PHY 610 QFT, Spring 2015

HW4 Solutions

1. 6.1 We are to explicitly evaluate the Feynman propagator in position space (6.34), which in our metric convention is

$$D_F(x,0) = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+m^2-i\epsilon} e^{-ikx}.$$

Let us first perform the k^0 integral. (The following discussion is essentially identical to that of Schwartz, p. 76, but going backwards.) Writing the denominator as $-k_0^2 + \vec{k}^2 + m^2 - i\epsilon = -(k_0 + \sqrt{\vec{k}^2+m^2-i\epsilon})(k_0 - \sqrt{\vec{k}^2+m^2-i\epsilon})$, we see that there are two simple poles at $\pm\sqrt{\vec{k}^2+m^2-i\epsilon}$, one with positive real part and displaced slightly below the real axis, and one with negative real part displaced slightly above the real axis (see fig 6.1, p. 76). We wish to use the residue theorem of complex analysis to perform this integral. For $x^0 > 0$, the exponential factor e^{-ikx} becomes small when k_0 has large imaginary part, so we can close the integral in the upper half plane. This means we pick up the residue of the pole at $k_0 = -\sqrt{\vec{k}^2+m^2-i\epsilon}$. Conversely, for $x^0 < 0$, we have to close the ingral in the lower half plane, picking up the residue of the other pole (as well as a minus sign for a clockwise contour). Thus

$$\begin{aligned} D_F(x,0) &= \int \frac{d^3k}{(2\pi)^3} \left(\theta(x^0) i \frac{ie^{-i\vec{k}\vec{x}-ix^0\sqrt{\vec{k}^2+m^2-i\epsilon}}}{-2\sqrt{\vec{k}^2+m^2-i\epsilon}} + \theta(-x^0)(-i)\frac{ie^{-i\vec{k}\vec{x}+ix^0\sqrt{\vec{k}^2+m^2-i\epsilon}}}{2\sqrt{\vec{k}^2+m^2-i\epsilon}} \right) \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i\vec{k}\vec{x}-i|x^0|\sqrt{\vec{k}^2+m^2-i\epsilon}}}{2\sqrt{\vec{k}^2+m^2-i\epsilon}}. \end{aligned}$$

In 3+1 dimensions, the angular integral can be performed easily, since there is conveniently a $\sin\theta$ in the measure to go with the $e^{-ikr\cos\theta}$ in the integrand:

$$\begin{aligned} D_F(x,0) &= \int_0^\infty k^2\, dk\, \frac{1}{8\pi^2} \int_{-1}^1 d\cos\theta\, \frac{e^{-ikx\cos\theta - i|x^0|\sqrt{\vec{k}^2+m^2-i\epsilon}}}{\sqrt{\vec{k}^2+m^2-i\epsilon}} \\ &= \int_0^\infty dk\, \frac{k^2 e^{-i|x^0|\sqrt{\vec{k}^2+m^2-i\epsilon}}}{4\pi^2\sqrt{\vec{k}^2+m^2-i\epsilon}} \frac{\sin(kx)}{kx}. \end{aligned}$$

This expression is actually a Bessel function. (Bessel functions are solutions to Laplace's equation.) One way to see this is to invoke Lorentz invariance to simplify the integral. For timelike separations x, rotate to a frame where $x^\mu = (x^0, 0)$, so that the $\sin(kx)/kx$ factor in the integrand becomes unity. Then,

$$\begin{aligned} D_F(x,0) &= \int_0^\infty dk\, \frac{k^2 e^{-i|x^0|\sqrt{\vec{k}^2+m^2-i\epsilon}}}{4\pi^2\sqrt{\vec{k}^2+m^2-i\epsilon}} \\ &= \frac{m^2}{4\pi^2} \int_{1-i\epsilon}^{\infty-i\epsilon} dy\, \sqrt{y^2-1+i\epsilon}\, e^{-im|x^0|y} \\ &= \frac{im}{8\pi|x^0|} H_1^{(1)}(m|x^0|), \qquad \text{(timelike)} \end{aligned}$$

where on the second line we have changed variables to $y = \sqrt{k^2+m^2-i\epsilon}/m$, and on the last line we have used the following integral representation for the Hankel function $H_\nu^{(1)}$,[1]

$$H_\nu^{(1)} = -i\frac{2(-a/2)^\nu}{\pi^{1/2}\Gamma(\nu+1/2)}\int_1^{\infty-i\epsilon} e^{-iat}(t^2-1)^{\nu-1/2}\,dt, \qquad \Re(\nu)>1/2, a>0.$$

