PHY 303
Solutions to Assignment 2

1.26 a) Let \( v_0 \) be the velocity transferred to the puck by the kick.

Then in \( S \), we have:
\[
\begin{align*}
x(t) &= 0 \\
y(t) &= v_0 t
\end{align*}
\]

b) Let \( x', y' \) be the coordinates of the puck in the frame \( S' \) and consider the diagram:

\[
\begin{array}{c}
y' \\
\downarrow \\
vt \\
\downarrow \\
x'
\end{array}
\]

\( x' < 0 \) in this picture.

Since the origins of \( S \) and \( S' \) coincide at \( t=0 \), \( S' \) is displaced by \( vt \) relative to \( S \) at time \( t \). Thus, we have:
\[
\begin{align*}
x'(t) &= x(t) - vt = -vt \\
y'(t) &= y(t) = v_0 t
\end{align*}
\]
c) Similarly, since $S$ and $S''$ coincide at $t=0$, we have that $S''$ is displaced by $\frac{1}{2}at^2$ relative to $S$ (the initial velocity of $S''$ is 0).

\[ x''(t) = x(t) - \frac{1}{2}at^2 = -\frac{1}{2}at^2 \]

\[ y''(t) = y(t) = v_0t \]

Note that the trajectory in the $S''$ frame satisfies

\[ \left(\frac{y''}{y_0}\right)^2 = -\frac{2}{a}x'' \]

i.e. it is a parabolic motion.

The motion of the puck in the frames $S$ and $S''$ is linear:

\[ +x' = +v' + \dot{\gamma}x' = \infty \]

\[ \gamma' = \infty \]

\[ R = \infty \]
Since the puck has no force acting on it, we conclude that $S$, $S'$ are inertial frames but $S''$ is not.

1.40 a) $v_0 =$ initial speed of cannon ball

The motion along $x$ is not accelerated, hence:

$$x(t) = v_0 \cos \theta t$$

(assume $x(0) = y(0) = 0$)

The motion along $y$ undergoes gravitational acceleration:

$$y(t) = v_0 \sin \theta t + \frac{1}{2} at^2$$

Since gravity always acts downward,

$$a = -g$$
\[ y(t) = v_0 \sin \theta - \frac{1}{2} gt^2 \]

2) Let \( r(t) \) = distance from cannon ball to the cannon.

Then \( r^2(t) = x^2(t) + y^2(t) \). We want \( r(t) \) to be increasing during the whole motion.

The cannon ball hits the ground at time \( t_f = \frac{2v_0 \sin \theta}{g} \).

Hence, we want to impose:

\[ \dot{r}(t) = \frac{dr}{dt} > 0 \quad \text{for all} \quad t \in [0, \frac{2v_0 \sin \theta}{g}] \]

So we compute \( \dot{r}(t) \):

\[ r^2 = x^2 + y^2 \Rightarrow 2 \dot{r} = 2\dot{x} + 2\dot{y} \]

\[ \Rightarrow \dot{r} = \frac{\dot{x} \dot{y}}{\dot{y}} > 0 \quad \forall t. \]

Since \( \dot{r}(t) \) is always positive, it is enough to ensure that \( x\dot{x} + y\dot{y} \) is always positive, now:

\[ I(t) \equiv x\dot{x} + y\dot{y} = (v_0 \cos \theta)(v_0 \cos \theta) \]

\[ + (v_0 \sin \theta - \frac{1}{2} gt^2)(v_0 \sin \theta - gt) \]
\[ I(t) = v_0^2t - \frac{1}{2}gt^2\cos\theta - gt^2v_0\sin\theta + \frac{1}{2}gt^2 \]

\[ = v_0^2t - \frac{3}{2}gt^2\cos\theta + \frac{1}{2}gt^2 \]

\[ = t\left(v_0^2 - \frac{3}{2}gtv_0\sin\theta + \frac{1}{2}gt^2\right) \]

Since \( t > 0 \), we only need to worry about making \( I(t) = v_0^2 - \frac{3}{2}gtv_0\sin\theta + \frac{1}{2}gt^2 \) positive over \( (0, \frac{2v_0\sin\theta}{g}) \).

To do that, we find the minimum of \( I(t) \) over that interval and make sure it is positive:

\[ \frac{dI}{dt} = g^2t - \frac{3}{2}gv\sin\theta = 0 \text{ at } t = t_{\min} \]

\[ \Rightarrow t_{\min} = \frac{3}{2} \frac{v_0\sin\theta}{g} \in (0, \frac{2v_0\sin\theta}{g}) \]

So we know the minimum occurs during the motion!

\[ \Rightarrow I_{\min} = v_0^2 - \frac{3}{2}g\left(\frac{3}{2} \frac{v_0\sin\theta}{g}\right)v_0\sin\theta + \frac{1}{2}g^2\left(\frac{3}{2} \frac{v_0\sin\theta}{g}\right)^2 \]

\[ = v_0^2 - \frac{9}{4}v_0^2\sin^2\theta + \frac{1}{2} \frac{g}{4}v_0^2\sin^2\theta \]

\[ = v_0^2\left(1 - \frac{9}{8}\sin^2\theta\right) > 0 \]

\[ \Rightarrow \sin^2\theta < \frac{8}{9} \Rightarrow \theta < \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \approx 70.5^\circ \]

\[ \Rightarrow \theta_{\text{max}} = 70.5^\circ \]
3.4 a)

When the two liobos have run at speed \( v \) (relative to the car) to the opposite end, they have already transferred momentum \( m_b v \) to the car. Therefore, their speed relative to the ground when they jump off is \( v = u \) and not just \( -u \):

