Encounters with Supergravity

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Plan : Something new & Something Old

OLD
1) Intro : Why I failed to invent SUGRA
2) Killing Vectors & Killing Spinors
3) de Sitter, Anti de Sitter &

whatever happened to Twistor theory

NEW
4) Special Holonomy, Branes, Calibrations

& Cohomogeneity one metrics

INFINITESIMAL SUPERSYMMETRY

(Killing Spinor)

v

INFINITESIMAL ISOMETRY

(Killing Vector)

$$\delta_K g_{\mu\nu} = \underset{K}{\pounds} g_{\mu\nu} = 0 \iff \boxed{K_{\mu;\nu} + K_{\nu;\mu} = 0}$$

Killing's Eqns.

$$\delta_\epsilon \psi_\mu = 0 \iff \boxed{\begin{aligned} &\hat{\nabla}_\mu \varepsilon = 0 \\ &\hat{\nabla}_\mu \varepsilon = \nabla_\mu \varepsilon + E_\mu \varepsilon \end{aligned}}$$

SUSY ALG $\Rightarrow$

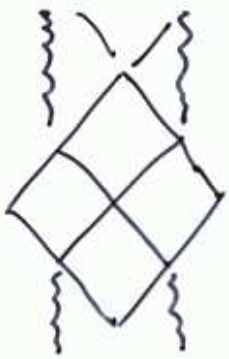
$\boxed{K^\mu = \bar{\varepsilon}\,\gamma^\mu \varepsilon}$ is Killing vector

Dirac's Lemma : K^μ is everywhere future directed timelike or null.

$K^0 = \psi^\dagger \psi > 0$ in all frames

$\Rightarrow$ Any BPS solution must be locally stationary & may admit at most a degenerate horizon

✗ $Q = 0$



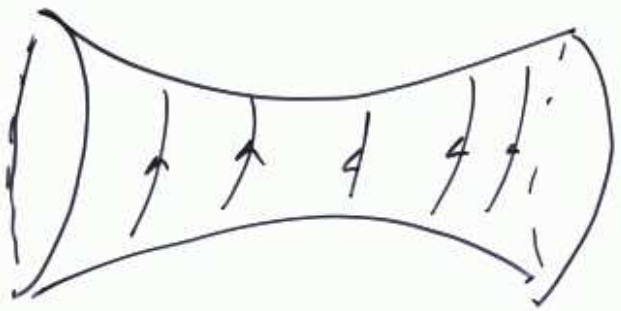
✗ $M > |Q|$

✓

De Sitter v. Anti-de Sitter

$E^{4,1}$

COSMOLOGICAL EVENT HORIZON



$E^{3,2}$

GLOBALLY STATIC

$$-(X^0)^2+(X^1)^2+(X^2)^2+(X^3)^2+(X^4)^2 = 1$$

✗

$$-(X^0)^2+(X^1)^2+(X^2)^2+(X^3)^2-(X^4)^2 = -1$$

✓

Wigner's Problem

$\forall\ H \in \mathfrak{L}(SO(4,1)) \quad \exists\ g \in SO_0(4,1)$ s.t.

$$-H = g H g^{-1}$$

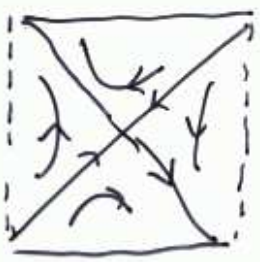
$\Rightarrow$ no positive energy $\Rightarrow$ no susy

$$X^0 = \sqrt{1-r^2}\sinh t \qquad X^i = r n^i,\ n^i \in S^2$$
$$X^4 = \sqrt{1-r^2}\cosh t$$

$$ds^2 = -(1-r^2)\,dt^2 + \frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$L_{04} = \frac{\partial}{\partial t}$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \in SO_0(4) \subset SO(4,1)$$



g interchanges inside & outside of cosmological horizon.