Meanwhile, for spacelike separations x, rotate to a frame where $x^\mu = (0,\vec{x})$, then

$$\begin{aligned} D_F(x,0) =& \int_0^\infty dk\;\frac{k\sin(kx)}{4\pi^2 x\sqrt{\vec{k}^2+m^2-i\epsilon}} \\ =& \frac{m}{4\pi^2 x}K_1(mx), \end{aligned} \qquad \text{(spacelike)}$$

where we have used another integral representation of the modified Hankel function $K_\nu(a) = i^{\nu+1}(\pi/2)H_\nu^{(1)}(ia)$. Note that K_1 is simply H_1 at imaginary arguments (up to some constant factors), so this spacelike case could have been obtained from the timelike case by analytic continuation.

Finally, for lightlike x, notice that, writing $2i\sin(kx) = e^{ikx}-e^{-ikx}$, the numerator of the integrand consists of the exponentials $e^{-i\omega_{\vec{k}}|x^0|\pm ikx}$, which, since $x^\mu x_\mu = 0$, approaches 1 as $\left|\vec{k}\right| \to \infty$. Therefore, at large k, the integrand is of order 1, so $D_F(x,0)$ diverges. Since this is an ultraviolet divergence, we can compute it in the massless limit (ie. consider the region with $m/k \ll 1$). Then

$$\begin{aligned} D_F(x,0)|_{m=0} =& \frac{-i}{8\pi^2 x}\int_0^\infty dk\;(e^{-ik(|x^0|-x)} - e^{-ik(|x^0|+x)}) \\ =& \frac{-i}{8\pi x}(\delta(x-\left|x^0\right| - \delta(x+\left|x^0\right|)) \\ =& \frac{-i}{4\pi}\delta(x^\mu x_\mu). \end{aligned} \qquad \text{(lightlike)}$$

Putting this all together, and restoring Lorentz invariance, the final result for the Feynman propagator is

$$D_F(x,0) = \theta(-x^2)\frac{im}{8\pi\sqrt{-x^2}}H_1^{(1)}(m\sqrt{-x^2}) + \theta(x^2)\frac{m}{4\pi^2\sqrt{x^2}}K_1(m\sqrt{x^2}) + \frac{-i}{4\pi}\delta(x^2).$$

In the timelike region, $H_1^{(1)}$ is an outgoing wave, while in the spacelike region, K_1 is exponentially decaying.[2] This is indeed what we expect for a solution to the wave equation.

In the $m^2 \to 0$ limit, using the Bessel function asymptotics $K_1(z) = 1/z + O[z]$, $H_1^{(1)}(z) = -2i/\pi z + O[z]$, or by computing the integral form of D_F directly, we obtain the conformal propagator

$$D_F(x,0) = \frac{1}{4\pi^2 x^2} + \frac{-i}{4\pi}\delta(x^2) = \frac{1}{4\pi^2(x^2+i\epsilon)},$$

where the useful identity $1/(a+i\epsilon) = \text{p.v.}\,1/a - i\pi\delta(a)$ of distributions has been used.

[1] Bessel functions satisfy a wealth of identities. For more information, see eg. Arfken and Weber's Mathematical Methods for Physicists.

[2] A useful mnemonic for the asymptotic behaviors of the Bessel functions is that J behaves like cos, N like sin, $H^{(1)} = J+iN$ like e^{iz} and $H^{(2)} = J-iN$ like e^{-iz}. Then $K(z) \sim H^{(1)}(iz)$ is exponentially damped and $I(z) \sim H^{(2)}(iz)$ grows exponentially.