So, the momentum conservation equation reads:

\[
p_i = 0 = p_f = 2m_b (v - u) + m_F c v
\]

momentum of two liobos    momentum of flatcar.

\[
\Rightarrow v \left(2m_b + m_F c\right) = 2m_b u
\]

\[
\Rightarrow v = \frac{2m_b u}{2m_b + m_F c}
\]
b) In this case, we need to apply momentum conservation to each stage separately:

\[ p_i = 0 = p_f = m_f (v_1 - u) + (m_b + m_h) v_1 \]

\[ \Rightarrow v_1 = \frac{m_f u}{2m_b + m_f} \]

In the second stage, we restrict our system to include only the flatcar and the hobo who has not jumped yet. This means that we now have non-vanishing initial momentum.

\[ p_i = (m_b + m_h) v_1 = p_f = m_f v_2 + m_b (v_2 - u) \]

\[ \Rightarrow v_2 (m_b + m_h) = m_f u + (m_b + m_h) v_1 \]

\[ \Rightarrow v_2 = \frac{m_f u}{m_b + m_h} + v_1 \]
\[ v = \frac{2mv_0}{2mv_0 + m_f} = mv_0 \left( \frac{1}{2m_f + mv_0} + \frac{1}{2m_f - mv_0} \right) \]

Note that in part a), we got:

\[ v < v_z \]

Therefore, \( v < v_z \) and jumping one after the other gives a greater final speed to the flat cars.

3.12 a) This is a simple application of the rocket equation:

\[ v = v_0 = v_{ex} \ln \left( \frac{m_0}{m} \right) \]

\( v_0 = 0 \) since the rocket accelerates from rest.

When the rocket has burned all of its fuel, its remaining mass is:

\[ m = m_0 - 0.6m_0 = 0.4m_0 \]

\[ \Rightarrow v = v_{ex} \ln \left( \frac{m_0}{0.4m_0} \right) = v_{ex} \ln \left( \frac{10}{4} \right) \approx 0.91v_{ex} \]
b) Let $v_1$ be the final speed of the rocket after the first burning stage, then:

$$v_1 = v_{ex} \ln \left( \frac{m_0}{m_o - 0.3m_o} \right) = v_{ex} \ln (\frac{10}{7})$$

As we jettison the tank, the mass of the rocket goes from $0.7m_o$ to $0.6m_o$. This does not change the momentum of the rocket because the tank is simply released, not expelled out.

Let $v_2$ be the final speed of the rocket after the last burning stage:

$$v_2 = v_1 + v_{ex} \ln \left( \frac{0.6m_o}{0.6m_o - 0.3m_o} \right)$$

$$= v_{ex} \left( \ln \left( \frac{10}{7} \right) + \ln (2) \right) \approx 1.05 \, v_{ex}$$

$\Rightarrow$ Burning the fuel in two separate stages gives a faster final speed.

3.36

The force acts on the dumbbell for a short time at.
The impulse given to the system is:
\[ \Delta \vec{p} = \vec{p}_f - \vec{p}_i = \vec{F} \Delta t = \vec{F} \cos \alpha \Delta t + \vec{g} \sin \alpha \Delta t \]

The final linear momentum of the system is the same as if it consisted of a mass \(2m\) located at the center of mass:

\[ \vec{p}_F = 2m \vec{v}_{CM} = \vec{F} \cos \alpha \Delta t + \vec{g} \frac{\sin \alpha \Delta t}{2m} \]

velocity of the center of mass.

\[ \Rightarrow \vec{v}_{CM} = \vec{F} \cos \alpha \Delta t + \vec{g} \frac{\sin \alpha \Delta t}{2m} \]

The torque acting on the dumbbell is:

\[ \vec{\tau} = \vec{r} \times \vec{F} \]

\[ \Rightarrow \vec{\tau} = -\vec{g} b \sin \alpha \]

Note that \(\vec{\tau}\) points in the \(-\vec{g}\) (into the paper) direction, which means the torque tends to spin the dumbbell clockwise.

We are computing the torque about the center of mass.
The amount of angular momentum transferred during time $\Delta t$ is:

$$\Delta \mathbf{L} = \mathbf{L}_f - \mathbf{L}_i = Z \Delta t$$

But by definition of the moment of inertia, we also have:

$$\mathbf{L}_f = I \mathbf{\omega}$$

$$\Rightarrow \mathbf{\omega} = \frac{Z \Delta t}{I} = -\frac{bF \sin \theta \Delta t}{2mb^2} \hat{z}$$

i.e. $\mathbf{\omega} = \frac{F \Delta t \sin \theta}{2mb}$, clockwise.

In the subsequent motion, the overall translation of the center of mass and rotation of the masses about the center of mass are decoupled:

Since there are no external forces, this motion goes on indefinitely.
Immediately after the impulse has been transferred, the left mass is given a velocity:

\[ \vec{V}_{IL} = b \omega \hat{y} + \vec{v}_{CM} \]

\[ = \frac{F_0 t \cos \alpha \hat{x}}{2m} + \frac{F_0 t \sin \alpha \hat{y}}{m} \]

and the right mass:

\[ \vec{V}_{IR} = -b \omega \hat{y} + \vec{v}_{CM} \]

\[ = \dot{x} \frac{F_0 t}{\Delta t} \cos \alpha \]

\[ \hat{y} \]

\[ b \omega \hat{y} \]

\[ \hat{x} \]

\[ -b \omega \hat{y} \]

Whole system has additional velocity \( \vec{v}_{CM} \)

Note that the above only work just after the impulse has been given.