensio

$$T_{\alpha\beta}{}^c = 0 \,,\quad T_{\alpha\beta}{}^\gamma = 0 = T_{\alpha\bar\beta}{}^{\bar\gamma} \,,\ \ldots$$

The Bianchi identities then solve for the torsions and curvatures in terms of a few superfields and give the equations of motion.

i.e $$T_{\alpha\beta}{}^{\gamma} \sim \delta_{(\alpha}{}^{\gamma} \Lambda_{\beta)}$$

The Bianchi

$$D_a T_{\beta\gamma}{}^{\delta} + \ldots = 0$$

$$\sim \quad \sigma^a D_a \Lambda + \ldots = 0$$

We identify

$$\Lambda_\alpha|_{\theta=0} = \lambda_\alpha(x^\mu), \ldots$$

$$\sim \quad \sigma^a D_a \bar{\Lambda} = -\frac{i}{16.2} \sigma^{abc} \Lambda (\bar{\Lambda} \sigma_{abc} \Lambda) + \frac{i}{5.192} \sigma^{abcde} \bar{\Lambda} G_{abcde}$$

$$G_{abcde} = {}^{*}G_{abcde} - \bar{\Lambda} \sigma_{abcde} \Lambda \qquad \text{etc}$$

where

$$G_{\mu_1 \ldots \mu_5} = 5 \partial_{[\mu_1} A_{\mu_2 \ldots \mu_5]} + 60i \left(A_{[\mu_1 \mu_2} \partial_{\mu_3} A^{*}_{\mu_4 \mu_5]} - c.c \right) - 20 \bar{\psi}_{[a} \sigma_{bcd} \psi_{e]}$$

Howe Wait.

Scalars in Supergravity Theories

The two scalars of $N = 4$, $D = 4$ supergravity belong to the coset $\frac{SU(1,1)}{U(1)}$ Cremmer, Scherk Ferrara.

The non-linear realization of G with subgroup H; given $g(x^\mu) \in G$ the theory is invariant under (Callan Coleman Wess Zumino)

$$g(x^\mu) \longrightarrow \underset{\in G \text{ rigid}}{g_0}\, g(x^\mu) \underset{\in H \text{ local}}{h(x^\mu)}$$

The Cartan forms transform as

$$v \equiv g^{-1} dg \longrightarrow h^{-1} v h$$

Let the generators of G be T^a and $H_I \in H$ then we can use the local H invariance to choose

$$g(x^\mu) = \exp \phi^a(x^\mu) T_a$$

If G involves P^μ then

$$g = e^{x^\mu P_\mu} e^{\phi^a(x^\mu) T_a}.$$

Previous Results on Supergravity Symmetries

— The scalars in supergravity theories belong to cosets

— IIB has 2 scalars in $\frac{SU(1,1)}{U(1)}$ (Schwarz West)

— D = 11 supergravity reduced on a torus

	G	H	
D = 10, IIA	$SO(1,1)/\mathbb{Z}_2$	I	Campbell West, Giani Panici, Huq Naimazie
D = 9	$GL(2)$	$SO(2)$	
⋮			
D = 6	$E_5 \cong SO(5,5)$	$USp(4) \times USp(4)$	
D = 5	E_6	$USp(8)$	
D = 4	E_7	$SU(8)$	
D = 3	E_8	$SO(16)$	

(Cremmer, Julia, Sch de Wit Nicolai)

— The gauge fields can also be included in a non-linear realization

Cremmer, Julia, Lu

Kac-Moody Algebras

Given a generalized Cartan matrix A_{ab} such that

(i) $A_{aa} = 2$

(ii) A_{ab} for $a \neq b$ are negative integers or zero

(iii) $A_{ab} = 0 \iff A_{ba} = 0$

and a set of Chevalley generators E_a, F_a, H_a which obey the Serre relations

$$[H_a, H_b] = 0 \; ; \; [H_a, E_b] = A_{ab} E_b$$

$$[H_a, F_b] = - A_{ab} F_b, \qquad [E_a, F_b] = \delta_{ab} H_a$$

and

$$\underbrace{[E_a, [E_a, \ldots [E_a}_{1 - A_{ba} \text{ factors}}, E_b] \ldots] = 0$$

plus relations for F_a's.