Actually SUGRA $\overset{?}{\Rightarrow}$ STRONG ENERGY CONDITION

WHICH EXCLUDES ACCELERATION

Twistor Eqn

$$\left(\nabla_\mu \gamma_\nu + \nabla_\nu \gamma_\mu\right)\varepsilon = \frac{1}{n} g_{\mu\nu}\, \gamma^\lambda \nabla_\lambda \varepsilon$$

$$\nabla\!\!\!/_\mu \varepsilon = \frac{1}{2a}\gamma_\mu \varepsilon$$

$$\frac{1}{a^2} + \frac{\Lambda}{3} = 0$$

$\Lambda < 0$; a real : $SO(3,2) \subset SO(4,2)$

$\Lambda > 0$; a pure imag : $SO(4,1) \subset SO(4,2)$

$\Lambda < 0 \longrightarrow \varepsilon \gamma^\mu \varepsilon$ Killing Vector

$\Lambda > 0 \qquad\quad \bar{\varepsilon} \gamma^\mu \varepsilon$ Conformal Killing Vector

Twistor Theory is contained within Conformal Supergravity.

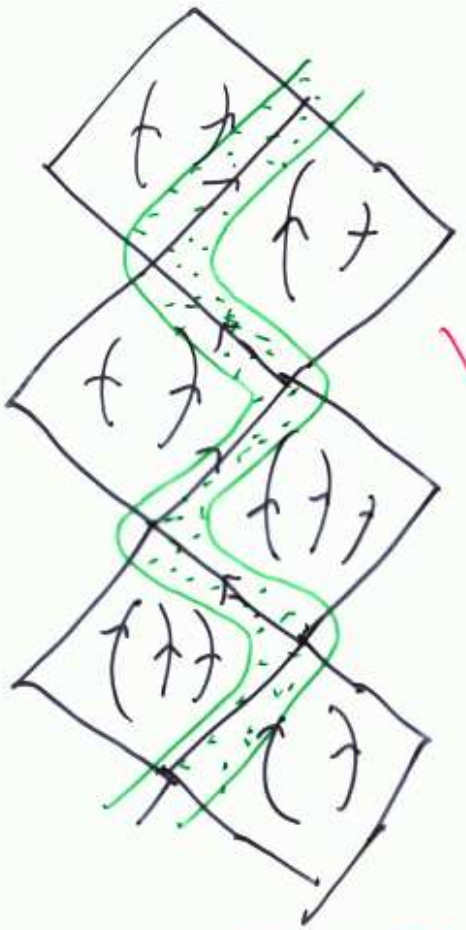
D3 - Branes or (M2 or M5)

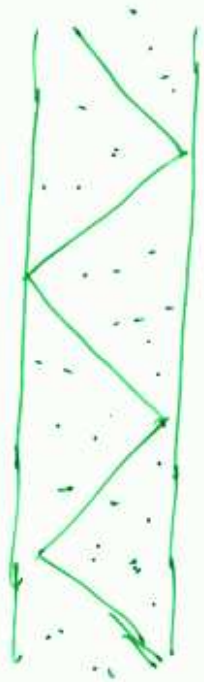
$$ds^2 = -H^{-1/2}(-dt^2 + d\underline{x}_3^2) + H\, d\underline{y}_6^2$$

$$\nabla_y^2 H = 0$$

$$F_5 = d\left(\frac{dt \wedge dx_1 \wedge dx_2 \wedge dx_3}{H}\right) + *\left(d(\dots)\right)$$

$$H = 1 + \frac{1}{|\underline{y}|^4}$$





No horizon geometry in $AdS_5 \times S^5$

Spatial interpolation between 2 maximal susy vacua

$\frac{1}{2}$ BPS.

M5 is like D3 non singular

M2 is like Reissner-Nordström : singular horizon

Special Holonomy

Killing spinors of D3 are constructed from constant or parallel spinors of $\mathbb{E}^6$

(M2 $\to \mathbb{E}^8$, M5 $\to \mathbb{E}^5$ etc..)

Idea i): replace $\mathbb{E}^n$ by curved mfd M^n admitting constant spinors

$$\nabla \epsilon = 0.$$

(ii) add additional NS & RR fluxes

$$d * F = i F_3 \wedge \bar{F}_3$$

$$\nabla^2 H = |F_3|^2$$

(Fractional branes)

need to find (M, g) & harmonic forms

Programme of wk. with Gelici, Liu & Pope)

Special & Exceptional Holonomy

Berger 1955 M irreducible & not symmetric

$Hol(M, g)$

$U(\frac{n}{2}) \subset SO(n)$: Kähler.

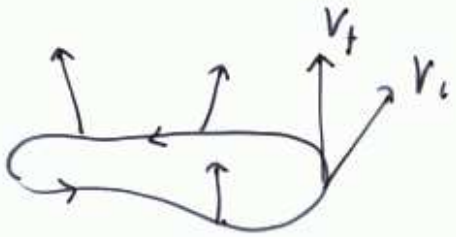
$SU(\frac{n}{2}) \subset SO(n)$: Calabi-Yau

$Sp(\frac{n}{4}) \subset SO(n)$: HyperKähler.