6.2 The advanced and retarded propagators are (6.27)

$$D_\pm(x,0) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-i\vec{k}\vec{x}}e^{\pm i\omega_k x^0}\theta(\mp x^0).$$

To write them as d^4k integrals, we want to write the integrand, $e^{\pm i\omega_k x^0}\theta(\mp x^0)/2\omega_k$, as a dk_0 integral, as we did for the Feynman propagator above. The integral

$$-\frac{1}{2\pi i}\int \frac{e^{ik_0x^0}dk_0}{k_0^2-\omega_k^2},$$

has two poles at $\pm\omega_k$. We have to specify which way the contour goes round the poles, in order that we use complex analysis to compute the integral. In the Feynman propagator, we chose a contour that went below the pole at $-\omega_k$ and above the pole at ω_k (see fig 6.1). For positive x^0, we close the integral in the upper half plane, and the residue of the pole at $-\omega_k$ gave us exactly

$$-\theta(x^0)\frac{1}{2\pi i}\int \frac{e^{ik_0x^0}dk_0}{k_0^2-\omega_k^2+i\epsilon} = \theta(x^0)\frac{e^{-i\omega_k x^0}}{2\omega_k},$$

which is what we desire. A similar argument holds for the advanced propagator, so

$$D_\pm(x,0) = \int \frac{d^4k}{(2\pi)^4}\frac{-ie^{-ikx}}{k^2+m^2-i\epsilon}\theta(\mp x^0) = \theta(\mp x^0)D_F(x,0).$$

Note that $D_\pm(x,0)$ are not Lorentz invariant. (In particular, they are not inverses of the Klein-Gordon operator, and so they should really not be called "propagators".)

Remark

Most other sources usually define the advanced and retarded Green functions to be the expectation value of the commutator,

$$\tilde{D}_\pm(x,y) := \theta(\mp(x^0-y^0))\langle 0|[\phi_0(x),\phi_0(y)]|0\rangle.$$

In this case, the integrand will contain both $\theta(\mp x^0)(e^{\pm i\omega_k x^0}+e^{\mp i\omega_k x^0})/2\omega_k$ terms, which arise as the residue of *both* poles at $\pm\omega_k$. Then, there is a simpler way of writing the d^4k integral, by placing the contour below (resp. above) both poles for the retarded (advanced) propagator. This may be achieved by displacing $k_0 \mapsto k_0 - i\epsilon$ ($k_0 \mapsto k_0 + i\epsilon$), so

$$\tilde{D}_\pm(x,y) = \int \frac{d^4k}{(2\pi)^4}\frac{-ie^{-ikx}}{-(k_0\pm i\epsilon)^2+\vec{k}^2+m^2}.$$

(Exercise: check that this indeed yields the desired expression!) There is no need for an explicit θ term, so this definition of the advanced and retarded Green functions is indeed Lorentz invariant.

2. The idea is to compare two methods of computing the commutator $[\varphi(x),\dot{\varphi}(y)]$; using canonical quantization and the Lehmann-Källén exact propagator. It is straightforward to show that, using canonical quantization, we have at equal times

$$Z_\varphi[\varphi(x),\dot{\varphi}(y)] = [\varphi(x),\Pi(y)] = i\delta^3(\mathbf{x}-\mathbf{y}).$$

Meanwhile, take the y^0 derivatives of (13.12), (13.13) to obtain, at equal times,

$$\begin{aligned}\langle 0|\varphi(x)\dot{\varphi}(y)|0\rangle =& \int \frac{d^3k}{(2\pi)^3 2\sqrt{\vec{k}^2+m^2}} i\sqrt{\vec{k}^2+m^2} e^{ik(x-y)} + \int_{4m^2}^{\infty} \rho(s) ds \int \frac{d^3k}{(2\pi)^3 2\sqrt{\vec{k}^2+s}} i\sqrt{\vec{k}^2+s} e^{ik(x-y)} \\ =& \frac{i}{2}\delta^3(\mathbf{x}-\mathbf{y})\left(1+\int_{4m^2}^{\infty}\rho(s)\, ds\right).\end{aligned}$$

Hence

$$\langle 0|[\varphi(x),\dot{\varphi}(y)]|0\rangle = i\delta^3(\mathbf{x}-\mathbf{y})\left(1+\int_{4m^2}^{\infty}\rho(s)\, ds\right),$$

and comparing the two expressions, we conclude that

$$Z_\varphi = \left(1+\int_{4m^2}^{\infty}\rho(s)\, ds\right)^{-1}.$$

The Lehmann-Källén exact propagator yields an easy way to compute the wavefunction normalization.