Then the Kac-Moody algebra is given by the multiple commutators

$$[E_{a_1}, [E_{a_2} \ldots, [E_{a_{n-1}}, E_{a_n}] \ldots]$$

plus multiple commutators of F_a's.

It is uniquely determined by A_{ab}

If A_{ab} is a positive definite matrix we get a finite dimensional semi-simple Lie algebra

Example $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

The Chevalley generators are

$$E_1, E_2, H_1, H_2, F_1, F_2$$

The Serre relations are

$$[H_1, E_2] = -E_2 \quad \text{etc}$$

and

$$[E_1, [E_1, E_2]] = 0 \quad \text{etc.}$$

The resulting algebra is

E_1, E_2, $[E_1, E_2]$ } positive roots
(α_1) (α_2) $(\alpha_1 + \alpha_2)$

H_1, H_2 Cartan subalgebra

F_1, F_2, $[F_1, F_2]$ } negative roots
$(-\alpha_1)$ $(-\alpha_2)$ $(-\alpha_1 - \alpha_2)$

ie $SU(3)$.

D = 11 Supergravity as a Non-linear Realization

hep-th/2000-147

field e_μ^a, $A_{a_1 a_2 a_3}$, $A_{a_1 \cdots a_6}$

generator $K^a{}_b$, $R^{a_1 a_2 a_3}$, $R^{a_1 \cdots a_6}$.

These obey an algebra G_{11} which contains $GL(11)$ $(K^a{}_b)$, P^a and

$$[K^a{}_b, R^{c_1 \cdots c_3}] = \delta^{c_1}_b R^{a c_2 c_3}{}_{+ \cdots}$$

$$[R^{c_1 \cdots c_3}, R^{c_4 \cdots c_6}] = 2 R^{c_1 \cdots c_6}$$

We consider the non-linear realization of G_{11} with subgroup $SO(1,10)$ i.e

$$g = e^{x^m P_m} e^{h_a{}^b(x) K^a{}_b} e^{\{\frac{1}{3!} A_{c_1 \cdots c_3} R^{c_1 \cdots c_3} + \frac{1}{6!} A_{c_1 \cdots c_6} R^{c_1 \cdots c_6}\}}$$

- carry out simultaneous non-linear realization with conformal group $SO(2,11)$.

$\rightsquigarrow$ bosonic equations of motion of $D = 11$ supergravity i.e

$$F_{a_1 \cdots a_3} = \frac{1}{7!} \epsilon_{a_1 \cdots a_{11}} F^{a_4 \cdots a_{11}}$$

The Borel Subalgebra of E_7.

hep-th 105270

Restricting the indices to $i = 5, \ldots, 11$ in G_{11} gives the subalgebra

$$D\,,\quad K^i{}_j\,,\quad R^{i_1 i_2 i_3}\,,\quad S_j = \frac{1}{6!}\epsilon_{j i_1 \cdots i_6} R^{i_1 \cdots i_6}$$

This is the Borel subalgebra of E_7

The adjoint of E_7 breaks to $se(7)$ as

$$\begin{aligned} 133 = {} & 48\,(K^i{}_j) + 1\,(D) && GL(7) \\ & + 35\,(R^{i_1 i_2 i_3}) + \overline{7}\,(S_j) && \text{+ve roots} \\ & + \overline{35}\,(R^{i_1 \cdots i_3}) + 7\,(S^i) && \text{-ve roots} \end{aligned}$$

$D = 11$ supergravity is invariant under the Borel subgroup of E_7. If we increased the local subgroup so as to include $SU(8)$ in the restriction we would have a full E_7 symmetry.

Is D=11 Supergravity invariant under a Kac-Moody Symmetry?

G_{11} is not a Kac-Moody algebra, but we can

(i) use an alternative description of D=11 supergravity

(ii) enlarge the local symmetry.
($g \to g_0 g^h$ so does not affect field content).

such that the resulting closure with the conformal algebra is a Kac-Moody algebra

The local subalgebra should be the one invariant under the Cartan involution.

$$g = \prod_a e^{H_a \phi_a} \prod_{\substack{\alpha \text{ +ve} \\ \text{root}}} e^{\phi_\alpha E_\alpha}$$

Identification of Kac-Moody Algebra. $\mathcal{G}$.