$G_2 \subset SO(7)$: Exceptional

$Spin(7) \subset SO(8)$

$Sp(\frac{n}{4}) \cdot Sp(1) \subset SO(n)$: Quaternionic



$V_f = R V_i$

$R \subset Hol \subseteq SO(n)$

Group generated by parallel transport.

$d\theta + \theta \wedge \theta = 0$

$R = d\theta + \theta \wedge \theta$: Lie algebra valued 2-form

values in $Hol(M, g)$

Holonomy grps which preserve one or more spinors

			susy's
$SU(\frac{n}{2})$	:	2 covariant spinors	$N=2$
$Sp(\frac{n}{4})$	:	$n+1$ " "	$N=n+1$
G_2	:	1 " "	$N=1$
$Spin(7)$	:	1 " "	$N=1$

i) $\nabla_\mu \varepsilon = 0 \Rightarrow Ric = 0$.

[not true for Lorentz signature!]

ii) Hol is subgroup of $SO(n)$ leaving invariant a spinor

Eg. spin rep of $SO(7)$: $8 = 1 + 7$

↑ preserved by $G_2 \subset SO(7)$

where in fact G_2 preserves the structure constants of the octonions.

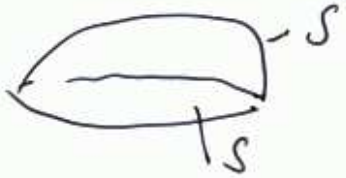
Calibrating Forms

$$\bar{\epsilon}\,\gamma_{\mu_1\dots\mu_p}\epsilon = \Phi_{\mu_1\dots\mu_p} = \frac{1}{p!}\Phi_{\mu_1\dots\mu_p}\,dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p}$$

$$\nabla\epsilon = 0 \Rightarrow \nabla\Phi = 0 \Rightarrow d\Phi = 0.$$

in fact $\Phi[\text{tangent plane}] \le |\text{tangent plane}|$

now



$$\int_S \Phi = \int_{S'} \Phi$$

S is calibrated if $\Phi|_S$ = volume form

$$\Rightarrow \int_S \Phi = \mathrm{vol}(S) \le \mathrm{vol}(S')$$

S minimizes volume in its homology class.

SUSY cycles & Calibrating forms

BI actions have K-symmetry

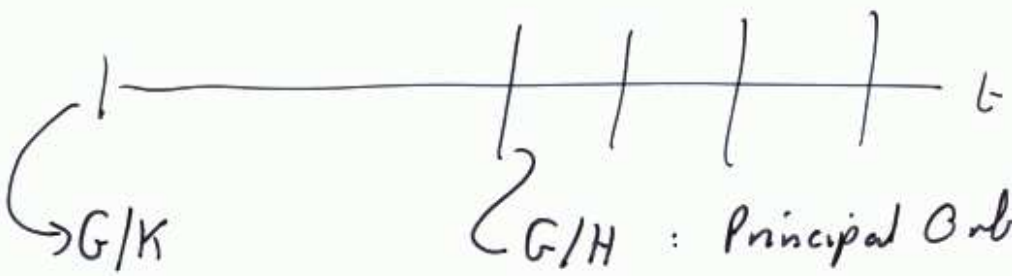
K symmetry ⇒ SUSY ⇒ Calibrated

$SU(\frac{n}{2})$	:	ω ← Kähler, Ω ← holomorphic $\frac{n}{2}$-form ⇒ holomorphic cycles or SLAGS ; $\omega\|_S = 0$; $\mathrm{Re}\Omega\|_S = \mathrm{vol}$. SLAG's have dimension $\frac{n}{2}$
G_2	:	Φ ← associative 3-form — associative 3-fold $*\Phi$ — co-associative 4-fold
$Spin(7)$	:	$\Xi = *\Xi$ self-dual 4-form Cayley 4-fold

Cohomogeneity one examples

G acts by isometries with co-dimension one principal orbits.

$$M = G/K \sqcup (\mathbb{R}_+ \times G/H)$$

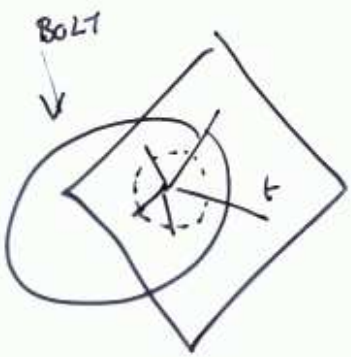


degenerate orbit
(BOLT)