3. 1.2 We are to verify that the first and second quantized formulations of nonrelativistic quantum mechanics are equivalent. Let us compute $H|\psi,t\rangle$ in second quantized language. First, consider the single particle terms $H_1 = \int d^3x\, a^\dagger(\mathbf{x})(-\nabla^2/2m + U(\mathbf{x}))a(\mathbf{x})$. We use the (anti)commutation relations to move $a(\mathbf{x})$ past the $a^\dagger$s in $|\psi,t\rangle$, yielding n delta functions at each of the positions. Each delta function $\delta^3(\mathbf{x}-\mathbf{x}_j)$ kills the $\int d^3x$ integral, setting $\mathbf{x}$ to $\mathbf{x}_j$. (For the spatial derivatives, partially integrate twice, yielding $\nabla^2_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{x}_j) = \delta^3(\mathbf{x}-\mathbf{x}_j)\nabla^2_{\mathbf{x}_j}$.) Finally, we can move the $a^\dagger(\mathbf{x})$ from the front to the jth position. Note that the number of $a^\dagger$s we have to anticommute the $a^\dagger(\mathbf{x})$ and $a(\mathbf{x})$s through are the same, so this holds for both bosons and fermions. Thus

$$\begin{aligned}H_1|\psi,t\rangle =& \sum_{j=1}^{n}\int d^3x \prod d^3x_j\, (-)^{j+1} a^\dagger(\mathbf{x})\left(-\frac{1}{2m}\nabla^2 + U(\mathbf{x})\right)\delta^3(\mathbf{x}-\mathbf{x}_j) \\ & \psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)a^\dagger(\mathbf{x}_1)\ldots \cancel{a^\dagger(\mathbf{x}_j)}\ldots a_\dagger(\mathbf{x}_n)|0\rangle \\ =& \sum_{j=1}^{n}\left(\frac{1}{2m}\nabla^2_{\mathbf{x}_j} + U(\mathbf{x}_j)\right)\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)|0\rangle.\end{aligned}$$

Similarly, for the two particle term $H_2 = 1/2\int d^3x\, V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})$, (anti)commuting the as past the $a^\dagger$s yields delta functions setting $\mathbf{x}$ and $\mathbf{y}$ to any pair of positions $\mathbf{x}_j, \mathbf{x}_k$, with $j \neq k$, and the $a^\dagger(\mathbf{x}), a^\dagger(\mathbf{y})$ then take the original spots, with the same number of (anti)commutations required. (Notice that this is only true for this specific order, $a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})$!) Each pair (j,k) is counted twice, so the result is

$$H_2|\psi,t\rangle = \int \prod d^3x_j \sum_{j<k}^{n} V(\mathbf{x}_j - \mathbf{x}_k)\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)|0\rangle.$$

Therefore

$$H|\psi,t\rangle = \int \prod d^3x_j \tilde{H}\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)|0\rangle,$$

where $\tilde{H}$ is the first quantized hamiltonian. It is clear from this equation that the Schrödinger equations in the first and second quantized pictures are equivalent.

1.3 Notice that $N, a^\dagger$ and a satisfy the harmonic oscillator algebra, $[N, a^\dagger(\mathbf{x})\} = a^\dagger(\mathbf{x})$, $[N, a(\mathbf{x})\} = -a(\mathbf{x})$.[3] Therefore, N simply counts the number of particles, which is to say, given a chain with n creation operators $a^\dagger$s and m annihilation operators as, $[N, a^\dagger a a a^\dagger a \ldots] = [N, a^\dagger] a a a^\dagger a \ldots + a^\dagger [N, a] a a^\dagger a \ldots + \ldots = (n-m) a^\dagger a a a^\dagger a \ldots$. (Note that N is a bosonic operator even if a is fermionic.) Since H is particle number conserving (its terms have equal numbers of as and $a^\dagger$s), it follows that $[N, H] = 0$.

[3] $[A, B\}$ denotes the graded bracket, which is the anticommutator if A and B are fermionic, and the commutator otherwise.