We split the generators in G_{11} into

$$G_{11}^{+} = \{ K^{a}{}_{b} \quad a<b, \quad \ldots R^{c_1 \cdots c_3}, R^{c_1 \cdots c_6} \}$$

$$G_{11}^{0} = \{ H_a \equiv K^{a}{}_{a} - K^{a+1}{}_{a+1}, a = 1, \ldots 10, \quad D = \sum_a K^{a}{}_{a} \}$$

The remaining generators are $K^{a}{}_{b}$, $a > b$ and are in the local subgroup

We expect that $G_{11}^{+} \subset \mathcal{G}_{+}$

Now G_{11}^{+} is generated by

$$E_a = K^{a}{}_{a+1}, \; a = 1, \ldots, 10 \text{ and } E_{11} = R^{9101 1}$$

and identify these as the simple positive roots of $\mathcal{G}$.

We expect that $G_{11}^{0} \subset H$. and so a rank 11 algebra.

After choice of an appropriate basis for H we can determine $\mathcal{G}$ from

$$[H_a, E_b] = A_{ab} E_b .$$

Now E_a for $a = 1, \ldots, 10$ are the simple roots of A_{10} and E_a for $a = 5, \ldots, 11$ are the simple roots of E_7.

Given these embeddings the unique Kac-Moody algebra & has a Cartan subalgebra

$$H_a, a = 1 \ldots, 10 \quad , H_{11} = K^9_9 + K^{10}_{10} + K^{11}_{11} - \frac{1}{3} D$$

and is E_{11}

G_{11} does not contain E_8. Under $Sl(8)$ the adjoint of E_8 decomposes as

$$248 \longrightarrow \underbrace{63 + 1}_{K^i{}_j,\ D} + \underbrace{56 + \overline{28}}_{GL(8)} + \underbrace{8}_{R^{k_1 \cdots k_3}\ \epsilon_{k_1 k_2 \cdots} R^{\cdots}\ ?} + \underbrace{\overline{56} + 28 + \overline{8}}_{\text{negative roots}}$$

positive roots

$i, j = 4, \cdots 11$

where is the 8?

We can modify the G_{11} algebra by adding a new generator $R^{a_1 \cdots a_8, b}$

- gives Borel subalgebra of E_8

$$R^{c_1 \cdots c_3} \quad R^{c_1 \cdots c_6} \quad ; \quad K^a{}_b \ , \ R^{a_1 \cdots a_8, b}$$

$$A_{c_1 \cdots c_3} \quad A_{c_1 \cdots c_6} \qquad\qquad h_a{}^b \qquad h_{a_1 \cdots a_8, b}$$

dual formulation of 3-form

dual formulation of gravity.

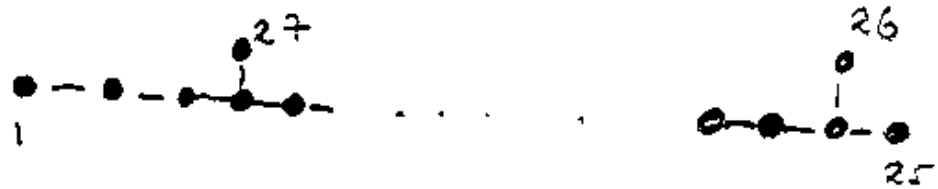
Conclusions

- We have argued that $D = 11$ supergravity is invariant under E_{11}

- A similar calculation implies that IIA and IIB are also E_{11} invariant.

($D = 11$ to IIA requires enlarged G_{11} algebra)

- The low energy effective action of the $D = 26$ closed bosonic string $(h_a{}^b, \phi, B_{a_1 a_2})$ has a corresponding Kac-Moody algebra of rank 27.



- requires a new formulation of gravity which should explain Geroch symmetries.

- Conjecture that M-theory is invariant under E_{11}