$K \supset H$

$K/H = S^d$

Globally M^n is an $\mathbb{R}^{d+1}$ bundle over G/K
↑ zero section



Example $M = T^* S^p$: $G = SO(p+2)$

$K = SO(p+1)$

$H = SO(p)$

$\dim M = 2p$

$p = 3$ is "deformed conifold"

Stenzel metric

Einstein Eqns

$$ds^2 = dt^2 + g_{ab}(t)\, e^a(x) e^b(x)$$

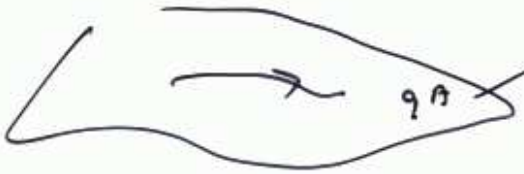
$a = 1 \ldots \dim G/H$

e^a G covariant vielbein

Cartan-Maurer Forms $\quad g = h \oplus p$

$\uparrow e^a$ are a basis of p

$g_{ab}(t)$ is a curve of G invariant metric on G/H



"Superspace of Wheeler"

$$L = T - V \qquad V$$

$\uparrow$ Ricci scalar of G/H

$$G^{AB}_{DeWitt}\, \frac{dq^A}{dt}\, \frac{dq^B}{dt} - V(q^A)$$

$\uparrow$ DeWitt metric (Lorentzian signature!)

$$T = T_0\, g^{-1}\dot{s}\, s^{-1}\dot{s} - \frac{1}{2}\left(T_0\, s^{-1}\dot{s}\right)^2$$

2nd order ODES plus on constraint

$\Rightarrow$

$$\boxed{T + V = E}$$

DIFFICULT TO SOLVE

Special Holonomy & Superpotentials

We find that imposing Special holonomy

$\Rightarrow$ 1st order eqns which themselves imply the 2nd order eqns.

There are many ways of obtaining 1st order ode's but in all cases we have found a superpotential

$$V = -G^{AB}\frac{\partial W}{\partial q^A}\frac{\partial W}{\partial q^B}$$

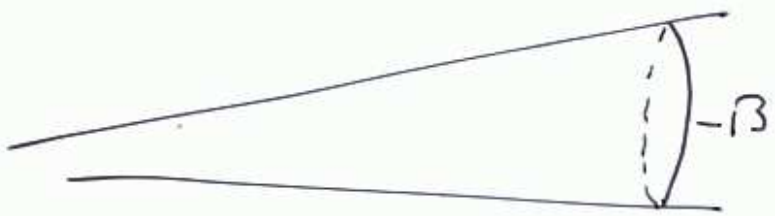
$$\Rightarrow \quad \frac{dq^A}{dt} = G^{AB}\frac{\partial W}{\partial q^A}$$

In a number of cases these may be solved explicitly & completely (e.g. Spin(7) 3 funct cases) In others they must be integrated numerically.

Behaviour nr. infinity

We typically encounter 2 types of asymptotics although other possibilities can certainly arise if we drop the cohomogeneity one assumption

(i) AC



Asymptotically Conical

Ricci flat cone

$$ds^2 = dt^2 + t^2 g^\infty(G/H)$$

where $g^\infty(G/H)$ is Einstein metric

with Killing spinors

Einstein Sasaki	5-dim	Calabi-Yau	6
e.g. Nearly Kähler	6-dim	G_2	7
Weak G_2	7-dim	Spin 7	8
Tri-Sasaki	8-dim	HK	8
Einstein Sasaki	7-dim	Calabi-Yau	8

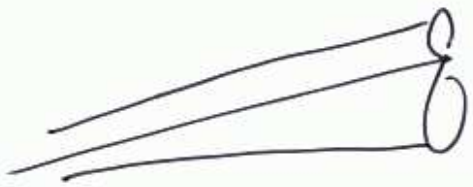
In 8-dimensions we can have

Spin(7)	$\cong$ Spin(7)	1
SU(4)	$\subset$ Spin(6)	2
Sp(2)	$\cong$ Spin(5)	3

susy's

(11) Asymptotically Locally Conical ALC

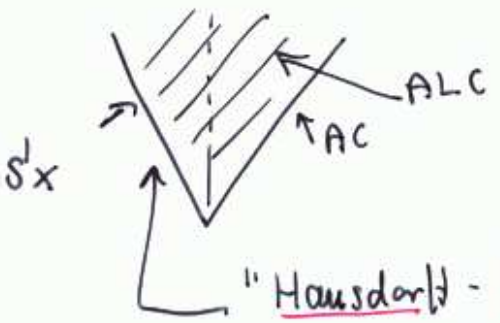
This is more generic & favoured by the eqns.



$$S^1 \times \frac{R \times (G/H \times SO(?))}{AC}$$

a circle of constant length nr. ∞.

There allows an M-Theory interpretation

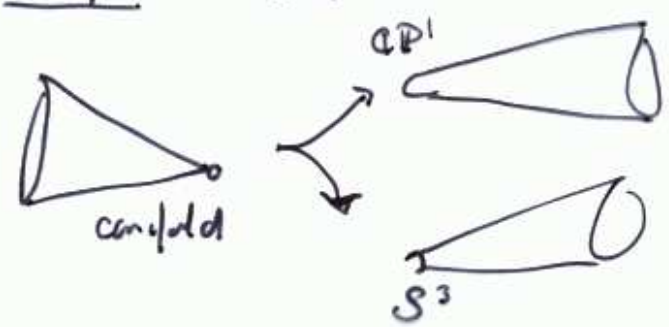


Typical moduli space

"Hausdorff-Gromov limit in which a circle splits off as a product.

Calibrated BOLTS We typically find that the BOLT or degenerate orbit is a calibrated submfd.

Example 6-dim C.Y. cone over $T^{(1,1)} \cong S^1$ bdle over $S^2 \times S^2$



$O(k_1) \oplus O(k_2)$ on $\mathbb{CP}^1$ cplx line bundle.

stenzel: BOLT is SLAG.

Orientifold Planes & D6-Planes

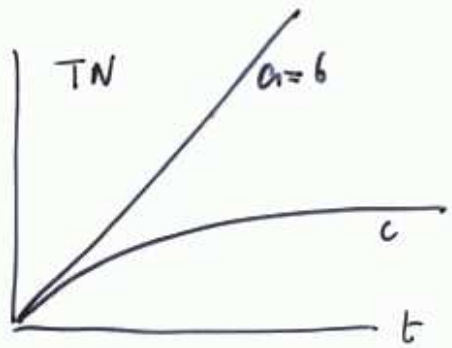
Taub-NUT v Atiyah-Hitchin

D6 O6

+ve mass −ve mass

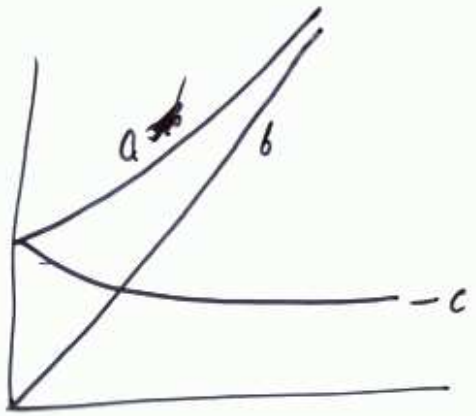
$$ds^2 = -dt^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2$$

$d\sigma_1 = \sigma_2 \wedge \sigma_3$ etc.



NUT

+ve mass $\mathbb{R}^4$



BOLT

$S(S^2)$

Asymptotically AH ~ TN(−ve mass)/CP

$P: (r, \theta, \phi) \to (r, \pi-\theta, \phi+\pi)$

$C: \phi \to -\psi$ ← M-theory circle.

New Spin 7 examples

Old cone

Bryant Salamon AC example on $S^+(S^4)$

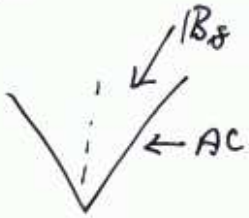
↑ chiral spinors

S^4 : Cayley 4-fold

$G = SO(5)$

New examples : i) ALC on $S^+(S^4)$

(explicit) (with parameters)



(ii) isolated ALC cone on

$$\mathbb{R}^8 \to \mathbb{A}_8$$

(iii) ALC example on ; $\mathbb{C}_8$

bundle over $\mathbb{CP}^2 \leftarrow S^2$

$\downarrow$

S^4

$$G/H \simeq SO(5)/SO(3) \cong S^7$$

wrapped orientifolds